20 The Addition Of Angles Theorem For Rotations

20.1 Products of Rotations

Suppose p and q are rotations (each is then direct). Thus $s = pq$ is also direct and is a rotation or translation. But in Section 18.5 we observed that a translation can just as well be considered as a rotation (through $0°$, with infinitely distant centre).

Theorem 20.1 (Addition of angles). *If rotations p, q have angles α, β respectively, then $s = pq$ is a rotation with angle $\gamma = \alpha + \beta$. In particular, if $\alpha + \beta$ is any integral multiple of $360°$, then s is a translation.*

Remark. The proof below contains a procedure for determining the centre of the new rotation s, or the vector in case s happens to be a translation.

Proof . Suppose that p and q have centres A and B, respectively.

Case 1–*Equal Centres.* Note that $s = pq$ fixes A, since p and q each do so. It is now clear from the definition of rotation that s must itself be a rotation with centre A and angle $\alpha + \beta$.

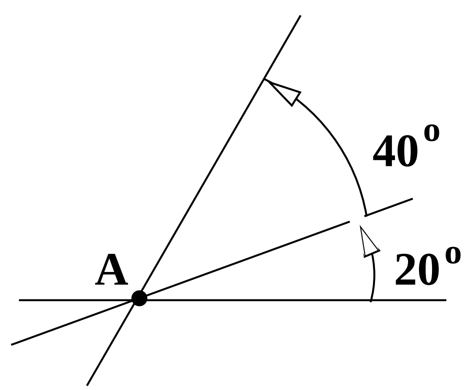


Figure 64: Rotations with the same centre.

In the above figure, for example, $\alpha = 20°$, $\beta = 40°$, and so s has the same centre and angle $\gamma = \alpha + \beta = 60°$.

Case 2–*Different Centres.* On the other hand, when p and q have different centres, so that $A \neq B$, then it is not at all obvious where the new centre C for pq is located.

To understand what is going on, we first let m_2 be the line through A and B (see Figures 65 and 66). Next draw the line m_1 through A such that the angle *from* m_1 *to* m_2 equals $\frac{\alpha}{2}$. Finally draw the line m_3 through B such that the angle *from* m_2 *to* m_3 equals $\frac{\beta}{2}$. As usual, we let r_j be the reflection in line m_j.

It is important to draw these various angles in the correct sense. For example, in Figure 66 we have $\alpha = +90°$, so that $\frac{\alpha}{2} = +45°$ (anti-clockwise). Also, $\beta = +60°$, so that

$\frac{\beta}{2} = +30°$ (again anti-clockwise). On the other hand, in Figure 65, we have $\alpha = +90°$ (anti-clockwise) whereas $\beta = -90°$ (clockwise).

Now let's proceed with the proof. It follows from Theorem 18.1 that

$$p = r_1 r_2 \text{ and } \mathrm{q} = \mathrm{r}_2 \mathrm{r}_3 \ ,$$

so that

$$s = pq = r_1 r_2^2 r_3 = r_1 r_3 \quad .$$

Note that lines m_1 and m_3 may or may not intersect in some point C, so that we must consider two subcases.

Subcase 2.1: Lines m_1 and m_3 are parallel (see Figure 65).

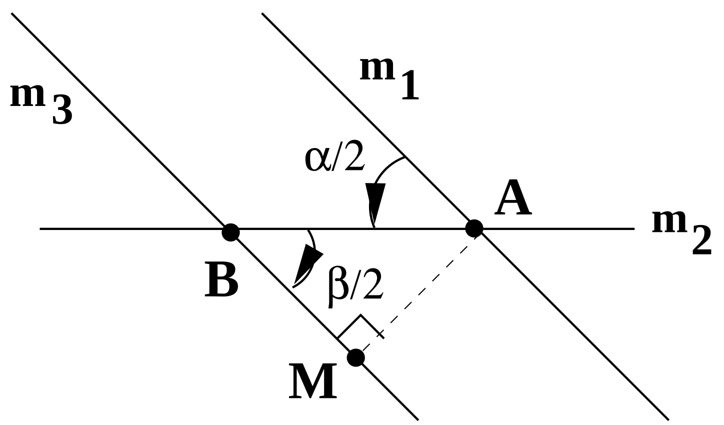


Figure 65: A product of rotations which is a translation.

Here we conclude at once from Theorem 18.3 that $s = r_1 r_3$ is a translation with vector $2\vec{AM}$, where M is the foot of the perpendicular from A to m_3. Note that the length of segment AM is given by

$$AM = AB \cdot |\sin(\frac{\beta}{2})| \quad .$$

We must also determine which angles α and β will produce this situation. By Theorem 5.2, the alternate angles at A and B must be equal. Here we tacitly assume that all angles are measured in the positive (anti-clockwise) sense: negative angular measurement is an innovation needed to cope with rotations. Thus we really have $\frac{\alpha}{2} = -\frac{\beta}{2}$, so that

$$\alpha + \beta = 0° \quad .$$

You might experiment with other angles, say $\alpha = 200°$ and $\beta = 160°$. You should then convince yourself of the following

Conclusion : m_1 is parallel to m_3 and $s = pq$ is a translation precisely when $\alpha + \beta$ is a multiple of $360°$ (that is, $\alpha + \beta = 0°, \pm 360°, \pm 720°$, etc.).

Subcase 2.2: Lines m_1 and m_3 intersect in some point C (see Figure 66). By our work in the previous subcase, we observe that $\alpha + \beta$ is *not* a multiple of $360°$.

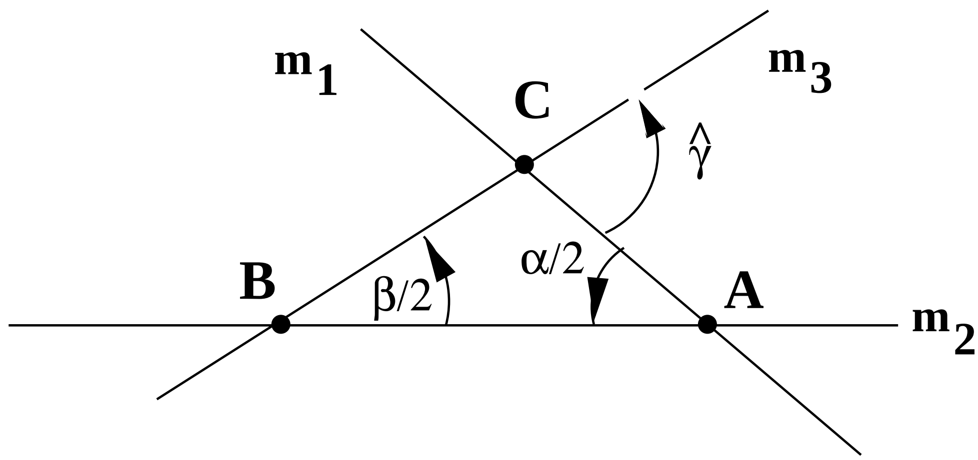


Figure 66: A product of rotations which is a proper rotation.

Now let $\hat{\gamma}$ be the angle *from* m_1 *to* m_3. By Theorem 5.5 (concerning the exterior angle for a triangle), we have

$$\hat{\gamma} = \frac{\alpha}{2} + \frac{\beta}{2} \quad .$$

Thus, by Theorem 18.1, we conclude that

$$s = pq = r_1 r_3$$

is a rotation with centre C and angle

$$\gamma = 2\hat{\gamma} = \alpha + \beta \quad .$$

In other words, we have verified that the angles add as required, and moreover we may determine the centre C following the diagram in Figure 66. //

Corollary 20.2 *The product of two half turns h_A and h_M is the translation t with vector $2\vec{AM}$.*

Proof. This is a special case of the theorem, since

$$180° + 180° = 360° \quad .$$

But h_A fixes A so that $t = h_A h_M$ maps A to B where M is the midpoint of segment AB. Thus t has vector [12]

$$\vec{AB} \equiv 2\vec{AM} \quad .$$

[12]There is actually a very subtle point here concerning the correspondence between vectors and translations, which we explore in the next section.

20.2 Examples

Let $AB = 4$cm., let p be a $30°$ rotation at A, q a $-70°$ rotation at B, and w a $-30°$ rotation at B.

(a) Determine $s = pq$. This is a rotation with angle $30° - 70° = -40°$.

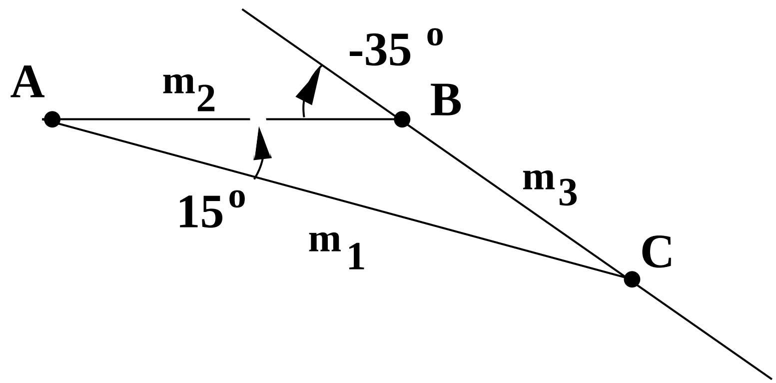


Figure 67:

Now construct the (clockwise) angle $-\frac{70°}{2} = -35°$ from AB to a new line m_3. Then with C on m_3 construct the (anticlockwise) angle $\angle CAB = \frac{30°}{2} = 15°$ from CA to AB. Hence pq has centre C.

(b) Find pw. The rotation pw has angle $30° - 30° = 0°$ and hence is a translation. In fact, if $w : A \to D$, then $pw : A \to D$ as well, so that pw must have vector $\vec{AD}$ (see Figure 68). There is no real need to display the reflections *once* we know that the result is a translation.

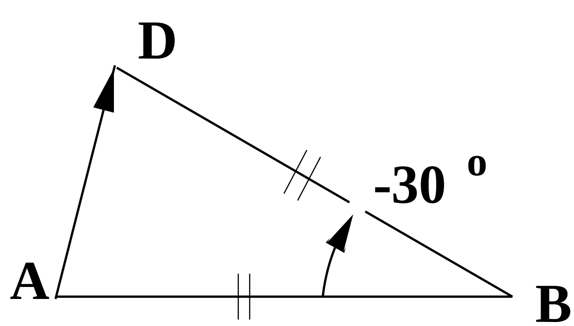


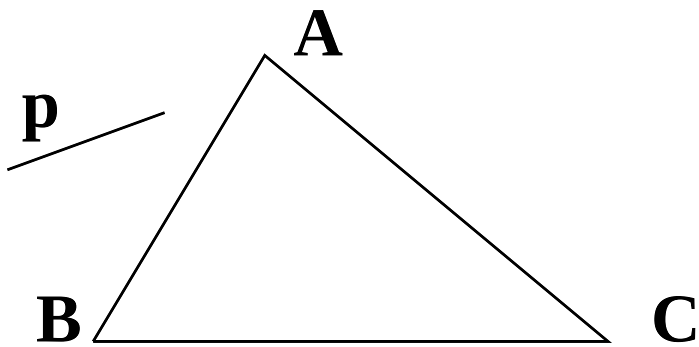
Figure 68:

In § 21, we shall look more closely at products of rotations and translations.

20.3 Some geometrical problems best solved by using isometries.

The following problems are most easily done using isometries.

1. In $\triangle ABC$ inscribe a line segment equal and parallel to a given segment p.



2. If squares are erected externally on the sides of a parallelogram, then their centres are the vertices of a square.

3. Construct an equilateral $\triangle ABC$ such that a given point P is 2 units from A, 3 from B, 4 from C.

4. Given any three parallel lines p, q, r, construct an equilateral $\triangle ABC$ with A on p, B on q, C on r.

5. Let A be one of the common points of two intersecting circles. Construct through A a line which is cut by the circles in two equal chords.

6. Prove that the midpoints of the edges of any quadrilateral form a parallelogram.

7. $\triangle ABC$ is scalene (all angles less than 90°). Where on side AB should a light source be placed so that a beam reflected successively by the other two sides returns to the source?

20.4 More Isometry Review Problems

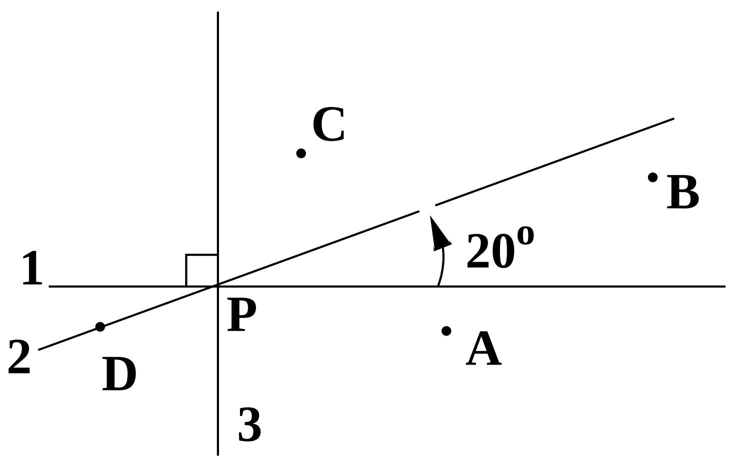
Instructions

- When asked to describe an isometry, state what type it is along with all pertinent information:

reflection	- give the mirror
rotation	- give the centre and angle (recall - clockwise angles are negative)
translation	- give the vector $\vec{AB}$
glide	- give the axis and shift vector.

- In the isometry problems below, r_i generally refers to the reflection in the line m_i, which we often simply label just i.

1. Copy the following figure accurately before answering the questions.



   (a) Where does $s = r_1 r_2$ send A, B, C and D? (Call the images A', B', etc.)

   (b) Use a ruler to compare AC with $A'C'$, and a protractor to compare $\angle ABC$ with $\angle A'B'C'$. How accurately did you determine A', B', C'?

   (c) Where does $u = r_2 r_1$ send A? Does $r_1 r_2 = r_2 r_1$?

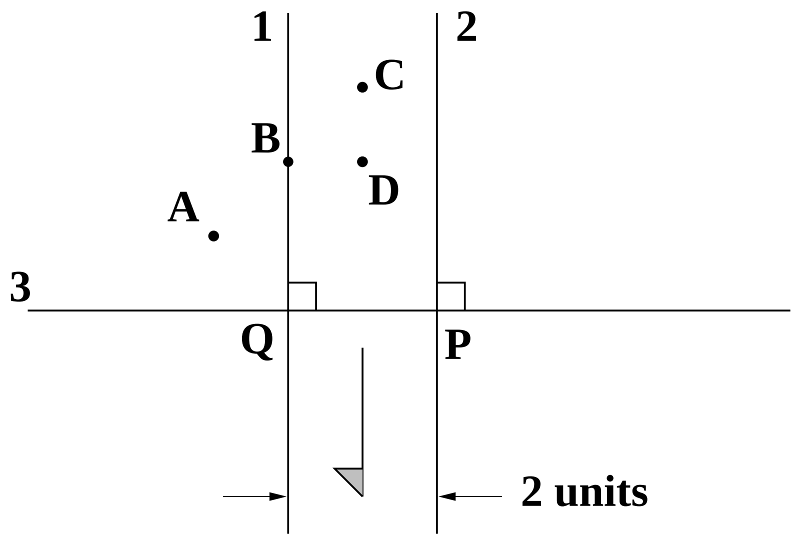
   (d) What isometry is $s = r_1 r_2$? What about $u = r_2 r_1$?

   (e) Write s^{-1} in terms of r_1 and r_2.

   (f) Determine the isometries s^2 , r_2^2 and s^9.

   (g) Describe $r_1 r_2 r_3$ and $r_1 r_2 r_1$.

2. Copy the following figure accurately before answering the questions.



(a) Describe $h = r_1 r_3$ and likewise $r_3 r_1$.
Why does $r_1 r_3 = r_3 r_1$? (Remember, this usually isn't true - see part (c) in question 1.)

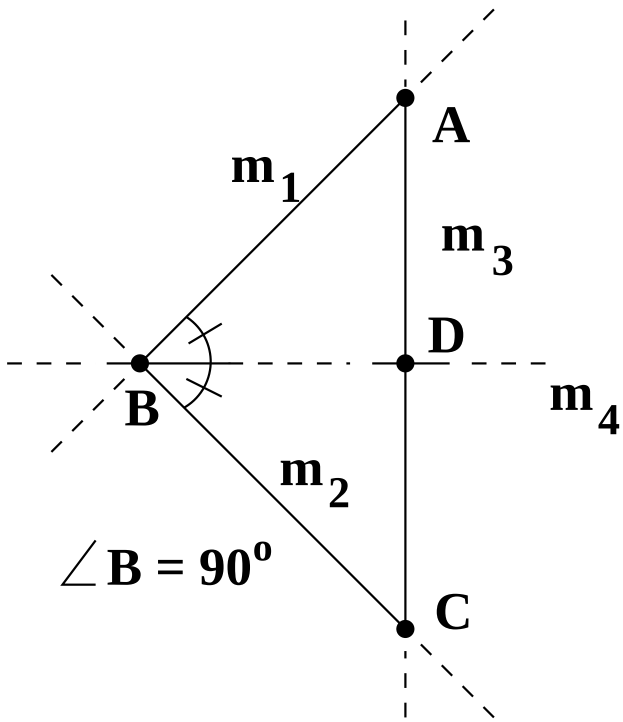
(b) What is $r_1 r_2$?

(c) What pattern results if we apply r_1 and r_2 repeatedly (in any order) to the 'flag' motif which is shown?

(d) Let $q = r_1 r_2 r_3$. Where does q send A, B, C, D? What isometry is q? Is it direct or opposite? Does it fix any points?

(e) What pattern results if we apply q repeatedly to the indicated motif?

3. Suppose lines m_1 , m_2 , m_3 enclose an isosceles right triangle $\triangle ABC$:



(a) Describe r_1r_2, r_1r_3 and r_2r_3.

(b) Find a mirror m_5 such that $r_1r_2 = r_5r_4$, where mirror m_4 bisects angle B as shown.

(c) Determine $q = r_1r_2r_3$.
Hint:

(i) From (b) substitute r_5r_4 for r_1r_2.

(ii) Why does $r_4r_3 = r_3r_4$?

(iii) Thus show $q = r_5r_3r_4 = tr_4$ where $t = r_5r_3$.

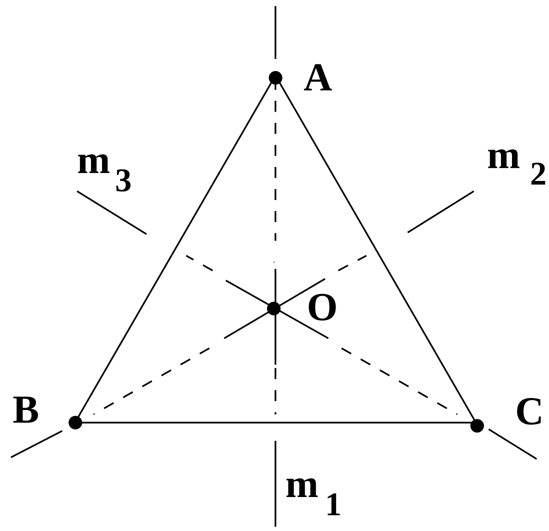
(iv) What is t?

4. As in problem 3, determine $q = r_1r_2r_3$, where mirrors m_1, m_2, m_3 are the sides of an equilateral triangle.

5. Let s be a $+60°$ rotation with centre A and let q be a $-60°$ rotation with centre B. Determine sq.

6. Let s be a rotation through α with centre A and let u be a rotation through $-\alpha$ with centre B. Determine su.

7. An equilateral triangle (denoted $\{3\}$) is clearly symmetric by reflections r_1, r_2, r_3 in lines m_1, m_2, m_3 through the centre O.



In order to clearly describe all symmetries we put labels - say $A, B, C-$ near the vertices of the triangle and keep these labels as well as lines m_1, m_2, m_3 **fixed in position**.

Thus, the reflection

$$r_1 : \begin{cases} A & \to & A \\ B & \to & C \\ C & \to & B \end{cases}$$

Note carefully that $B \to C$ means the vertex *initially* in position B ends up *finally* in position C.

(a) How many symmetries does this $\{3\}$ possess?

(b) Describe each symmetry by completing the following chart:

Name	Description	Effect
r_1	reflection in m_1	$r_1 : \begin{cases} A & \to & A \\ B & \to & C \\ C & \to & B \end{cases}$
	etc.	

(c) Describe $s = r_1 r_2$. What is s^{-1}?

– continued next page –

(d) Using the chart we may describe $q = r_1 r_3$ as follows:

$$\begin{array}{ccccc} & r_1 & & r_3 & \\ A & \to & A & \to & ? \\ B & \to & C & \to & ?? \\ C & \to & B & \to & ??? \end{array} \qquad \text{(Careful!)}$$

or, omitting the intermediate step

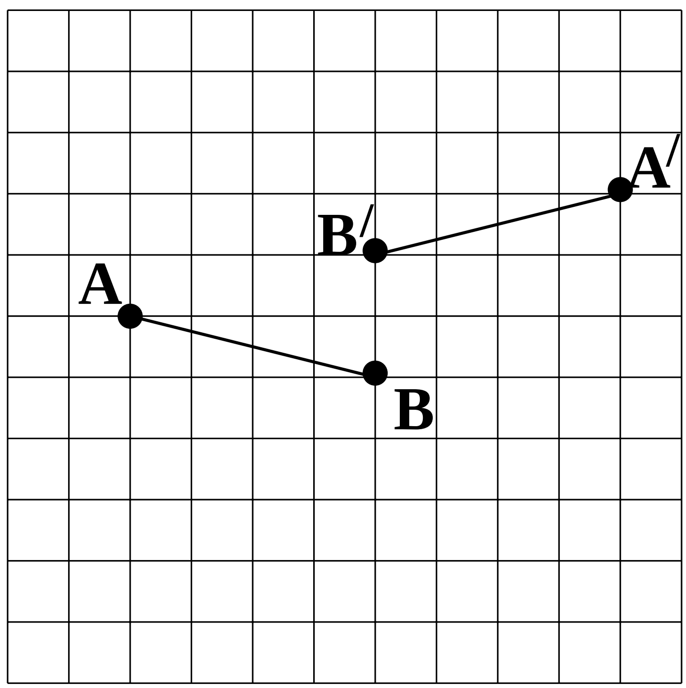
$$q = r_1 r_3 : \begin{cases} A & \to \quad ? \\ B & \to \quad ?? \\ C & \to \quad ??? \end{cases}$$

Thus identify q from your chart.

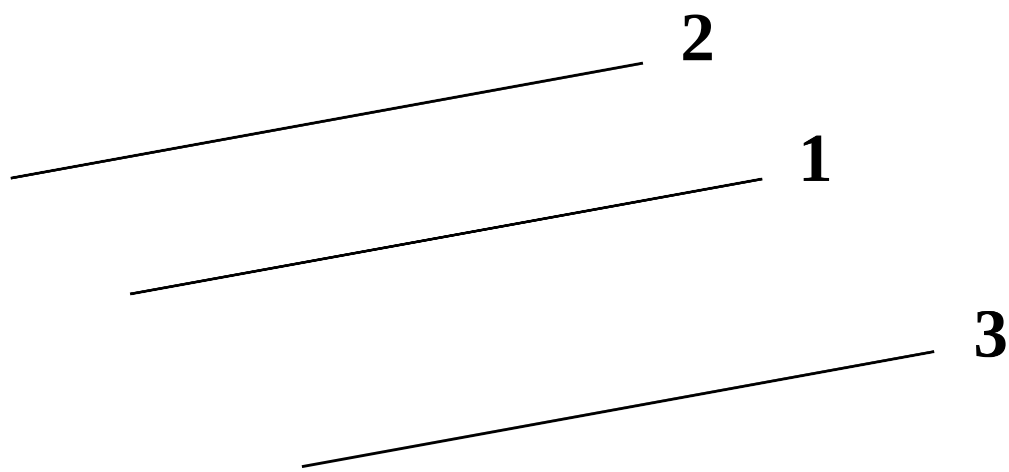
(e) Use this method to identify $w = r_1 r_2 r_1 r_3 r_1 r_2 r_3 r_1$.

(f) Write out a multiplication table for the symmetry group.

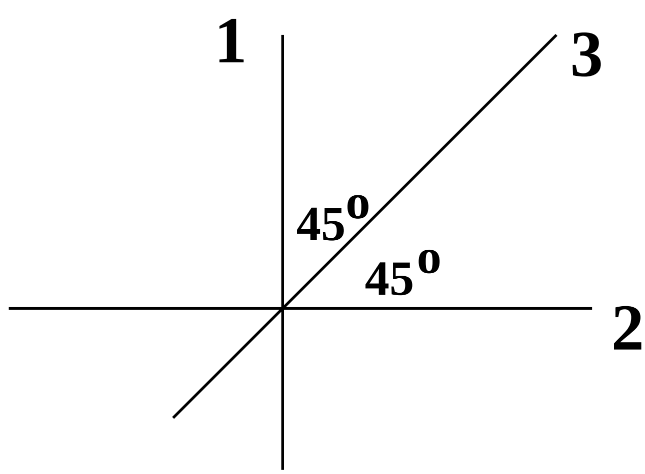
8. What isometries could possibly map both $A \to A'$ and $B \to B'$?



9. Describe $q = r_1 r_2 r_3$, where r_i is the reflection in line labelled i:



10. Let r_i be the reflection in the line labelled i:



Briefly explain your answers to the following questions:

(a) Does $r_1 r_3 = r_3 r_1$?

(b) Does $r_1 r_2 = r_2 r_1$?

(c) Does $r_2 r_3 = r_3 r_1$?

(d) Let $q = r_2 r_1 r_3$. Is q direct or opposite?

(e) What type of isometry is q? Describe q completely.

11. In the rectangle below, PQ is twice as long as PS:

Using the addition of angles theorem for rotations (20.1), find and describe pq where:

(a) $p = 60^\circ$ rotation at P.
$q = 37^\circ$ rotation at P.

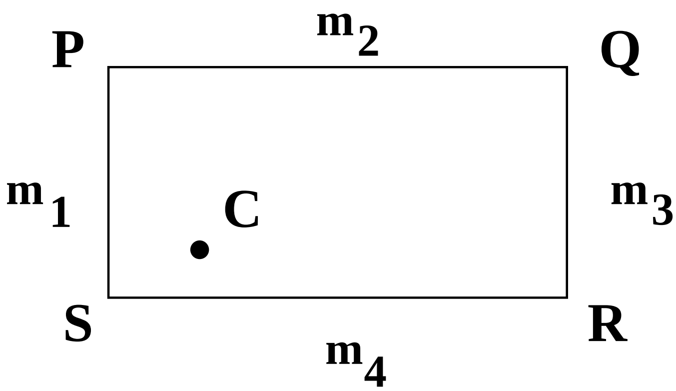
(b) $p = 45^\circ$ rotation at S.
$q = -45^\circ$ rotation at S.

(c) $p = 90^\circ$ rotation at P.
$q = 90^\circ$ rotation at Q.

(d) $p = 90^\circ$ rotation at P.
$q = -90^\circ$ rotation at Q.

(e) $p = 180^\circ$ rotation at P.
$q = 180^\circ$ rotation at R.

12. Review problem 1 (c) on page 36: shoot the cue ball C so that it hits mirrors m_4, m_3, m_2, m_1 in that order then returns to its original spot.



(a) Let $w = r_1 r_2 r_3 r_4$ and say $w : C \to C'$. Show that you should shoot for C'.

(b) But $w = pq$ where

$$\begin{aligned} p &= r_1 r_2 \\ q &= r_3 r_4. \end{aligned}$$

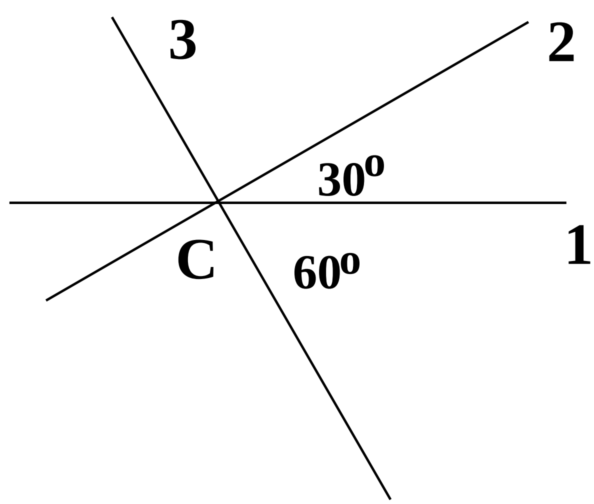
Describe the isometries p and q fully.

(c) Use one of the results in problem 11 to say what type of isometry w is.

(d) Thus briefly describe to a 'man on the street' how to solve this billiard table problem.

(e) How far has the ball travelled when it returns to its original position?

13. (a) Which isometry is $q = r_1 r_2 r_3$, where r_i is the reflection in line i?

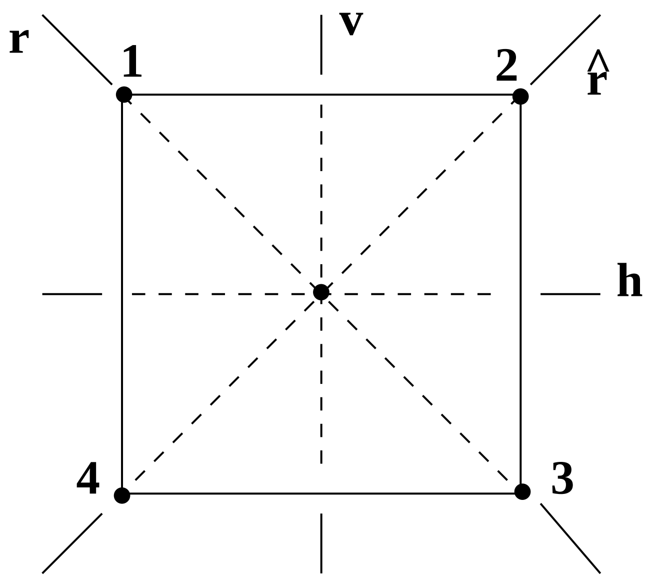


(b) Describe $s = r_1 r_3$. Describe r_3^{-1}.

14. Let h be a half-turn with centre C and let r be a reflection in line m.
Determine and describe the isometry $q = hr$. Distinguish between the cases when C does or does not lie on m.

15. The square shown here has eight symmetries:

1	= the identity
s_1	= 90° rotation about the centre
s_2	= 180° rotation about the centre
s_3	= 270° rotation about the centre
h	= reflection in a horizontal line
v	= reflection in a vertical line
r	= reflection in the 'down'diagonal
$\hat{r}$	= reflection in the 'up' diagonal.



Much as in problem 7, we suppose that the positions labelled 1, 2, 3, 4 are fixed. If you like, think of the square resting on a table, and paste labels 1, 2, 3, 4 on the table (*not* on the square). Likewise, the mirrors of reflection are fixed on the table, too. The square is movable, however.

Thus, for example, s_1 moves the vertex in position 1 to position 4, etc. which we indicate by

$$s_1 : \begin{bmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 1 & 2 & 3 \end{bmatrix} \quad \text{or better still,} \quad s_1 : \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix} .$$

(a) Likewise describe the remaining 7 symmetries.

(b) Use this scheme to identify $s_1^2 h$. Hint: $s_1^2 : 1 \to 3$ and $h : 3 \to 2$, so

$$s_1^2 h : \begin{bmatrix} 1 & & \\ & & \dots \\ 2 & & \end{bmatrix} .$$

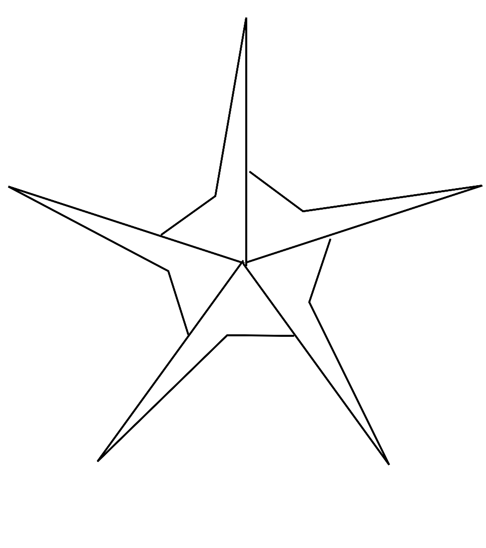
(c) Fill in the multiplication table for the symmetry group of the square (see below).

second term

first term

	1	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	**h**	**v**	**r**	$\hat{\mathbf{r}}$
1								
$\mathbf{s_1}$								
$\mathbf{s_2}$								
$\mathbf{s_3}$								
h								
v								
r								
$\hat{\mathbf{r}}$								

16. (a) Determine the symmetry group G for the figure below.
    (b) Write out its multiplication table.



21 A Closer Look At Translations And Vectors

21.1 The Translation Group T

We shall see here that the collection **T** of all translations is a group in its own right. Thus, **T** is a subgroup of the full Euclidean group **Isom**. However, unlike **Isom**, the translation group **T** is *commutative* (or *abelian*: see Theorem 21.3 below). For an alternate and very elegant treatment of these matters, we refer to [19, sections 1.5, 2.1, 2.2].

Now let us recall Definition 11: each directed line segment (i.e. vector) $\vec{AB}$ describes a translation t which moves each point P through the same distance and direction as $\vec{AB}$, thus along a line parallel to the line through A and B.

It is a subtle, though important fact, that 'equivalent' vectors describe equal translations:

Theorem 21.1 *Suppose that vectors $\vec{AB}$ and $\vec{CD}$ have the same length and sense along parallel lines, and let t and u be the corresponding translations. Then $t = u$.*

Proof (refer to Figure 69). Consider an arbitrary point P and suppose $t : P \to P'$, so that by definition we have $PP' \| AB$ and $PP' = AB$. But we are given $AB \| CD$, so that $PP' \| CD$, by Theorem 5.1. We are also given $AB = CD$ so that $PP' = CD$. Thus the translation u, with vector $\vec{CD}$ also maps P to P'. Since P is a general point, we conclude that $t = u$. //

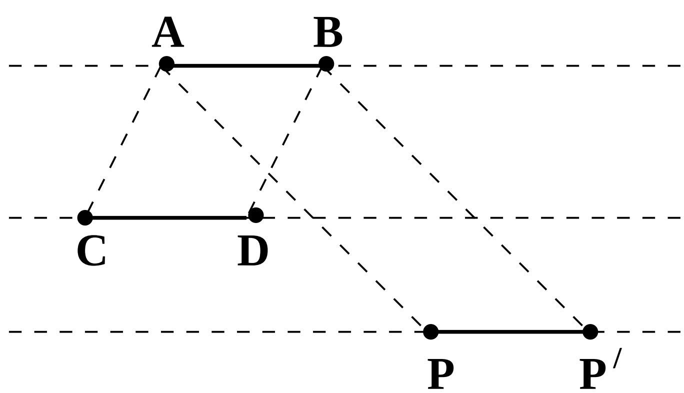


Figure 69: Equivalent vectors and their translation.

Corollary 21.2 *If t has vector $\vec{CD}$ and $t : A \to B$, then vector $\vec{AB}$ just as well describes the translation t.*

We now see why we are justified in saying that vectors $\vec{AB}$ and $\vec{CD}$ are 'equal' if they have the same length and direction, since then they certainly do describe equal translations.

It is useful to think of these two vectors as issuing the same translation command. On the other hand, since AB and CD usually will be quite different segments, the word 'equal' is a bit misleading. Thus, let us say instead that $\vec{AB}$ and $\vec{CD}$ are *equivalent*, and write $\vec{AB} \equiv \vec{CD}$.

Another way to assert that $\vec{AB} \equiv \vec{CD}$ is to demand that $ABDC$ be a parallelogram (see Figure 70). Here we must allow *degenerate* parallelograms which have 'collapsed' onto a line.

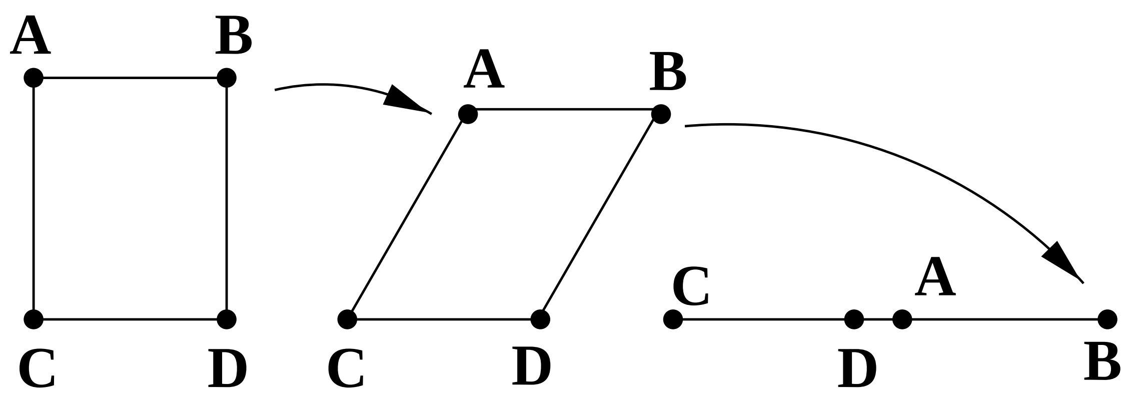


Figure 70: Ordinary and degenerate parallelograms.

This way looking at equivalent vectors is more in the spirit of *affine geometry*, which avoids mention of congruence, but which does have the usual notion of parallelism. Note that we defined a parallelogram in Definition 3 without ever mentioning congruence.

As innocuous as it seems, Theorem 21.1 has some important consequences. To begin with, we can clarify a small point in the proof of Corollary 20.2 : The product of two half turns h_A and h_M is the translation t with vector $2\vec{AM}$.

Indeed, we know from Theorem 20.1 that $t = h_A h_M$ is a translation. And certainly $t : A \to B$, where M is the midpoint of AB. Thus, by Corollary 21.2, t is described by the vector $\vec{AB} \equiv 2\vec{AM}$.

Now consider any two translations t and u with vectors $\vec{AB}$ and $\vec{EF}$ respectively, and construct parallelogram $ABDC$ with $\vec{BD} \equiv \vec{EF}$, as in Figure 71. Suppose MN and PQ bisect the sides of the parallelogram; note that MN and PQ must cross at the midpoint L of diagonal AD.

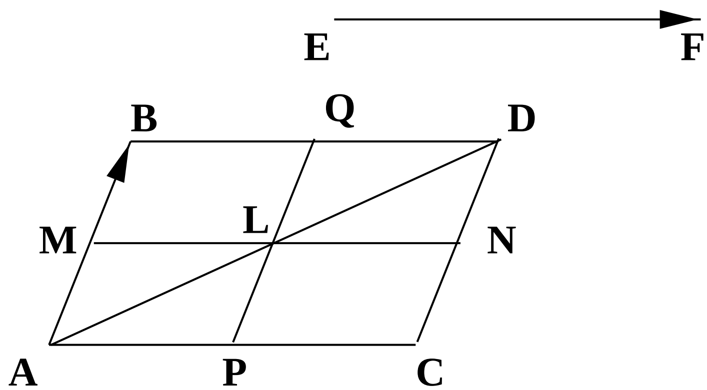


Figure 71: Multiplication of translations is closed and commutative.

Thus $t = h_A h_M = h_P h_L$ and $u = h_M h_L = h_A h_P$. Hence,

$$tu = (h_A h_M)(h_M h_L) = h_A h_M^2 h_L = h_A h_L$$

is a translation with vector $2\vec{AL} \equiv \vec{AD}$. Similarly,

$$ut = (h_A h_P)(h_P h_L) = h_A h_L = tu \quad .$$

In short, the product of two translations is a translation, and moreover these two translations commute. Recall as well that the identity 1 is a translation, with vector $\vec{o} \equiv \vec{AA}$, for any point A. Also, the inverse of translation t is itself the translation with vector $\vec{BA} \equiv -\vec{AB}$. We may summarize these facts in the following

Theorem 21.3 *The collection* $\mathbf{T}$ *of all translations is a commutative subgroup of the full isometry group* **Isom***.*

In fact, we can say a little more about how $\mathbf{T}$ acts on the plane, rather as in Corollary 19.5:

Corollary 21.4 *The translation group* $\mathbf{T}$ *acts* sharply transitively *on the points of the plane. In other words, if* A *and* B *are any points of the plane, then there is exactly one translation mapping* A *to* B*.*

Proof. Clearly the translation t with vector $\vec{AB}$ does the job. But could some other translation u with vector $\vec{CD}$ also work? No! By Corollary 21.2, u must equal t. //

21.2 Vector Operations, Position Vectors and Vector Spaces

Vectors and the operations of scalar multiplication and vector addition were first introduced in an informal way in Section 15.3. That intuitive approach is more formally justified in Theorem 21.3 and the remarks preceding it.

For instance, we conclude from Figure 71 that the parallelogram method for adding vectors is just the thing needed to force vectors to *add* in exactly the same manner as the corresponding translations *multiply*. In short, the collection of vectors should and does behave as a commutative group.

On the other hand there is this small but tiresome issue concerning equivalent (rather than equal) vectors: we should consider vectors $\vec{AB}$ and $\vec{CD}$ as equivalent (written $\vec{AB} \equiv \vec{CD}$) if they issue the same translation command.

One very useful way to simplify matters is work with *position vectors*. First of all, choose any convenient point as an *origin*. Given this choice, every point P in the plane is described by exactly one position vector $\vec{p} = \vec{OP}$.[13] In other words, by fixing O as the common initial point for all vectors, every point P is described by exactly one vector $\vec{p}$ (see Figure 72). In particular, the origin itself is described by the zero vector $\vec{o} = \vec{OO}$. Also, every vector $\vec{q} = \vec{OQ}$ has an inverse (or negative) $-\vec{q} = \vec{OQ'}$, where Q' is chosen so that O is the midpoint of segment QQ'.

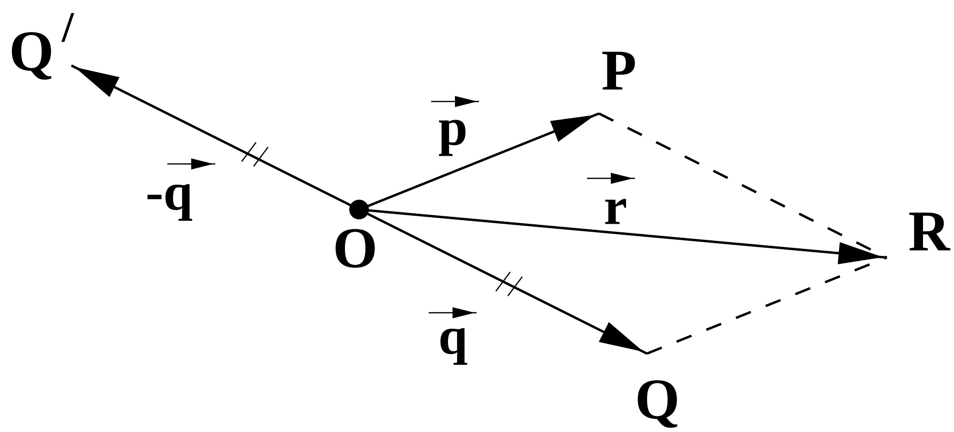


Figure 72: Position vectors.

Of course, we still use the paralleogram rule for vector addition—just ensure that the resulting vector is positioned with O as initial point. We see from Figure 72 that

$$\vec{p} + \vec{q} = \vec{r} = \vec{OR} \ ,$$

if OR is the diagonal of parallelogram $OPRQ$. (Again we must allow degenerate parallelograms.)

[13] By fixing an origin in this way we may use '=' instead of '≡' without ambiguity.

Now let $\mathbf{V}$ denote the collection of all position vectors $\vec{p}$. The upshot of the above discussion is that $\mathbf{V}$ is a commutative group, which therefore has the following properties, in which $\vec{p}$, $\vec{q}$, $\vec{r}$ denote position vectors:

VS-1. There is a zero vector $\vec{o}$ such that for any vector $\vec{p}$

$$\vec{p} + \vec{o} = \vec{p} \quad .$$

VS-2. For any vector $\vec{p}$ there is a vector $-\vec{p}$ such that

$$\vec{p} + (-\vec{p}) = \vec{o} \quad .$$

VS-3. For any two vectors $\vec{p}$ and $\vec{q}$, $\vec{p} + \vec{q} = \vec{q} + \vec{p}$.

VS-4. For any three vectors $\vec{p}$, $\vec{q}$ and $\vec{r}$,

$$(\vec{p} + \vec{q}) + \vec{r} = \vec{p} + (\vec{q} + \vec{r}) \quad .$$

Next let's consider scalar multiplication (also introduced in Section 15.3), as it pertains to position vectors. Suppose that γ is any real number and $\vec{p} = \vec{OP}$ is any position vector. Then the vector $\gamma\vec{p} = \vec{OP'}$ lies along the same line as $\vec{OP}$, but is $|\gamma|$ times as long, and is pointed in the opposite direction if $\gamma < 0$.

Scalar multiplication behaves and interacts with vector addition according to the following rules, in which we take $\vec{p}$, $\vec{q}$ to be any position vectors, and γ , λ to be any real scalars:

VS-5 (Mixed Associativity). $(\gamma\lambda)\vec{p} = \gamma(\lambda\vec{p})$.

VS-6 (Mixed Distributivity). $(\gamma + \lambda)\vec{p} = \gamma\vec{p} + \lambda\vec{p}$; $\gamma(\vec{p} + \vec{q}) = \gamma\vec{p} + \gamma\vec{q}$.

VS-7 (Multiplication by 1). $1\vec{p} = \vec{p}$.

The only law whose proof requires much verification is the second mixed distributive law. In Figure 73 we suppose $\gamma > 0$, which is typical enough. Thus $\vec{p} = \vec{OP}$, $\gamma\vec{p} = \vec{OP'}$, $\vec{q} = \vec{OQ}$ and $\gamma\vec{q} = \vec{OQ'}$, where

$$\frac{OP'}{OP} = \gamma = \frac{OQ'}{OQ} \ .$$

Now complete the parallelogram $OP'R'Q'$ and draw $PR \| P'R'$ with R on the diagonal OR'. Thus by Theorem 7.4,

$$\frac{OR'}{OR} = \frac{OP'}{OP} = \gamma = \frac{OQ'}{OQ} \ ,$$

so that $RQ \| R'Q'$. Therefore, $OPRQ$ is a parallelogram, so that

$$\vec{p} + \vec{q} = \vec{r} = \vec{OR} \ ,$$

and thus

$$\begin{aligned} \gamma(\vec{p} + \vec{q}) &= \gamma\vec{r} = \vec{OR'} \\ &= \vec{OP'} + \vec{OQ'} = \gamma\vec{p} + \gamma\vec{q} \ . \end{aligned}$$

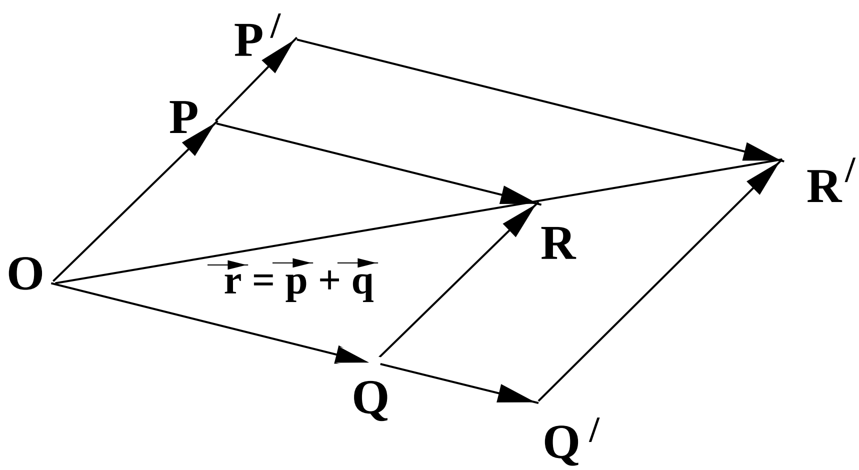


Figure 73: A mixed distributive law with $\gamma = 3/2$.

The remaining properties for scalar multiplication follow even more easily.

Any set equipped with two operations satisfying properties **VS-1** through **VS-7** is called a *vector space*. Thus the set **V** of position vectors is a vector space.

Properties **VS-1** through **VS-7** actually imply all sorts of algebraic facts that we might take for granted: the zero vector is unique, additive inverses are unique, $0\vec{p} = \vec{o}$, $-1\vec{p} = -\vec{p}$, etc. This last item is rather subtle: we are asserting that scalar multiplication by -1 has the effect of producing the additive inverse of $\vec{p}$. We won't pursue the details here: you may consult any standard text on linear algebra or modern algebra.

21.3 Parametrized Lines

The two basic vector space operations suffice to describe linear objects, such as the line m through points A and B. Let these points have position vectors $\vec{a} = \vec{OA}$ and $\vec{b} = \vec{OB}$ as in Figure 74.

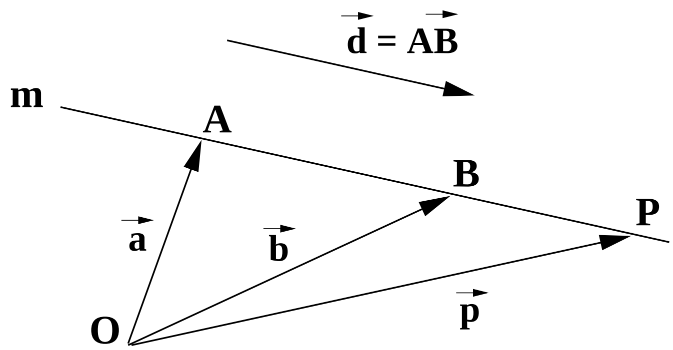


Figure 74: Parametrizing the line.

The vector $\vec{d} = \vec{AB}$ is called the *direction vector* for the line m. Note that $\vec{d} = \vec{b} - \vec{a}$. Now each point P on line m defines a unique real number

$$\gamma = \frac{AP}{AB} \ ,$$

where we naturally take γ to be negative if segments AP and AB have opposite sense on the line. In other words,

$$\vec{AP} = \gamma \vec{AB} \ ,$$

so that P has position vector

$$\vec{OP} = \vec{OA} + \vec{AP} = \vec{OA} + \gamma \vec{AB} \ .$$

Emphasizing the dependence of $\vec{p} = \vec{OP}$ on γ, we thus have

$$\begin{aligned} \vec{p}(\gamma) &= \vec{a} + \gamma \vec{d} \\ &= \vec{a} + \gamma(\vec{b} - \vec{a}) \ , \quad (\gamma \in \mathbb{R}). \end{aligned}$$

We can think of $\vec{p}(\gamma)$ as a vector valued function which provides a parametric description of the line m. Suitable restrictions on the parameter γ will describe certain portions of the line. For example, the segment AB is described by taking $0 \leq \gamma \leq 1$ and the ray issuing from A (and away from B) is given by $\gamma \leq 0$.

21.4 Oblique Coordinates

Suppose $\triangle OAB$ is any triangle with one vertex at the origin. We will call the line OA the x-axis and OB the y-axis, although these lines need not be perpendicular (see Figure 75).

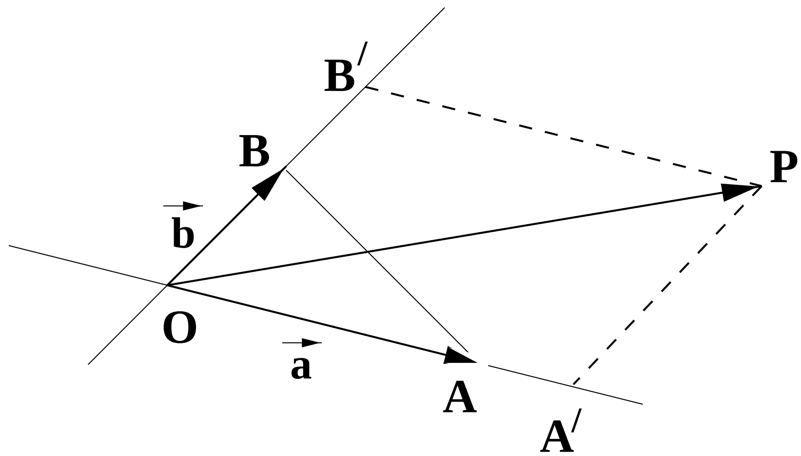


Figure 75: Setting up oblique coordinates.

Now let P be any point in the plane. By the parallelism axiom, there is a unique line through P which is parallel to OB and which thus meets the x-axis at some point A'. Likewise the unique line through P and parallel to OA meets the y-axis in some point B'. In short, $OA'PB'$ is a parallelogram with A' on the x-axis and B' on the y-axis.

Next let

$$x = \frac{OA'}{OA} \quad \text{and} \quad \mathrm{y} = \frac{\mathrm{OB}'}{\mathrm{OB}} \ ,$$

where again we take the ratio to be negative if the segments have opposite sense on their common line. Thus $OA' = x\,OA$ and $OB' = y\,OB$, and it follows that

$$\begin{aligned} \vec{p} &= \vec{OP} \\ &= \vec{OA'} + \vec{OB'} \\ &= x\,\vec{OA} + y\,\vec{OB} \\ &= x\,\vec{a} + y\,\vec{b} \ . \end{aligned}$$

The ordered pair of real numbers $[x, y]$ gives the *coordinates* of the point P with respect to the oblique coordinate system defined by $\triangle OAB$. For example, the origin O has coordinates $[0, 0]$, the point A has coordinates $[1, 0]$ and the point B has coordinates $[0, 1]$.

If $\triangle OAB$ has $OA = OB$ and $\angle BOA = 90°$, then x, y are ordinary Cartesian coordinates. We shall study these further in Section 23.