26 Inversive Geometry

An inversion q (in a circle μ) is a transformation which behaves very much like a reflection r (in a line m). Just as reflections (and the other isometries they generate) make Euclidean geometry 'tick', we shall see that inversions make inversive geometry 'tick'. In inversive geometry, the fundamental objects of study are circles and lines: nevertheless, we shall encounter a variety of beautiful theorems concerning these familiar objects that are likely to be totally new to you.

Inversive geometry also opens a door into the world of non-Euclidean geometry. Indeed, inversions in the Euclidean plane mimic the behaviour of reflections in the bizarre world of the non-Euclidean plane.

26.1 Inversions

Definition 19 *Suppose μ is a circle, with centre C and radius k. Now with respect this circle, we define the* inverse *of any point P (except for the centre C) to be the unique point P' on the ray CP such that*

$$(CP)(CP') = k^2$$

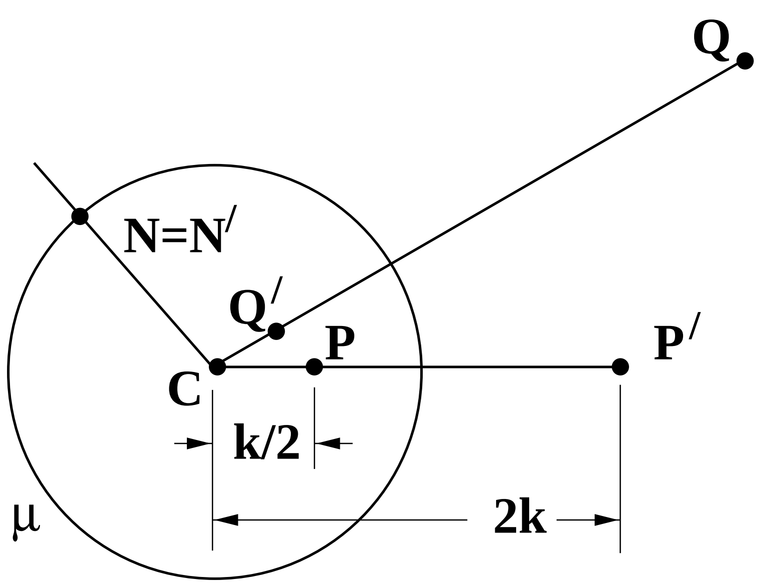


Figure 92: Inversion in the circle μ.

(a) It is important that you understand the several consequences of this definition. As you study these properties with the aid of the above diagram, try to determine the analogous statement for an ordinary reflection r.

(i) If P' is the inverse of P, then P is the inverse of P'. Thus $P'' = P$.

(ii) The inverse of the point N is $N' = N$ itself precisely when N lies on the circle μ.

(iii) Each point Q outside the circle has inverse Q' inside the circle. Each point $P \neq C$ inside the circle has inverse P' outside the circle. As P approaches C, the distance CP approaches 0, so that in compensation CP' must become very large.

(b) *The Problem with the Centre C.* The observations made in (a) do suggest that the mapping $P \to P'$ defines a transformation which behaves rather like an ordinary reflection. Yet there is one mild problem - a transformation q of the plane must without exception be defined for every point P of the plane, and each point P must itself be the image of some point Q.

Now look again at the centre C: it has no inverse, nor is it the inverse of any point outside the circle μ. There is a very convenient and useful remedy for this situation. We simply invent a new

point at infinity (denoted ∞)

to serve as the image of C.[18]

(c) *The Point at Infinity.* We shall agree upon the following reasonable properties of the point ∞ at infinity.

(i) We noted above that as $P \to C$ along any ray emanating from C, then $P' \to C' = \infty$ on this ray as well. Thus the point at infinity lies on every line through C; in fact we want the same point ∞ to work this way for all the different inversions in all possible circles of the plane. Let us summarize these first properties of the point at infinity as follows:

[18] You may object, 'How can you just invent a new point to suit your mathematical tastes?' My answer is that all mathematics is invention; we shall essentially accept another axiom that there is a point at infinity satisfying certain reasonable properties. (These are summarized in **Inf-1** to **Inf-8** in the text.) It is possible to show that the introduction of a single point at infinity does not contradict what we have achieved so far in geometry. As a bonus, we shall see that the point at infinity serves to unify and increase the elegance of many statements in the geometry of circles and lines.

Inf-1 There is a single point ∞ which lies on every line of plane. It is convenient to think of ∞ as being 'infinitely far removed' on all lines at once.

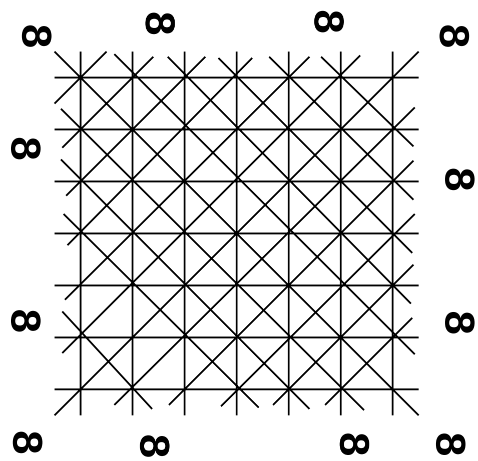


Figure 93: The point at infinity lies on *all* lines at once.

Inf-2 When we invert in any circle μ with centre C, each point has an inverse. In particular,

$$C' = \infty \text{ and } \infty' = C.$$

(ii) Draw a family of circles, each passing through a fixed point Q on a line b, but with centre P a point further and further out along the line b. As $P \to \infty$ along the line b, these circles look more and more like the line c perpendicular to b at Q.

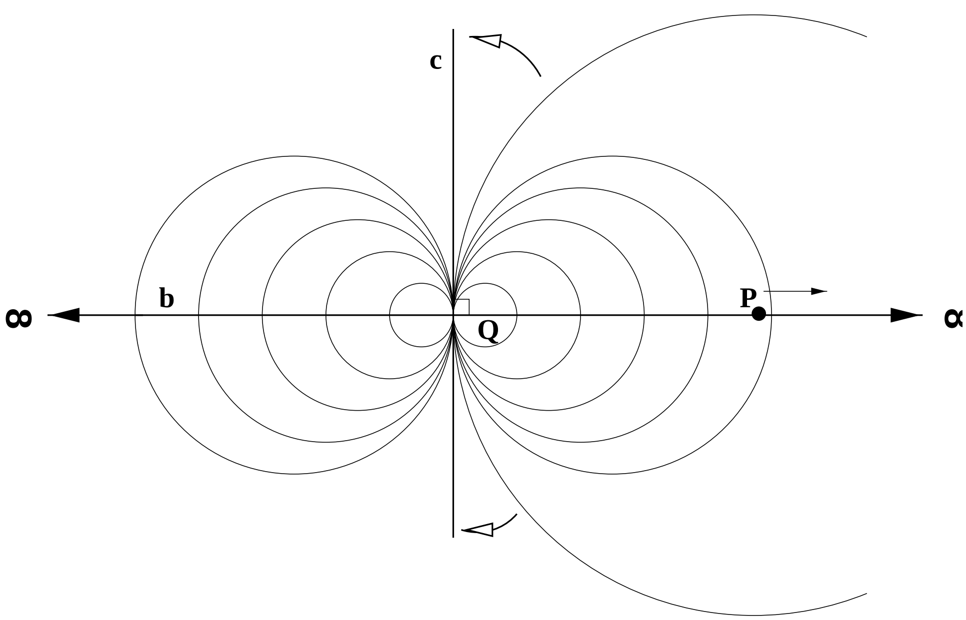


Figure 94: A pencil of tangent circles.

Thus, in some limiting sense, any straight line can (like c) be thought of as a *circle*.

Inf-3 Each straight line is a 'circle' (of infinite radius and with the point ∞ as centre). Thus any straight line both passes through ∞ and has ∞ as its centre.

Inf-4 We shall henceforth use the word *circle* (usually in italics) to denote either a straight line or an ordinary circle. Thus a *circle* passes through ∞ if and only if it is actually a straight line.

(d) The Euclidean plane, together with the point ∞ lying on all straight lines, is called the *inversive plane*. Thus each ordinary circle μ (with centre C and radius k) defines an *inversion*

$$q : P \to P' ,$$

which we now understand to be a *transformation of the whole inversive plane*.

26.2 Constructions

Using ruler and compasses we can construct the inverse P' of any point P in the circle μ. In order to understand these constructions and further inversive theorems, one must be familiar with the basic circle theorems, which are outlined as problems, starting on page 60.

Suppose μ is a circle with centre C and radius k.

(a) If P lies outside μ, construct the circle with diameter CP, meeting μ at A and B. Then P' lies on AB and CP.

(b) If P lies inside μ, construct the line b perpendicular to CP, meeting μ at A. Construct line t tangent to μ at A, and meeting CP at P'.

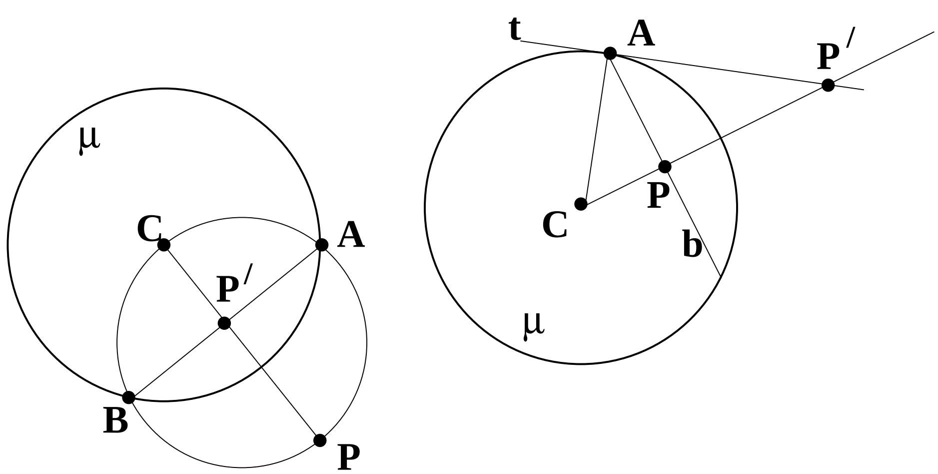


Figure 95: Ruler and compasses constructions for inverses.

26.3 Basic Theorems Concerning Inversion

(a) Many important inversive theorems concern the angles between two *circles*. If two *circles* μ and λ meet at the point A, then we measure the angle between μ and λ by means of the ordinary angle between the two tangent lines t_1 and t_2 at A.

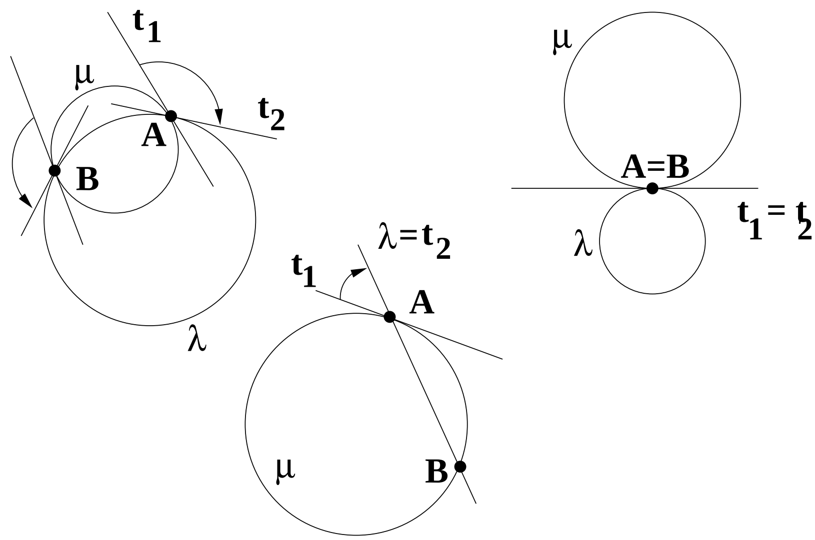


Figure 96: Angles between circles.

Remarks:

(i) By symmetry, if there are two points of intersection, then the two angles of intersection are equal (if we ignore the sense of the angles).

(ii) A straight line is considered to be tangent to itself at each of its points.

(iii) Suppose μ and λ meet in two points A and B. Now rotate just one of the circles about B so that A approaches B. Then in the limiting position, μ and λ become tangent at $A = B$. Furthermore, it makes sense to say that *tangent circles* meet at an angle $0°$.

(b) **Theorem 26.1** *Let P and P' be inverses with respect to the circle μ with centre C and radius k. Then every* circle *λ through P and P' is perpendicular to μ.*

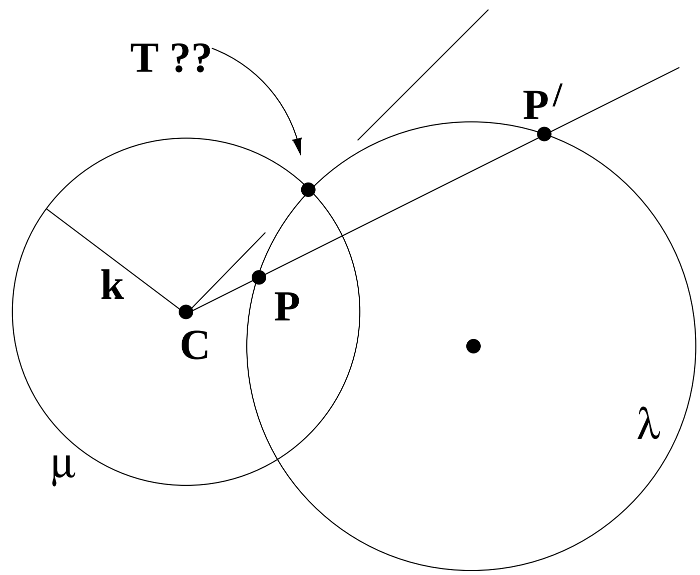


Figure 97: Inverses and perpendicular circles.

Proof. Let CT be tangent to λ at T. Although the figure is suggestive, we don't know for sure that T lies on μ. But by problem 3g on page 61, we have

$$(CT)^2 = CP\,CP' = k^2 \quad ,$$

so that $CT = k$, which means T is in fact on μ. But the tangent CT is perpendicular to the radius of λ at T and this radius must be perpendicular to circle μ. In other words, the two circles are perpendicular where they meet at T. //

Corollary 26.2 *The inverse of P in μ is the second point of intersection of* any *two circles λ_1 and λ_2 which both pass through P and are perpendicular to μ.*

Remarks: (i) Be sure to understand this theorem in the special case that λ is the straight line PP'. Also consider the case that $P = C$ and $P' = \infty$.

(ii) Note that Corollary 26.2 gives an alternative definition for inversion in a *circle* μ. The beauty of this definition is that it works just as well when μ is actually a straight line m. (Look again at problem 1 on page 38.

Inf-5 It makes sense to consider the reflection in the line m to be the inversion in the *circle* m. Thus each reflection is an *inversion.* Reflections are the only inversions which fix the point at infinity.

(c) **Theorem 26.3** (converse to Theorem 26.1) *Suppose μ and λ are perpendicular* circles*. Then any line through the centre C of μ, which meets λ, does so in inverse points P and P'.*

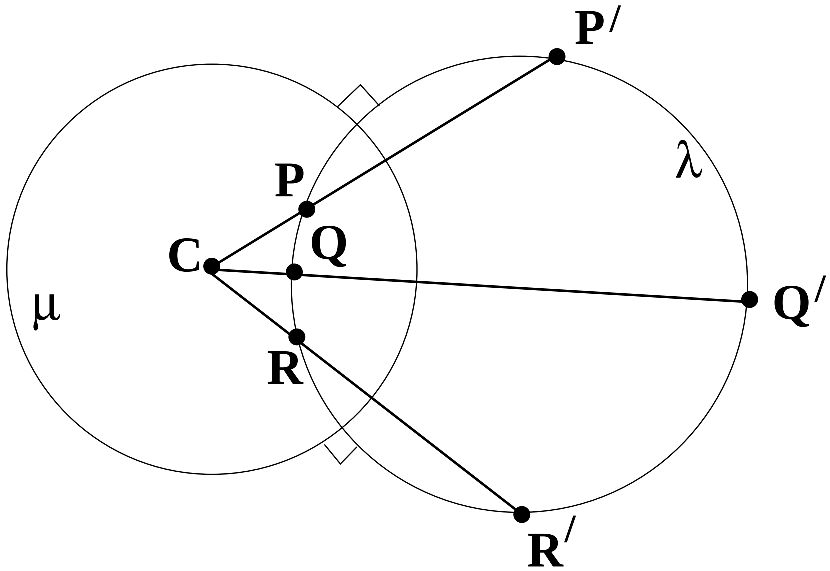


Figure 98:

Proof. This similarly uses problem 3g on page 61. //

Remarks: (i) Note the effect of the $\mu-$inversion on the arcs of the circle λ and on the points inside μ but outside λ.

(ii) Given two points A and B and a *circle* μ, we can now construct the *circle* λ through A and B which is perpendicular to μ.

(iii) Given a point A and two *circles* μ_1 and μ_2, we can also construct the circle through A which is perpendicular to both *circles*.

26.4 Inversion and Shape

(a) We can consider any isometry as acting on the inversive plane; we merely insist that each isometry fix the point at infinity. This is reasonable since each isometry q maps straight lines to straight lines; but since any two straight lines meet in ∞, we conclude that q must fix ∞.

Now isometries (reflections, glides, translations and rotations) by their very nature preserve the shapes and sizes of objects in the plane. (Recall that isometries preserve distance.) Despite some resemblance with reflections, inversions are very different. Inversions generally distort distance, although we shall see that they do preserve angles. We next investigate the effect that an inversion q has on a line or circle in the plane.

(b) **Theorem 26.4** *Let q be the inversion in the circle μ with centre C and radius k. Then*

(Inf-6) *each line through the centre C is mapped to itself by q. In particular the points C and ∞ on each such line are interchanged, as are certain other pairs of points. But the line as a whole is preserved.*

Proof. We need only note that q sends each point P on ray CP to another point P' on the same ray. //

(c) **Theorem 26.5** *Again let q be the inversion in the circle μ with centre C and radius k. Then*

(Inf-7) *each line n not passing through C is mapped by q onto a circle ν which does pass through C, and vice versa. More precisely, if the line CA is perpendicular to n with A on n, then ν has diameter CA' (where A' is the inverse of A).*

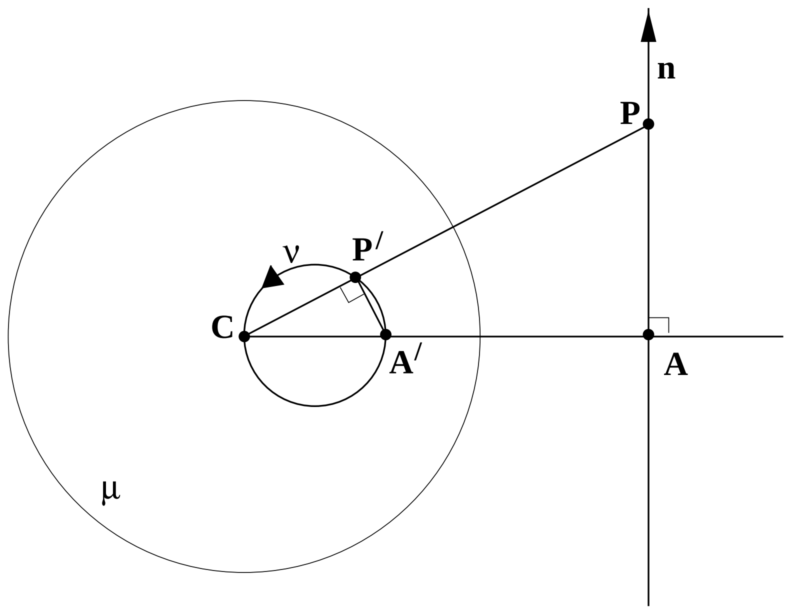


Figure 99:

Proof. Since $(CA)(CA') = k^2 = (CP)(CP')$, we conclude that

$$\frac{CA}{CP} = \frac{CP'}{CA'} \quad .$$

Since $\angle C$ is common, we have $\triangle CA'P' \sim \triangle CPA$. Therefore,

$$\angle CP'A' = \angle CAP = 90^\circ \quad .$$

Now compare problem 3b on page 61. It follows easily from this that P' must lie on the circle ν with diameter CA'. In short, the inverse image of line n (through ∞, but not through C) is a proper circle ν (through C, but not through ∞). //

(d) Theorem 26.5 describes the effect of q on circles passing through the centre C of inversion. The remaining circles are moved among themselves.

Theorem 26.6 *Once more let q be the inversion in the circle* μ *with centre* C *and radius* k.
(**Inf-8**) *Then each circle* λ *not passing through* C *is inverted into another circle* λ' *of the same type. In fact, the circles* λ *and* λ' *can be identical: any circle* λ *perpendicular to the circle of inversion is inverted into itself (as a whole— the points on such a circle are moved in pairs).*

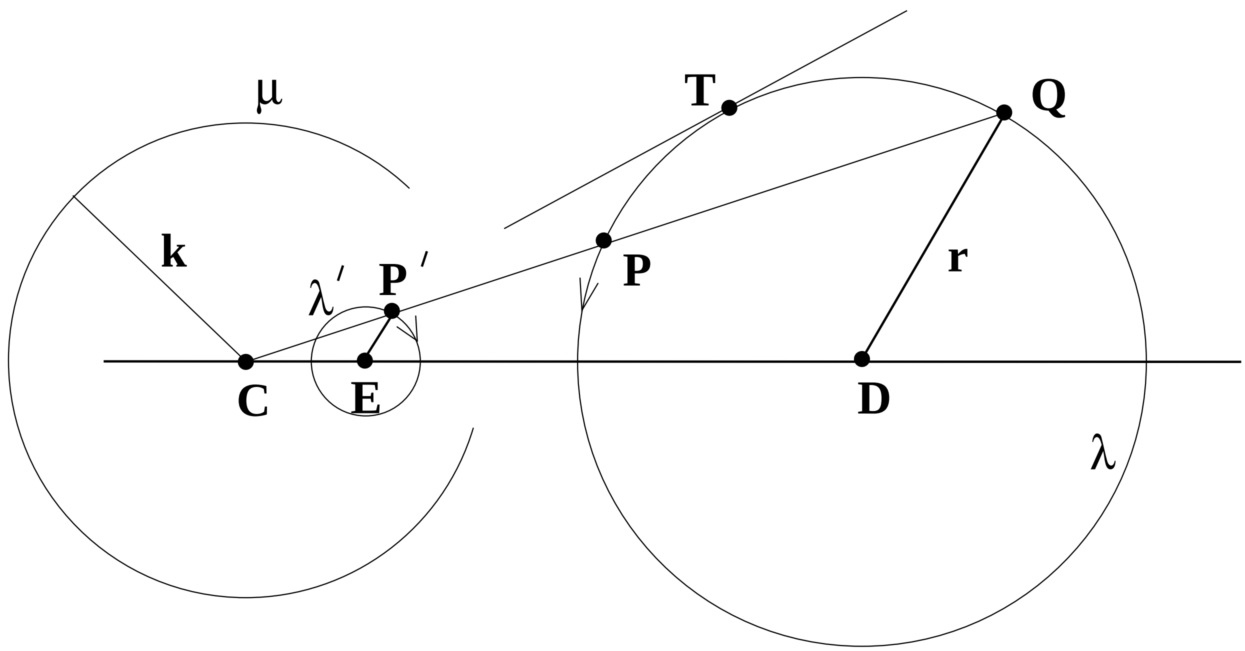


Figure 100:

Proof. Suppose λ has centre D and radius r, as in the figure. Let P be any point on λ, with inverse P' in μ. Let CP meet λ again at Q, and draw $P'E \| QD$, with E on line CD. Thus

$$\frac{CE}{CP'} = \frac{CD}{CQ} \quad .$$

Now as P moves along λ, P' moves along some curve λ', which we suspect is a circle, but maybe isn't! For example, perhaps it happens that E moves about on line CD as P moves along λ. Let's see what happens for sure.

In fact, from the equation just above,

$$CE = CD\frac{CP'}{CQ} = CD\frac{(CP')(CP)}{(CQ)(CP)} = \frac{(CD)k^2}{(CT)^2} \ ,$$

where CT is the (fixed!!) tangent from C to circle λ. (Again we use problem 3g on page 61 .) Thus E is a *fixed* point, and

$$P'E = \frac{r(CE)}{(CD)} = r' \quad \text{(another constant)} \ .$$

Hence, P' traces out the circle λ', with radius r' and centre E. It is worth noting that the new centre E is *not* the inverse image of D in μ. (Prove this!) //

(e) We can summarize these facts as:

Inf-9 Let q be the inversion in the *circle* μ with centre C and radius k. Then each *circle* is inverted into another *circle* (where a straight line is considered to be *circle* through ∞). The *circle* μ is fixed pointwise by q. Otherwise, a *circle* λ is mapped onto itself precisely when it is perpendicular to μ.

Remark: It is useful exercise to verify the truth of each of the above statements when *circle* means straight line. Remember that if μ is a straight line, then the inversion q is actually the reflection in the line μ.

(f) We can now explain the way in which inversions preserve angles, even though they distort distances. Suppose that two lines n and l meet at P. To unambiguously describe the resulting angle we shall in fact use directed lines: so put arrows on n and l, and let θ be the angle from the arrow on l to the arrow on n.

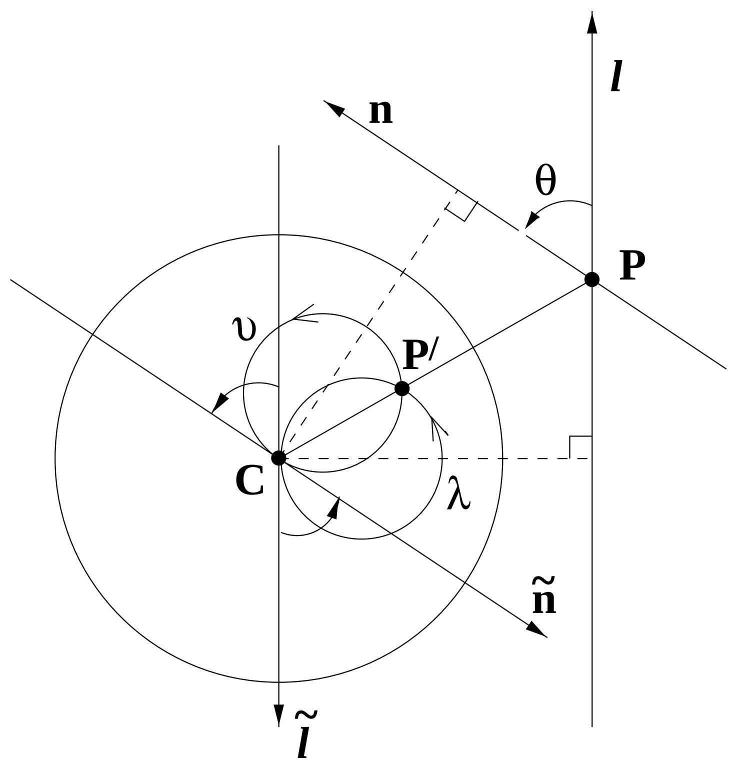


Figure 101:

Now invert to get circles λ and ν meeting at P' and at $C = \infty'$. Note that the tangents $\tilde{l}$ and $\tilde{n}$ at C are respectively parallel to l and n. Let a point Q move in the positive direction on l; the corresponding point Q' on λ tells us how we should assign a direction to the circle λ. Also, the tangents $\tilde{l}$ and $\tilde{n}$ have been directed in a consistent manner.

Now we easily see that

$$\begin{aligned}
\text{the angle } \theta \text{ from } l \text{ to } n \quad &= \text{the angle from } \tilde{l} \text{ to } \tilde{n} \\
&= \text{the angle from } \lambda \text{ to } \nu \text{ at } C \\
&= \text{the angle from } \nu \text{ to } \lambda \text{ at } P' \\
&\quad \text{(by symmetry).}
\end{aligned}$$

In short, any small angle at P is inverted into the equal angle of opposite sense at P'. A transformation which thereby preserves small angles (but not necessarily their sense) is said to be *conformal*.

Theorem 26.7 *Any inversion is a conformal mapping.*

There are many applications of this notion of conformal mapping; it is important not only in geometry but also in complex analysis and many other branches of mathematics.

The group generated by all possible inversions is called the *Möbius group*, denoted **Möb**. Thus every element of **Möb** is is a product of inversions; in fact, rather in the spirit of Corollary 19.6, it can be proved that each Möbius transformation u in **Möb** is a product of at most *four* inversions. Also, **Möb** acts sharply transitively on *clusters* of four mutually tangent *circles* in the inversive plane.

Every reflection is an inversion (in a straight line) so that every isometry is automatically a Möbius transformation: in other words, the Euclidean group **Isom** is a proper subgroup of **Möb**. In fact, one can prove that a Möbius transformation u is an isometry if and only if it commutes with the *antipodal map*

$$e : [x, y] \to \frac{-1}{x^2 + y^2}[x, y] \ .$$

26.5 Problems on Inversion

1. Let γ be the unit circle in the $x, y-$plane:

$$x^2 + y^2 = 1$$

   (a) Inversion in γ maps each point $P = (x, y)$ to its inverse $P' = (x', y')$. Write x' and y' in terms of x and y only.

   (b) Likewise describe the inversion in polar coordinates:

$$P = (r, \theta) \rightarrow P' = (r', \theta') \ .$$

   (c) Sketch and precisely describe the image (under inversion in γ) of the following curves:

      (i) the line $x = 3$

      (ii) the circle of radius 2 centred at $(1, 5)$

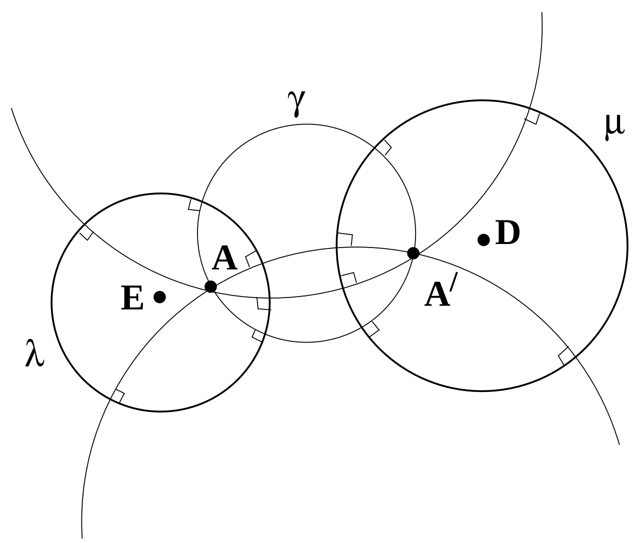
      (iii) $x^2 + y^2 = 2y$ (what curve is this, and where is it situated, anyway?)

      (iv) the parabola $y = \dfrac{1 - x^2}{2}$

      (v) the hyperbola $x^2 - y^2 = 1$

   (d) What are the proper names for the image curves in parts (iv) and (v) above?

2. You are given two non-intersecting circles λ and μ with centres E and D, respectively.



Our goal is to find two points A and A' such that *every* circle γ through A and A' is orthogonal to *both* λ and μ.

   (a) Give an obvious example of one such *circle* $\hat{\gamma}$. (Remember that in inversive geometry straight lines are considered to be circles of a special sort.)

   (b) To find A, A' we need only produce another such circle γ. Describe how to do this, and thus find A, A'. (This could be made into an R-C construction, but you need not perform the details. Just describe in words and sketches how to proceed in principle.)