11 A detour – we discuss functions, the key to modern mathematics

11.1 Mathematics in the 20th century.

In many ways the major theme of 20th century mathematics has been the interaction between certain sets with 'structure' and 'structure preserving' functions on these sets.

Unfortunately, the centrality of this powerful idea is obscured in some mathematics courses, such as Calculus, where functions basically become things to differentiate or integrate [8]. This narrow focus is regretable.

So let's back-track a little, reconsider the idea of function, then move to a more general point of view which will allow us to do some geometry in the modern way.

11.2 What are old-fashioned functions?

You have all encountered functions like

$$\begin{aligned} f(x) &= x^2+1 \quad , \quad x \in \mathbb{R} \\ g(x) &= 5 \quad , \quad x \in \mathbb{R} \\ h(x) &= \frac{1}{\sqrt{2-x}} \quad , \quad x < 2 \end{aligned} .$$

In each case, the function takes an input 'x', which is allowed to run over some domain, and by some 'rule' assigns to each input exactly one output $f(x)$ (or $g(x)$, or $h(x)$, respectively).

For example, the 'rule' for the function f is

'take any real number as input, square, then add 1 to produce the output'.

Although this is really what the function is all about, the verbal description is so cumbersome that we use an algebraic shorthand

$$f(x) = x^2 + 1, \quad x \in \mathbb{R}$$

to signify the same thing. Notice that exactly the same function is described by an alternative shorthand, such as

$$f(t) = t^2 + 1, \quad -\infty < t < \infty,$$

since the same 'rule' taking inputs to outputs is indicated. The actual variable used, x or t, is irrelevant; so these are called 'dummy variables'.

[8]You may have studied linear algebra, where functions called *linear transformations* are used in a more insightful way.

Each of the above functions is numerical, in that inputs and outputs are (real) numbers. Such functions are the stuff of high school algebra and Calculus.

11.3 Functions from a modern viewpoint.

One of the many marvelous insights of modern mathematics is that functions need not work on numbers, and thus need not be defined by 'formulas' at all. So let's give a modern [9]:

Definition 5 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are any sets. Then a* **function**

$$f : \mathcal{A} \to \mathcal{B}$$

is any rule which assigns to each element $p \in \mathcal{A}$ **exactly one** image *$p' \in \mathcal{B}$. We also say that p is a* preimage *of p'.*

This definition is very general, for the sets $\mathcal{A}$ and $\mathcal{B}$ might be very different and might have nothing to do with numbers.

In calculus, we would write

$$p' = f(p) \ ,$$

which somewhat awkwardly is read right-to-left: take input p, apply f, get output p'.

In *group theory*, the kind of modern algebra we shall use in geometry, we often avoid this awkwardness by using some sort of left-to-right notation, as in

$$f : p \to p' \qquad \text{or} \qquad p \xrightarrow{f} p' \ .$$

Some authors even write $(p)f = p'$ or $p^f = p'$, which I like, but would take too much getting used to here.

Exercise. Write in traditional notation the following functions:

(a) $f : x \to x^3 - x$;

(b) $x \xrightarrow{g} 2^x$;

(c) $(x)h = 2 - \dfrac{1}{x}$;

(d) $(x^h)^f$, where h and f are described in parts (c) and (a).

[9] Using set theory it is possible to avoid the rather vague word 'rule' in our definition. But we need not pursue such technicalities here.

11.4 Functions in Modern Geometry.

We shall use **E** to denote the Euclidean plane, or more accurately, the set of all points in the plane. Thus **E** is an infinite set, and to indicate a typical point P in the plane we write $P \in \mathbf{E}$.

Reformulating our definition in this special case, we say that a **function**

$$f : \mathbf{E} \to \mathbf{E}$$

(i.e. from the plane to itself) is any rule which assigns to each point P of the plane **E** exactly one *image* P' in **E** (for which P is a *preimage*).

11.5 Some examples.

Since our inputs are geometric objects (i.e. points) we require geometric, rather than algebraic, definitions.

1. **Reflection** r in line m – review Definition 1 in § 4.2.

   The Rule – for each point P in the plane

   $$r : P \to P' \quad ,$$

   where the image P' is the point on the line through P, perpendicular to m, but at an equal distance from m on the side opposite P.

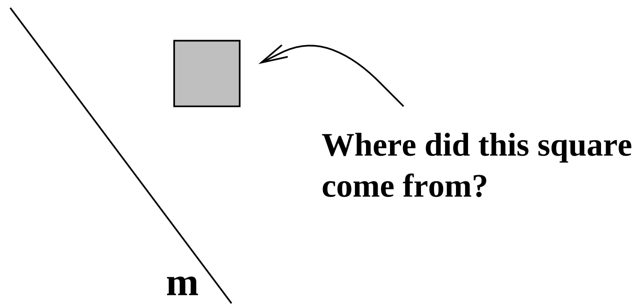
   Notice that the defining rule is (and must be) essentially geometric – there is no 'formula'. On the other hand, we can still do some simple algebra with the function:

   (a) $r : P \to Q$ implies $r : Q \to P$.
   This is proved by applying the rule first to P to get Q, then again to Q. In short, the mirror image of the mirror image is the original point.
   One could write $r : P \to P'$ implies $r : P' \to P$, but then one must systematically replace P by P' in the rule.

   (b) $r : A \to A$ (i.e. $A = A'$) implies A is a point on the mirror m.
   Such points are **fixed points** for this reflection.

   Another way to interpret (a) is to imagine that if you **know** the mirror image of some figure, then you can always **recover** the original.

Easy Exercise. The square shown is the mirror image in line m of some square S. Locate the original square S.



One reason reflections are so nice is that we can recover original figures from their images. Another is that reflections preserve distance – we proved this in Theorem 4.2.

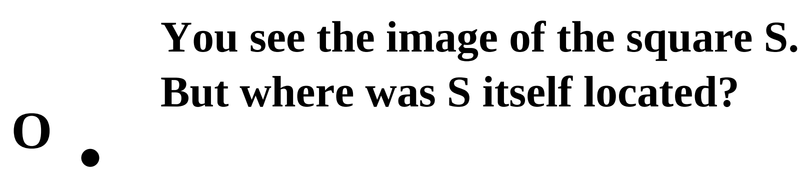
2. A simple, but destructive function.

   Fix any point O. The **constant** function k is defined by

   The Rule – each point P in the plane is mapped to the one point O:

$$k : P \to O.$$

   This easily defined function is actually quite destructive. It is impossible to recover any figure from its image. For example, here is the image of a certain square S:



   We have no way of knowing where S is located, how big it is, etc. The problem is that too many different points (all of them, in fact) have the same image. Intuitively, you might think of a sheet of paper being compressed so badly that you have no way of unfolding it.

3. The **identity** – a simple, but very useful, function.

   The Rule – The **identity** function

$$1 : P \to P$$

   fixes *all* points in the plane.

   Here, the image of our square is the square – there is no effort at all in recovering it.

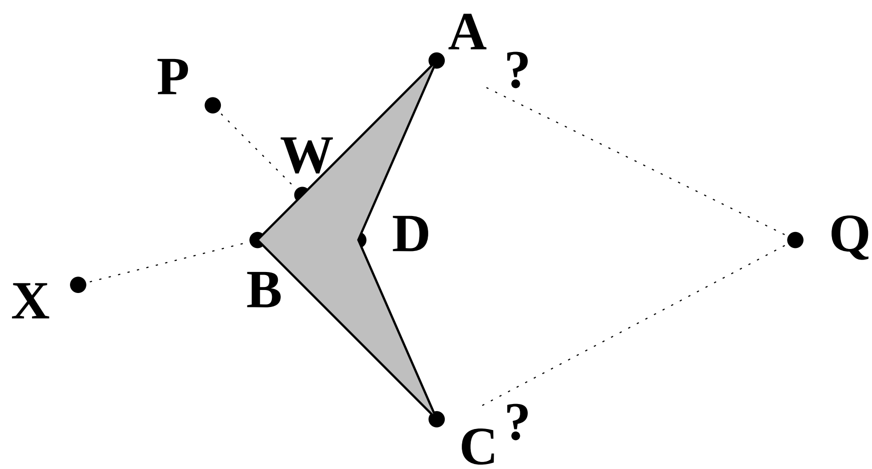
   It is important to remember here that r and 1 do not denote numbers; instead they are symbols representing some way of moving about points. We shall soon do more sophisticated algebraic manipulations with these symbols, but one must remember that there is only a limited analogy with the ordinary arithmetic of numbers.

4. A useful kind of function.

The next example illustrates an extremely useful technique in studying polygons, polyhedra and their relatives – *polytopes* – in still higher dimensions. Although we won't again use these functions, we take this opportunity to illustrate a subtle point:

to each input point P the rule must associate exactly one *output point P'.*

(a) Let $\mathcal{R}$ be the following **re-entrant** quadrangle $ABCD$ (including both area and boundary):



We attempt to define a function f by the following

'Rule': each point P in the plane is mapped to 'the' point P' in $\mathcal{R}$ which is closest to P.

But is this f a function? If you look carefully, you can check that

$$f : X \to B \quad \text{and} \quad P \to W.$$

Also $f : A \to A$, since A, being in $\mathcal{R}$, is zero distance from itself. In all these cases, there was exactly one output for the input.

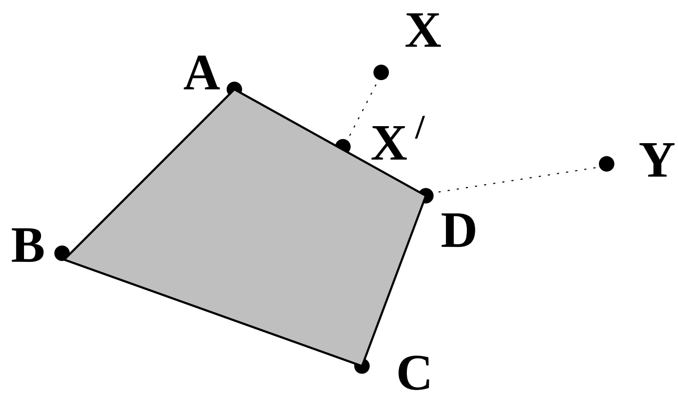
But where does f send the input Q? Since Q is equidistant from A and C, there is not exactly one output for the input Q, but rather two possibilities! Thus our 'Rule' does not define a function; and no function f is described by this 'Rule'.

(b) The reason we insist on 'exactly one output' is so that we should have control over the function. Then we can always be certain which output arises from a given input. Intuitively, we don't want one point spawning two (or more) points.

The jargon commonly used is that our so-called 'Rule' is not 'well-defined'. Indeed, for any Rule which purports to describe a function, we should check that the Rule is indeed **well-defined**. Often, as in the case of most functions in Calculus, this is quite clear, although you may recall difficulties for the inverse trigonometric functions.

(c) Now let $\mathcal{Q}$ be the following **convex** quadrilateral $ABCD$. Define a function g by

The Rule – for each point P in the plane, the image is the point P' in $\mathcal{Q}$ which is closest to P.



Thus $g : A \to A,\ \ X \to X',\ \ Y \to D$, etc.

Exercises.

(i) Why is this Rule well-defined?

(ii) If you know only the image of some figure, say a circle, can you recover the circle?

(iii) Draw a line segment of positive length, whose image is the constant point B.

(iv) Why might you call this g a 'gift-wrap' function?

12 Transformations – the special type of functions we now need

12.1 An intuitive look at transformations

We have examined various functions from the plane **E** to itself. There are very many such functions – far too many in fact. So we have to limit our discussion to the special kinds of functions most useful in geometry. The idea of **symmetry** gives us some clues as to what to look for.

The square in Figure 38 is a symmetrical looking object - its shape and position are preserved when we reflect, in say, the verticle line m through the centre C. We can similarly 'rotate' the square through 90° about C without changing its appearance.

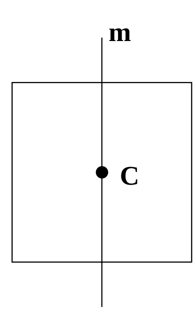


Figure 38: A reflection symmetry for the square.

A quick look reveals three other reflections and three other rotations which 'preserve' the square. Notice that we might consider a rotation through 90° as identical with one through $450° = 90° + 360°$, since the net effect on the square is the same in each case. In other words, what concerns us most is the final position of the square relative to its initial position - how we get there is of little concern in the study of symmetry.

Note as well that we can 'undo' the effect of each symmetry. For example, to recover the original state after reflection, we just reflect again. How do you recover from a 90° (i.e. anti-clockwise) rotation?

Now the rotation and reflection mentioned above preserve the shape of the square. But if we imagine the square to be made of rubber, then we could certainly change its shape, say by stretching it horizontally, so as to obtain a rectangle with twice the original width but with the same height (Figure 39):

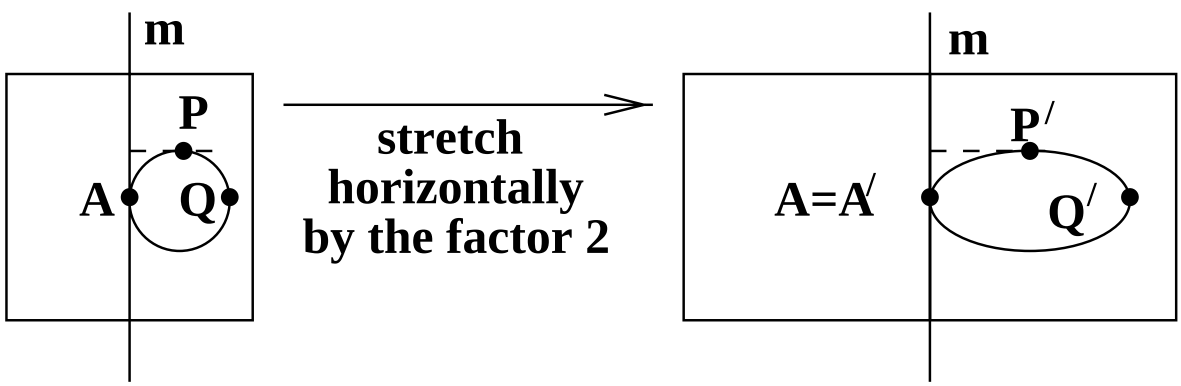


Figure 39: Stretching the square.

Points such as A on m are fixed (since they are pulled equally in opposite directions), but each other point P is moved horizontally to a new position P' twice as far from m. Notice that the circle is stretched into an ellipse. To revert to the original square shape just let the rubber snap back, i.e. 'shrink horizontally by the factor $\frac{1}{2}$.'

This property of 'recoverability' for all the functions just discussed is a crucial, and very desirable, thing. Soon we will formally give such nice functions the name *transformation*, but first we must investigate the idea of recoverability more clearly. This involves the ideas of 'product' and 'inverse'.

12.2 Some problems to motivate some algebra

You may already have tried the problems with the billiard table on page 36. There, a very natural thing to do is to bounce the cue ball off two or more consecutive banks. So to analyze the problem geometrically, we must 'multiply' reflections by applying them in succession.

Or consider what happens when a light beam bounces successively off two perpendicular mirrors:

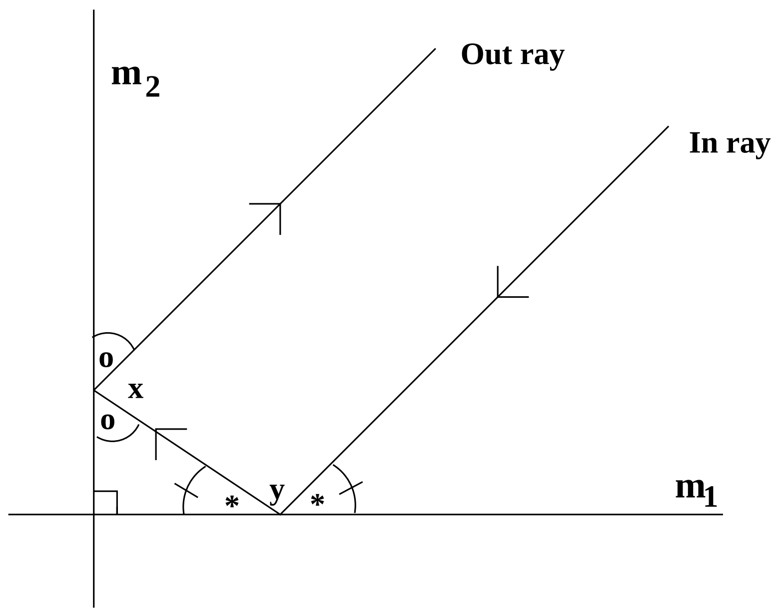


Figure 40: Consecutive reflections in perpendicular mirrors.

Problem: What is the net effect of reflections r_1 and r_2 in two perpendicular mirrors m_1 and m_2?

Consider *any* light ray hitting (and bouncing from) m_1 at angle $*$. (See Figure 40.) Thus the two equal angles at m_2 are $\circ = 90^\circ - *$, so that

$$\begin{aligned} x + y &= x + [180^\circ - 2*] \\ &= x + 2[90^\circ - *] \\ &= x + 2\circ \\ &= 180^\circ. \end{aligned}$$

Hence, the in- and out- rays are parallel by Corollary 5.3 (c). In other words, the light beam returns to its source regardless of how that source is positioned relative to the mirrors.

Remark. A configuration of mirrors similar to this is used in conjunction with lasers to measure distances (even to the moon!) very accurately.

12.3 Products of Functions

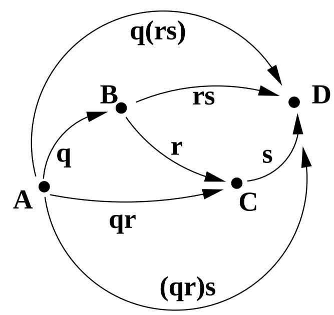
The previous problems suggest that it is worthwhile studying the net effect of two functions.

Definition 6 *Suppose q and r are any two functions from* **E** *to* **E** *.*

Rule – The net result of performing first q, then r, is a third function called their product *and written qr.*

(a) This kind of multiplication is *associative*:

$$(qr)s = q(rs) \quad .$$



Proof. First of all, (qr) takes A to C, so $(qr)s$ takes A to D. But q takes A to B, and (rs) takes B to D, so $q(rs)$ also takes A to D. Since $(qr)s$ and $q(rs)$ have the same net effect on any point A, which is typical of all points in **E**, we conclude that $(qr)s = q(rs)$. //

(b) However, it sometimes happens that $qr \neq rq$, so there is *no* commutative law:

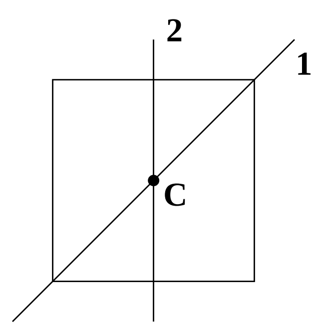


Figure 41: Non-commuting reflections.

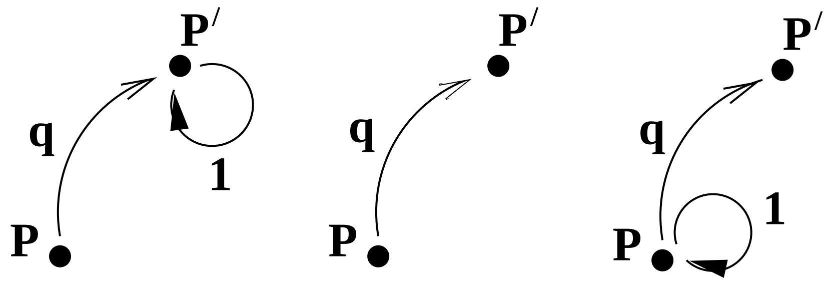
Here, if q is reflection in mirror 1, and r is reflection in mirror 2, then qr is the $+90°$ rotation about the centre C of the square, whereas rq is the $-90°$ rotation. You should check this with a cardboard model.

(c) Thus in such products of functions, the placement of brackets is irrelevant, but the *order* of terms cannot *usually* be changed. These rules become evident if we think of

$q =$ 'put on socks'
$r =$ 'put on shoes'
$s =$ 'put on overshoes'.

(d) **An important example: how the identity** 1 **got its name**: For any function $q : \mathbf{E} \to \mathbf{E}$, we have

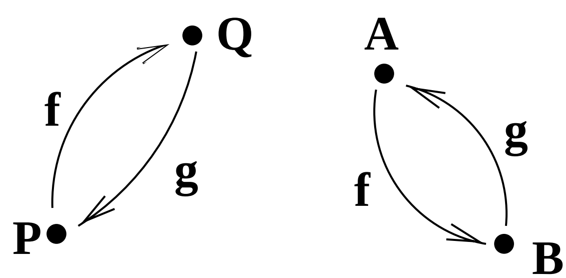
$$q1 = q = 1q \ .$$



This is very easy to prove: just look at the effect of applying $q1$ and $1q$ to a general point P.

12.4 Transformations – recovering from applying a function.

In order to completely recover from applying a function f, we must apply a 'recovery' function g which returns each point to its original position:



In other words, the product of the two functions must equal the identity:

$$fg = 1 \ .$$

Remember, the identity function 1 is the function which fixes each point P.

Now g, too, must have its own recovery function (we want to be able to recover from the recovery). A look at the diagram will convince you that only f will do the job, so we also want

$$gf = 1 \ .$$

These considerations finally motivate our

Definition 7 *A* transformation f *(of the plane* $\mathbf{E}$*) is any function* $f : \mathbf{E} \to \mathbf{E}$ *for which there exists a function* $g : \mathbf{E} \to \mathbf{E}$ *such that*

$$fg = 1 \ \text{ and } \ gf = 1 \ .$$

Notice that only one function g will undo the effect of f in this way: because if g and g_* were two such functions, we would have $fg = 1$, $gf = 1$ and $fg_* = 1$. Thus

$$\begin{aligned} g &= g1 \\ &= g(fg_*) \text{ (one assumption)} \\ &= (gf)g_* \text{ (by the associative law)} \\ &= 1g_* \text{ (another assumption)} \\ &= g_* \ . \end{aligned}$$

Any time an object is uniquely defined like this, it deserves a special name and notation:

Definition 8 *The* inverse *of any transformation f is the unique transformation g (coming from the definition of* transformation*) that neutralizes f. We write $f^{-1} = g$.*

In other words, if $f : P \to P'$, then $f^{-1} : P' \to P$, for any point $P \in \mathbf{E}$.

Here are some concrete examples:

(a)	$s =$	$+90°$ rotation of the square (anticlockwise)	$s^{-1} =$	$-90°$ rotation of the square (clockwise)
(b)	$r =$	reflection in m	$r^{-1} =$	r , since a reflection is neutralized by itself!

Thus we now know that every reflection is a transformation. We'll look more carefully at rotations and other motions in § 15 below.

Let's explore this new kind of algebra a bit more, before leaving this section.

Theorem 12.1 *Inverse transformations obey the following algebraic rules, for any transformations q, r on* $\mathbf{E}$*:*

(a) $qq^{-1} = q^{-1}q = 1$.

(b) The identity 1 *is a transformation; and* $1^{-1} = 1$.

(c) $(qr)^{-1} = r^{-1}q^{-1}$, **which might not equal** $q^{-1}r^{-1}$!

Proof. Part (a) is just the definition of q^{-1}, so doesn't need any proof.

To verify part (b), which should make sense, we need only check that $11 = 1 = 11$, which is clearly true. Remember that we proved that inverses are unique - so if we by good luck think of a function which satisfies the properties of the inverse, then that function is *the* inverse.

Part (c) is trickier, but really should make sense if you again think of $q =$ 'put on socks', and $r =$ 'put on shoes'. Here is the formal proof:

$$\begin{aligned} (qr)(r^{-1}q^{-1}) &= q(rr^{-1})q^{-1} \\ &= (q1)q^{-1} \\ &= qq^{-1} \\ &= 1. \end{aligned}$$

Since $r^{-1}q^{-1}$ does neutralize qr, it must equal $(qr)^{-1}$.//

Interpretation: Besides explaining an important calculation, part (c) also asserts that if q and r are transformations, each with its inverse, then the product qr also has an inverse, and so *is also a transformation.*

Warning: since q and r sometimes do not commute, i.e.

$$qr \neq rq \ ,$$

it sometimes happens that

$$(qr)^{-1} \neq q^{-1}r^{-1} \quad .$$

An easy extension of this inverse calculation is that for several transformations we have

$$(q_1 q_2 \cdots q_{n-1} q_n)^{-1} = q_n^{-1} q_{n-1}^{-1} \cdots q_2^{-1} q_1^{-1} \quad .$$

Remark. You have also encountered inverse functions in Calculus. There, pairs of inverse functions, like

$$f(x) = x^2 \ \text{, and } \ f^{-1}(x) = \sqrt{x}, \ \ (x \geq 0),$$

or like

$$f(x) = \ln(x) \ \text{, and } \ f^{-1}(x) = e^x,$$

are crucial, because we must be able to undo certain calculations if we are to solve significant problems. Likewise in geometry, we want to be able to undo motions so that the structure of the plane is not destroyed.

Exercises on Functions and Transformations. This question concerns two functions

$$f : \mathbf{E} \to \mathbf{E} \text{ and } g : \mathbf{E} \to \mathbf{E} .$$

of the plane to itself. (In fact, the same idea works for functions on other sets.) You are given that

$$fg = 1 ,$$

and *nothing else to work with.* You must not use any particular examples–your answers should be general. Remember that functions operate left to right.

(a) A function h is **onto** (or **surjective**) if it covers every possible candidate for an output. In other words, for each and every point Q in $\mathbf{E}$ we can manufacture an input P which is sent to Q by the function h.

Use the equation $fg = 1$ to prove that g is onto.

(b) A function h is **1–1** (or **injective**) if different inputs are always sent to different outputs.

Use the equation $fg = 1$ to prove that f is 1–1.

(Hint: think contradiction. Suppose P_1, P_2 *were* two different points with the *same* image under f. Investigate what happens if you apply f.)

The upshot of all this is that *transformations*, which we defined in class as being functions with inverses, could equally well be defined as functions which are both **1–1 and onto**. Synonyms for the same thing are **bijections** and **permutations**.

13 Transformation Groups

13.1 Putting all transformations into one algebraic package

Let's collect all transformations of the plane **E** into one enormous set, which we shall call **BIG**. Thus **BIG** contains the identity 1, all reflections (a different one for each of the infinitely many lines in the plane), all rotations, all stretches, and in fact, innumerable sorts of bizarre transformations which we haven't even dreamed of yet.

Nevertheless, we can say quite a lot about **BIG**. We have already verified the basic algebraic properties contained in the following:

Summary: The collection **BIG** of all transformations q, r, s, etc. on the plane **E** comes equipped with an operation, the *product* of transformations, satisfying:

1. The product qr is also a transformation in **BIG** [closure law].

2. $(qr)s = q(rs)$ [associative law].

3. There is an identity $1 \in$ **BIG** such that $q1 = 1q = q$.

4. Each $q \in$ **BIG** has an inverse q^{-1}, also in **BIG**, such that $qq^{-1} = 1 = q^{-1}q$.

In fact, the above four key algebraic properties indicate that the collection **BIG** of *all* transformations of the plane forms a *group*. Groups are central objects throughout much of modern mathematics, and we shall find them very useful in geometry. Here is a formal description.

13.2 Groups

A *group* G is any set $G = \{1, q, r, \ldots\}$ together with an operation (for example, a product qr) defined for all q, r in G such that:

G1. The product qr is also a member of G [closure law].

G2. There is a special identity element 1 such that $1q = q1 = q$ for all q in G.

G3. Each q in G has an inverse q^{-1} in G such that $qq^{-1} = q^{-1}q = 1$.

G4. An associative law holds: $(qr)s = q(rs)$.

Notice that the operation need not be *commutative* – sometimes $qr \neq rq$. On the other hand, in some (rather well behaved) groups the expected **commutative law** does hold:

Comm. $qr = rq$ for all q and r in the group.

Naturally, such a group is called **commutative** (or **abelian** [10]).

[10] Commutative groups are often called *abelian* groups, after the Norwegian mathematician Niels Henrik Abel (1802–1829).

13.3 Examples from Arithmetic

We'll later encounter all sorts of geometrical examples of this unexpected behaviour. But for now, we mention only a few examples of groups from arithmetic. It happens that each of these groups is commutative. In each case, you should check group properties **G1** – **G4**, as well as the special property **Comm**.

(a) $G = \{\text{integers}\}$; operation = addition.

(b) $G = \{\text{non-zero rationals}\}$; operation = multiplication.

(c) $G = \{\text{positive reals}\}$; operation = multiplication.

(d) $G = \{\text{residues (mod 7)}\}$
$= \{0, 1, 2, 3, 4, 5, 6\}$;
Operation = addition (mod 7).

(e) $G = \{1, \text{-}1\}$; operation = multiplication.

13.4 The group BIG

We now know that **BIG** is a group. In fact, **BIG** is too big — it is very infinite, and contains transformations which aren't of much use in doing geometry. We shall focus instead on much smaller groups of very special transformations, starting with isometries in the next section.

14 Isometries and Symmetries

14.1

Whereas the horizontal stretch *distorts* shape, any reflection *preserves* shape. There is a special name for shape-preserving transformations, of which the reflection is just one example.

Definition 9 *An* isometry *is a transformation q such that for any two points P, Q, with respective images P', Q' we have $PQ = P'Q'$.*

Remarks.

(a) In Theorem 4.2 we proved that reflections are isometries (see Figure 12).

(b) Since an isometry q preserves the mutual distances between the constituent points of a figure, it must preserve the shape and size of the whole figure. Thus a circle of radius 2 is mapped to another circle of radius 2. Likewise any straight line is mapped to another straight line.

14.2 Basic Isometry Properties

(a) Clearly, the identity $1 : P \to P$ is an isometry (of course, it isn't a reflection).

(b) If $q : P \to P'$ is an isometry, then so also is $q^{-1} : P' \to P$ an isometry:

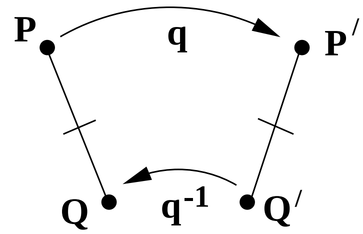


Figure 42: $PQ = P'Q'$, so $P'Q' = PQ$.

(c) If q and r are isometries, so is the product qr. (Since each preserves distances, together they do so.)

(d) Of course, isometries - like all transformations - obey the associative law:

$$(qr)s = q(rs).$$

But the commutative law sometimes fails: qr might not equal rq.

14.3 The Euclidean Group

We shall denote by **Isom** the collection of all isometries. Thus **Isom** contains the identity 1, all reflections, and other isometries discussed in Section 15.

We have verified just above that **Isom** satisfies the four defining properties of a group. Thus **Isom**, together with our way of multiplying isometries, is indeed a group. We shall study this important group in detail. For now, we note that **Isom** is still very infinite, and it is not commutative.

14.4 Symmetries

A *symmetry* of a particular object is any isometry which preserves the objects position (as well as its shape).

(a) *Example*: A stylized heart has two symmetries - the reflection r in m, and the identity 1.

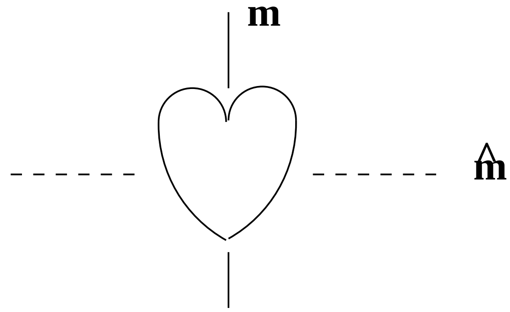


Figure 43: A heart with bilateral symmetry.

Of course, other reflections, such as $\hat{r}$ in line $\hat{m}$, do not preserve the *position* of the heart. For instance $\hat{r}$ turns the heart upside down.

(b) Thus only certain isometries are symmetries for a given object. This special class of symmetries – denoted **G** – is called the **symmetry group** of the object. Different objects may well have different symmetry groups: the kind of symmetry may vary, or one object may be more symmetrical than another.

(c) We now check that, for any plane figure at all, the collection **G** of symmetries does satisfy the four defining properties of a group.

   (i) It is clear that 1 is a symmetry, since the identity certainly fixes the position of any object.

   (ii) If q and r are symmetries what can we say about the product $p = qr$? Well, qr is certainly an isometry (it preserves *shape*), but does it preserve *position* as well? - Certainly it does. This is an interesting idea, because if we know two symmetries q and r of an object, we can generate a new symmetry qr by multiplying. Perhaps we hadn't noticed qr before. The way in which symmetries interact will say a lot about the nature of our pattern or object.

   (iii) Likewise, if q is a symmetry then so is q^{-1}.

   (iv) Again, **G** inherits the associative property – all transformations multiply associatively, so in particular, those in **G** do so.

So the collection **G** of all symmetries for a particular figure is indeed a group. A symmetry group may or may not be commutative, and it may or may not be finite.

For example, for the heart above, the symmetry group

$$\mathbf{G} = \{1, r\}$$

is both commutative and finite. We say that the order of **G** is 2: the **order** of a group is the number of elements in it. On the other hand, the Euclidean group **Isom** is uncountably infinite.

The order of a symmetry group thus provides a rough measure of the symmetry of an object.

15 The Four Species of Isometries

There are only four types of isometry: reflections, rotations, translations, and glides [3, ch.3].

15.1 Reflections

The *reflection* r in a line m was defined and shown to be an isometry in Section 4.2. For typical patterns see Figures 43 and 10.

Note that $r^{-1} = r$, so r^{-1} is also a reflection and $1 = rr^{-1} = r \cdot r = r^2$.

15.2 Rotations

Definition 10 *The* rotation *s with centre C and angle α fixes the point C. For each other point P choose P' so that $\angle PCP' = \alpha$ and $PC = P'C$:*

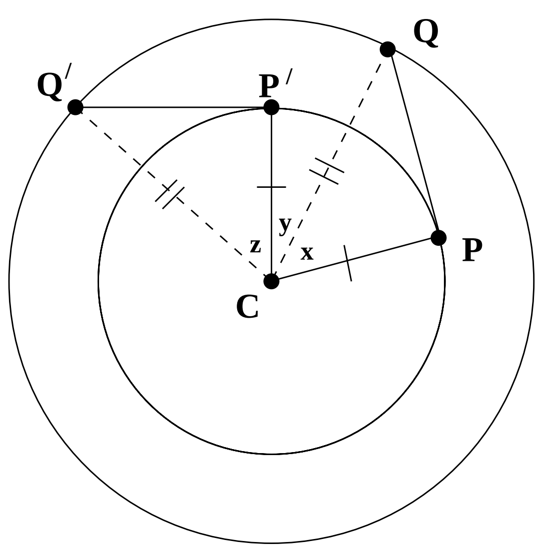


Figure 44: Rotations are isometries.

(a) *Verifying the isometry property.* Note that

$$\angle x + \angle y = \alpha = \angle y + \angle z,$$

so $\angle x = \angle z$. Moreover, by definition, $PC = P'C$, and $QC = Q'C$. Hence, by (s.a.s.), we have $\triangle PCQ \equiv \triangle P'CQ'$ and so $PQ = P'Q'$.//

(b) The rotation is anticlockwise if $\alpha > 0$, clockwise if $\alpha < 0$; thus s^{-1} is the rotation with centre C and angle $-\alpha$.

(c) The identity isometry 1 can be thought of as a rotation through 0°, ±360°, or through any multiple of 360° about any centre. For any other angle, s fixes only the centre C.

(d) *Typical Pattern.*

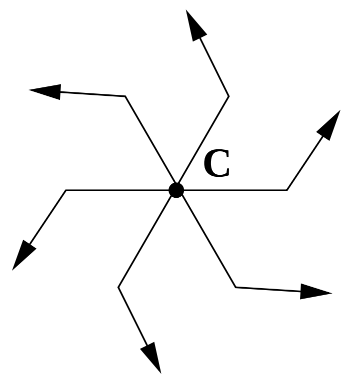


Figure 45: This pinwheel is symmetric by a 60° rotation about C.

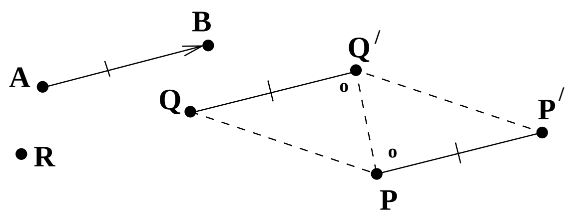
15.3 Translations

Definition 11 *Suppose that $\vec{AB}$ is a directed line segment with* initial *point A and* terminal *point B.*

Then $\vec{AB}$ represents a translation *t in which we require that t move each point P through a distance AB parallel to and in the same sense as $\vec{AB}$.*

A directed segment like $\vec{AB}$ is often called a *vector*. For the moment, this is little more than a convenient and suggestive terminology. In Section 21 we shall look more carefully at these ideas.

(a) Here is how a translation typically acts:



$$t: \begin{array}{l} A \longrightarrow B \\ R \longrightarrow Q \\ Q \longrightarrow Q' \\ P \longrightarrow P' \end{array}$$

(b) *Verifying the isometry property.* Since $PP' \| QQ'$, we have $\circ = \circ$ by Theorem 5.2. Thus by (s.a.s.), we conlude that

$$\triangle P'PQ' \equiv \triangle QQ'P$$

so that $P'Q' = QP$.//

(c) The inverse t^{-1} is the translation with the vector $\vec{BA}$, for to neutralize t we must shift the same distance in the opposite direction. We naturally write $\vec{BA} \equiv -\vec{AB}$, and call this vector the *negative* of $\vec{AB}$.

(d) If $A \neq B$, the translation t fixes no points. However, if $A = B$, then $t = 1$ fixes every point and we say that $\vec{AA} \equiv \vec{o}$ is the *zero vector*. Thus the identity 1 can be thought of as the translation with vector $\vec{o}$.

(e) Figures 46 and 47 indicate the sensible way to operate algebraically with vectors.

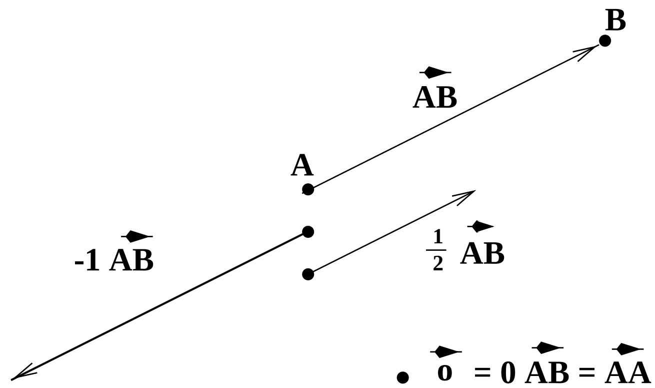


Figure 46: Scalar multiples of a vector.

For example, if γ is any real number and $\vec{AB}$ is any vector, then we can form a new vector $\gamma\vec{AB}$ under an operation called *scalar multiplication*. Here are some examples:

(i) $\frac{1}{2}\vec{AB}$ has $\frac{1}{2}$ the length of $\vec{AB}$ and the same direction.

(ii) $-2\vec{AB}$ has twice the length but the opposite direction.

(iii) $0\vec{AB} \equiv \vec{o}$, the zero vector.

(iv) $-1\vec{AB} \equiv -\vec{AB}$, the negative of $\vec{AB}$.

To compute the *vector sum* $\vec{AB} + \vec{QR}$, we shift $\vec{QR}$ parallel to itself so that $QRCB$ is a parallelogram. Then

$$\vec{AB} + \vec{QR} \equiv \vec{AB} + \vec{BC} \equiv \vec{AC}.$$

(In other words, to add vectors shift and complete the diagonal of the parallelogram.)

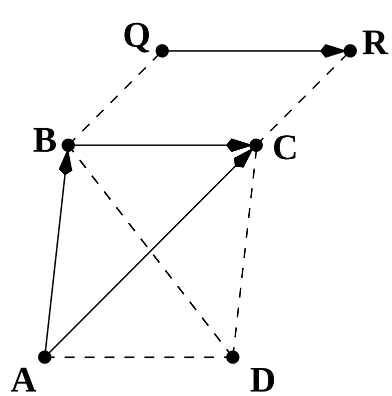


Figure 47: Vector addition and parallelograms.

Notice that

$$\vec{AB} - \vec{QR} \equiv \vec{DC} + \vec{CB} \equiv \vec{DB}$$

represents the other diagonal of parallelogram $ABCD$. The translations corresponding to these vectors multiply in essentially the same way as the vectors add. For now we will accept this as being intuitively reasonable. In Section 21 we shall pursue these ideas more formally.

(f) *Typical Pattern*: Shift the motif $\angle$ by repeatedly applying t or t^{-1}, where t:$A \to B$:

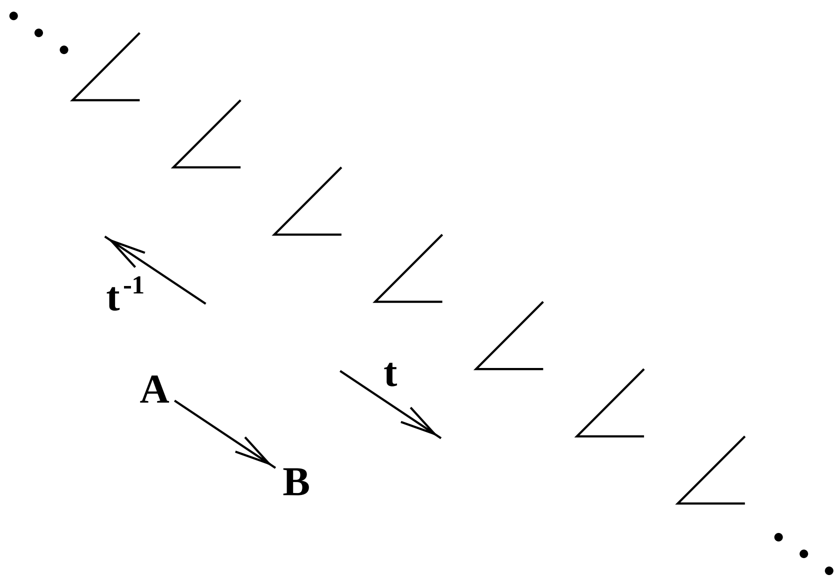


Figure 48: Translation symmetry.

(The 'dots' indicate that we should consider the pattern as extending infinitely far in either direction along a line.)

15.4 Glides

Definition 12 *A* glide *g is the net result of reflection in a line m together with some translation parallel to m. The line m is called the* axis *of the glide.*

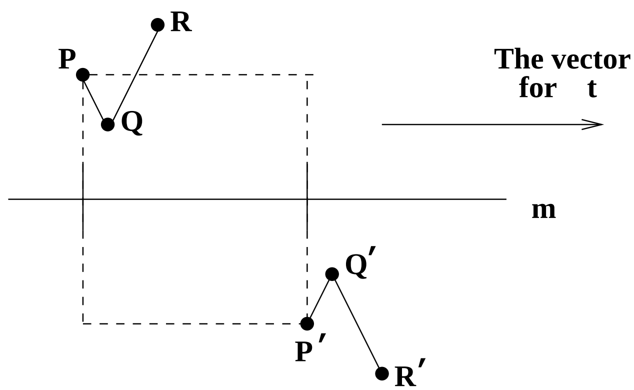


Figure 49: A glide.

(a) Thus $g = rt$ where r is reflection in m and the translation t has the indicated vector parallel to m. As the product of two isometries, g is itself an isometry [11].

(b) Note that while $rt : P \to P'$, it is also true that $tr : P \to P'$, for all points P in the plane. Hence, t and r commute (an unusual occurance), so $g = tr = rt$.

(c) Since $g^{-1} = (tr)^{-1} = r^{-1}t^{-1} = rt^{-1}$, (recall $r^{-1} = r$), we conclude that g^{-1} is also a glide with the same reflection component but with the inverse translation.

(d) Note that

$$\begin{aligned} g^2 &= (rt)(rt) \\ &= (tr)(rt) \\ &= t(r^2)t \\ &= (t1)t \\ &= t^2 \end{aligned}$$

Hence, g^2 is a translation with vector twice that of the component translation t.

(e) It could happen that the translation factor $t = 1$ (the translation with vector $\vec{o}$). In this case $g = r$ and we might say that g *degenerates* into a reflection. A *proper glide* will have $t \neq 1$.

[11] Our definition of glide may seem a cheat. Although we could give a 'point by point' definition rather as was done for reflections, rotations and translations, the present course is clear and efficient. Moreover, we see that by multiplying two known sorts of isometry, we might get something new. Indeed, a proper glide has some properties not shared by any of the other types of isometry, as you can check in Section 16.2.

(f) Glides typically generate 'footstep patterns'. Note how the translation g^2 shifts a left foot to a left foot, or a right foot to a right foot.

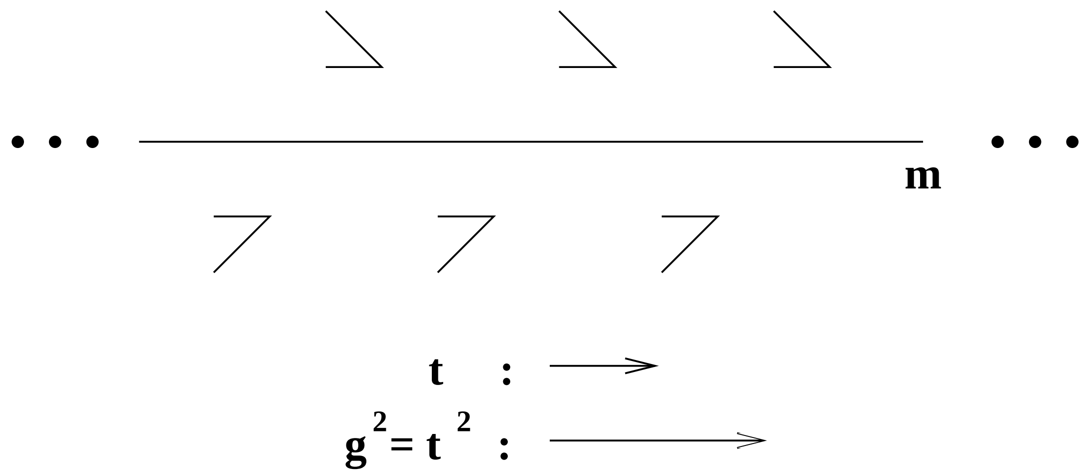


Figure 50: Glide symmetry.

16 Tracking Down Isometries

16.1

Rotations and *translations* are *direct* isometries, in that they take a triangle with clockwise orientation to a congruent triangle with clockwise orientation:

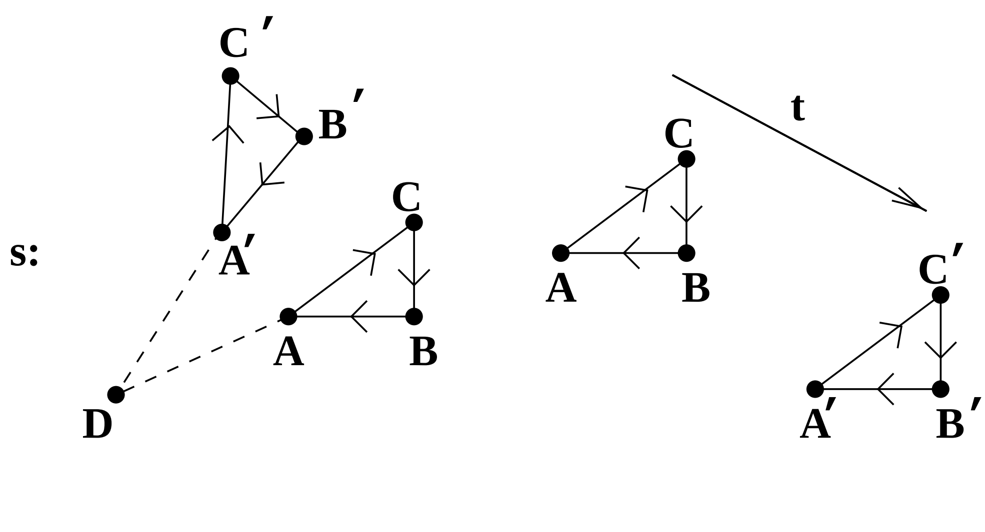


Figure 51: Direct isometries.

On the other hand, *reflections* and *glides* are *opposite* isometries, which take any clockwise oriented triangle to one with anti-clockwise orientation, and vice versa:

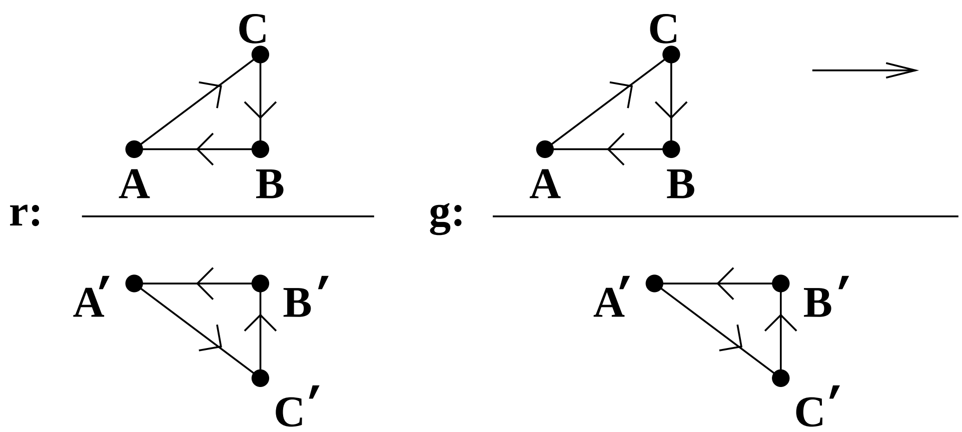


Figure 52: Opposite isometries.

Any time we apply an opposite isometry, we reverse the orientation of a figure. Thus, the product or two (or any even number) of opposite isometries must be a direct isometry. The product of a direct isometry q with any other isometry u is direct or opposite according as u is direct or opposite. We can summarize this interaction in the following table.

·	Direct	Opposite
Direct	Direct	Opposite
Opposite	Opposite	Direct

Multiplication Table for Direct/Opposite Isometries

Thus, for instance, the product of two reflections is direct and hence must be a rotation or translation. The product of 19 reflections (19 is odd) must be either a new reflection or a glide.

Notice also that direct and opposite isometries multiply in precisely the same way as the ordinary numbers $+1, -1$. Using some terminology from group theory, we say that there is a *homomorphism* from the full isometry group **Isom** onto the group $\{+1, -1\}$.

16.2

The main features of the four species of isometry are summarized below:

Type	Data required for a Full Description	Sense	Fixed Points
Rotation s	Centre C, angle α	Dir.	Only C if α is not a multiple of $360°$
Identity 1 (common)	angle $\alpha = 0$ or vector $\vec{AB} \equiv \vec{o}$		All points
Translation t	vector $\vec{AB}$	Dir.	None if $\vec{AB} \not\equiv \vec{o}$
Reflection r	mirror m	Opp.	all points on m
Glide g	axis m and translation vector $\vec{AB}$	Opp.	none if $\vec{AB} \not\equiv \vec{o}$; all pts. on m if $\vec{AB} \equiv \vec{o}$

17 A Hierarchy of Groups Acting on the Euclidean Plane

17.1 Examples from Geometry

(a) We saw in Sections 14.3 and14.2 that the collection of all isometries of the plane forms a group called the Euclidean group **Isom**.

(b) Any plane figure or pattern has a symmetry group **G** (see Section 14.4). Since **G** consists of special isometries, we say that **G** is a **subgroup** of the group **Isom** of all isometries.

For example, we recall that the stylized heart in Section 14.4 had two symmetries, so that its full symmetry group was $\mathbf{G} = \{1, r\}$.

The following diagram indicates the way that the increasing specialization of the type of transformation under discussion will lead to smaller and smaller subgroups.

The group **BIG** of all plane transformations: reflections, rotations, stretches, and far more horrible things.

$\downarrow$ (Specialize)

The Euclidean Group **Isom** (all isometries)

$\downarrow$ (Further specialize: consider some object or pattern)

The symmetry group **G** for an object (particular isometries)

18 Products of Reflections

18.1 Reflections in Intersecting Mirrors

Suppose r and $\hat{r}$ are reflections in m and $\hat{m}$, respectively. Then $p = r\hat{r}$ is direct and *must* be a rotation or translation. What exactly is this new isometry?

Theorem 18.1 *Suppose mirrors m, $\hat{m}$ intersect at A, where θ is the angle* from m to $\hat{m}$ *(*Figure 53*). Then $p = r\hat{r}$ is a rotation with centre A and angle $\alpha = 2\theta$.*

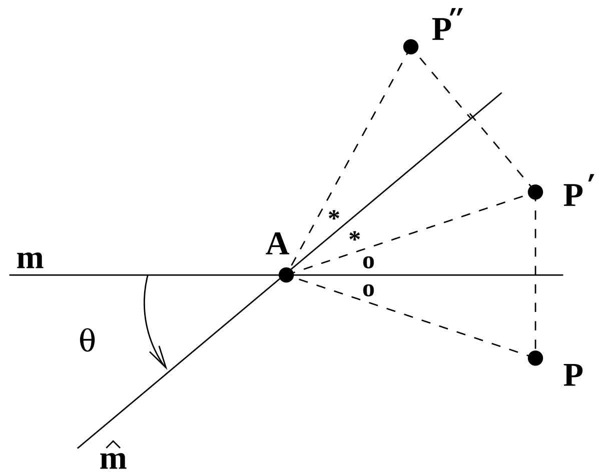


Figure 53: Two reflections in intersecting mirrors.

Proof: Consider a typical point P. Using the ideas from the proof of Theorem 4.2, we find $PA = P'A = P''A$, and $\circ = \circ$, $* = *$. Hence for *any* point P,

$$p = r\hat{r} : P \to P''$$

where $PA = P''A$ and $\angle PAP'' = 2(\circ + *) = 2\theta = \alpha$.

(The angle $\alpha = 2\theta$ is *constant* no matter where P is positioned.) Thus $r\hat{r}$ is a rotation with angle 2θ. //

Corollary 18.2 (One mirror free) *Let p be the rotation with centre A and angle α. Then p factors as a product*

$$p = r\hat{r}$$

of reflections in two mirrors m and $\hat{m}$ through A. Either m or $\hat{m}$ may be chosen at random, with the other adjusted to make with it the angle $\frac{\alpha}{2}$ or $-\frac{\alpha}{2}$ as required.

18.2 Some Examples

In the examples below, r_i denotes the reflection in mirror i.

(a) (i) If mirrors 1 and 2 are perpendicular at A, then $r_1 r_2$ is the $180°$ rotation with centre A.

(ii) **Definition 13** *The* $180°$ *rotation with centre* A *is called a* half-turn *and is denoted* h_A.

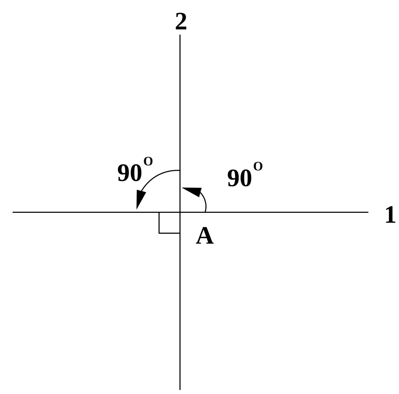
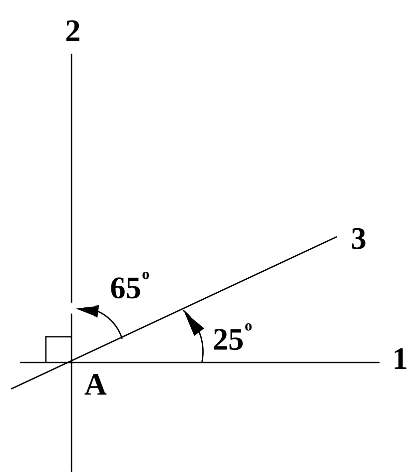


Figure 54: Perpendicular mirrors.

(iii) Note that the angle from mirror 2 to mirror 1 is also $90°$. Hence $r_2 r_1$ is also the $180°$ rotation at A, and so $r_2 r_1 = r_1 r_2$.

(iv) *Conclusion.* Two reflections *commute* if and only if their mirrors are perpendicular.

(b) *Example.*



(i) $r_1 r_2 = r_2 r_1$ is the $180°$ rotation h_A at A.

(ii) $r_1 r_3$ is the $+50°$ rotation at A.

(iii) Since the angle from 3 to 1 is $-25°$ (i.e. clockwise), $r_3 r_1$ must be the $-50°$ rotation at A.

(iv) Hence $r_1 r_3 \neq r_3 r_1$; indeed, the two mirrors are not perpendicular.

(v) $r_2 r_3$ is the $-130°$ rotation at A, which is precisely the same as the $230°$ rotation at A.

(vi) $r_3 r_2$ is the $+130°$ rotation at A.

(c) *Inverse Calculations.* The inverse of any product of reflections

$$p = r_1 r_2 \ldots r_k$$

is the product in reverse order:

$$p^{-1} = r_k \ldots r_2 r_1.$$

(i) Eg. $p = r_1 r_2 r_3$;

$$p^{-1} = r_3 r_2 r_1.$$

Indeed,

$$\begin{array}{rcccc} (r_1 r_2 r_3)(r_3 r_2 r_1) & = & r_1 r_2 r_3^2 r_2 r_1 & = & r_1 r_2 1 r_2 r_1 \\ & = & r_1 r_2^2 r_1 & = & r_1 1 r_1 \\ & = & r_1^2 & = & 1. \end{array}$$

(d) *Examples of Factoring.* Let s be a $30°$ rotation with centre A.

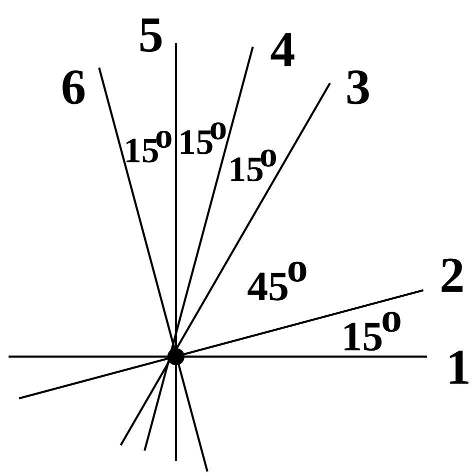


Figure 55:

(i) Write $s = r_1 r$: we must choose the mirror m for r so that the angle from 1 to m is $\frac{1}{2}(30°) = 15°$ (anticlockwise). Thus $m = m_2$, and $s = r_1 r_2$.

(ii) Write $s = rr_4$: the angle *from* m *to* 4 must be $\frac{1}{2}(30°) = 15°$, so m must be situated 15° clockwise from 4. Thus $m = 3$, and

$$s = r_3 r_4.$$

(iii) Write $s^{-1} = r_5 r$: the angle from 5 to m must be $\frac{1}{2}(-30°) = -15°$. Thus $m = 4$ and

$$s^{-1} = r_5 r_4.$$

Hence also

$$s = r_4 r_5.$$

(iv) s^2 is the 60° rotation at A. Thus we have $s^2 = r_3 r_5 = r_4 r_6$.

(v) Let us describe fully the isometry $q = r_1 r_2 r_5$.
Solution. q is a product of an odd number of reflections. Hence it is opposite and is either a glide or a new reflection. But each r_j fixes A, hence so does q. Thus q is a reflection in some mirror m though A. To describe q fully we must determine m exactly.
*Trick** In $q = r_1 r_2 r_5$ replace $(r_1 r_2)$ by a product $r\hat{r}$ which gives some cancellation. But we must then take $\hat{r} = r_5$ (remember we are free to choose one mirror). Thus,

$$\begin{aligned} r_1 r_2 &= 30° \text{ rotation at } A \\ &= ?r_5 \\ &= r_4 r_5 \text{ (we adjusted ? as required).} \end{aligned}$$

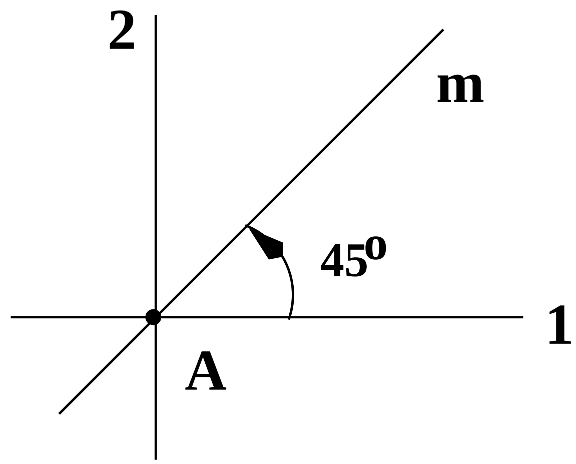
Hence,

$$\begin{aligned} q &= (r_1 r_2) r_5 \\ &= (r_4 r_5) r_5 \\ &= r_4 r_5^2 \\ &= r_4 1 \\ q &= r_4 \text{ !!} \end{aligned}$$

(vi) Similarly,

$$\begin{aligned}(r_5 r_3) r_2 &= (r r_2) r_2 \\ &= r1 \\ &= r\end{aligned}$$

where the mirror m for r is chosen so that the angle *from* 5 *to* 3 equals the angle *from* m *to* 2, which in turn equals $-30°$. Thus m is situated $+30°$ from mirror 2:



(vii) *Challenge*: Referring still to Figure 55, determine exactly the following isometries:

$$r_1 r_2 r_3 r_4$$
$$r_1 r_2 r_1 r_2 r_5$$
$$r_3 r_1 r_5 r_1 r_6 r_1 r_5$$

(e) In the above example we observe that the rotation $s = r_1 r_2 = r_3 r_4 = r_4 r_5$ factors in many ways. Perhaps you find it surprising that a rotation factors as a product of reflections (a different kind of isometry), and in many different ways at that. But there is an analogous situation in arithmetic. Replace *rotation* by *positive number*, and *reflection* by *negative number*, and consider:

$$\begin{aligned}+12 &= (-12)(-1) &&= (-2)(-6) \\ &= (-3)(-4) &&= (-5)(-2.4), \text{ etc.}\end{aligned}$$

Of course reflections—unlike numbers—do not usually commute.

18.3 Reflections in Parallel Mirrors

Not all mirrors m, $\hat{m}$ intersect at some point A. If we get rotations in the intersecting case, what might we get in the parallel case?

Theorem 18.3 *Suppose mirrors m, $\hat{m}$ are* parallel*, where $\vec{AB}$ is the vector running perpendicularly* from m to $\hat{m}$ *(*Figure 56*). Then $t = r\hat{r}$ is a translation with vector $2\vec{AB}$.*

Remark. The resulting translation vector is perpendicular to both m and $\hat{m}$ and is twice as long as the distance separating the mirrors.

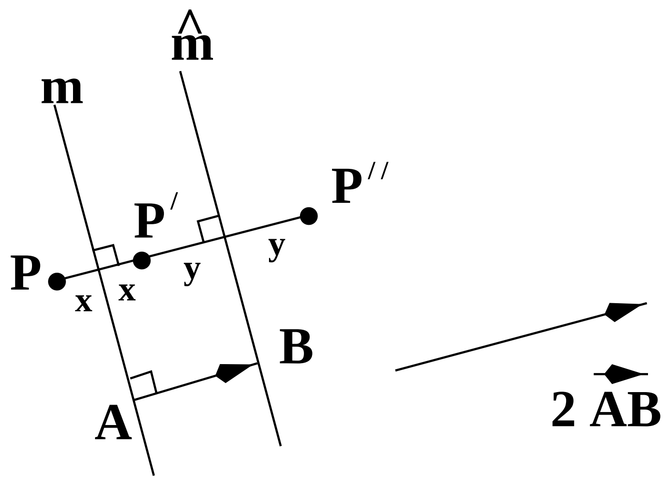


Figure 56: Two reflections in parallel mirrors.

Proof. Consider a typical point P. Then $t = r\hat{r}$: $P \to P''$, as shown. Note that PP'' is $\perp$ to m (or $\hat{m}$) and that the distance from P to P'' is

$$x + x + y + y = 2(x + y) \ ,$$

which is twice the distance from m to $\hat{m}$. Indeed the distance and direction through which P is moved is constant, regardless of the position of P. Thus $r\hat{r}$ must equal the indicated translation t. //

Corollary 18.4 (One mirror free). *Let t be the translation with vector $\vec{PQ}$. Then t factors as a product*

$$t = r\hat{r}$$

of reflections in parallel mirrors m, $\hat{m}$ both perpendicular to $\vec{PQ}$. Given this requirement, either of m or $\hat{m}$ may be chosen at random, with the other situated relative to it by means of the vector $\frac{1}{2}\vec{PQ}$ or $-\frac{1}{2}\vec{PQ}$ as required.

18.4 More Examples

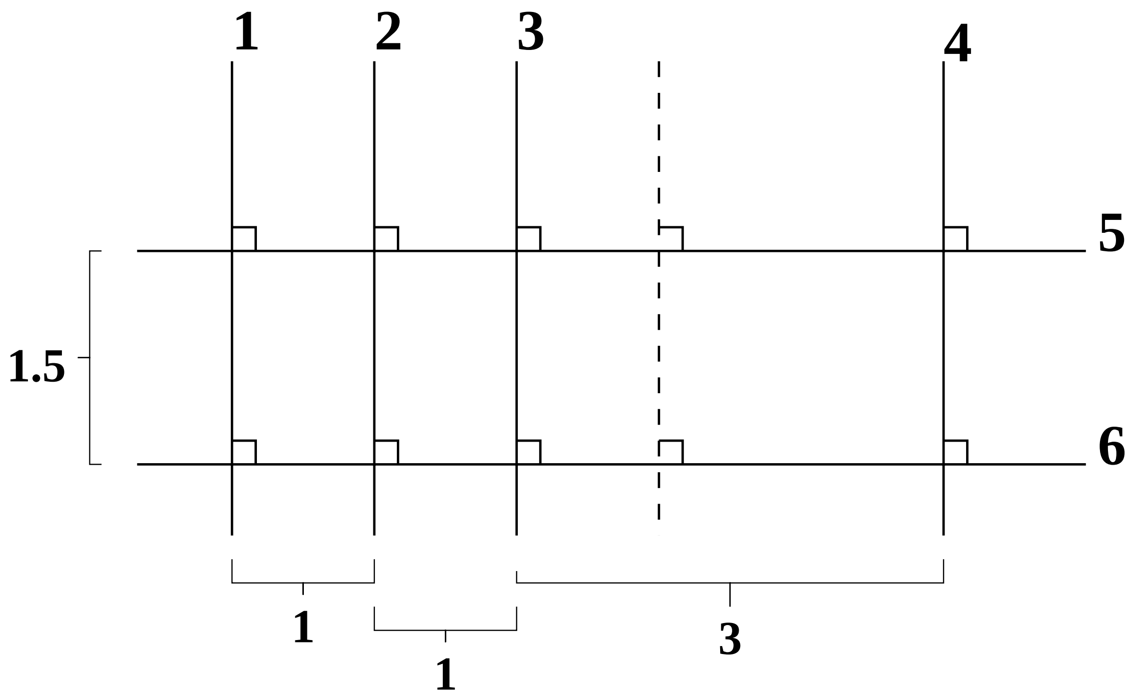


Figure 57: Several parallel or perpendicular mirrors.

(a) (i) $r_1 r_2 = r_2 r_3$ both equal the translation through 2 units right.

(ii) $r_6 r_5$ is the translation through 3 units up.

(iii) $r_4 r_3$ is the translation 6 units left.

(iv) Let t be the translation 8 units left. Write $t = r r_2$: well, the vector from m to mirror 2 must be $\frac{1}{2}(8) = 4$ units to the left. Thus m is 4 units to right of mirror 2, so $t = r_4 r_2$.

(v) Determine $q = r_1 r_3 r_4$. The result is opposite (product of *three* reflections), hence either a glide or new reflection. The mirrors $1, 3, 4$ are parallel, so we try to find some cancellation. Thus we try to write $r r_4 = r_1 r_3$, which is the translation 4 units right. So the vector *from m to* mirror 4 must equal the vector *from* mirror 1 *to* mirror 3, namely the vector through $\frac{1}{2}(4) = 2$ units right.
Thus m is vertical, 2 units left of 4 (see the dotted line), and

$$\begin{aligned} q &= r_1 r_3 r_4 &= (r r_4) r_4 \\ &= r(r_4^2) &= r \cdot 1 \\ &= r &, \end{aligned}$$

the reflection in the vertical line 2 units to the left of line 4.

(b) *Example* (Continued from above).

(i) Determine $q = r_1 r_2 r_5$.
Now $t = r_1 r_2$ is the translation through 2 units right, so that the corresponding vector is parallel to mirror 5. Hence $q = r_1 r_2 r_5 = t r_5$ is a glide through 2 units right, with axis line 5.

(ii) Thus a product of 3 reflections is sometimes a glide. Conversely, since every translation can be similarly factored, every glide can be written as a product of 3 reflections. Note, however, that a glide can be factored differently as a product of $5, 7, 9$ or more reflections.

18.5 Some Concluding Ideas

(a) Every isometry can be factored into a product of reflections, generally in many ways. The most economical way follows:

Isometry	Product of Reflections	Number of Reflections	Sense
reflection	r	1 (odd)	opposite
rotation	$r\hat{r}$	2 (even)	direct
translation	$r\hat{r}$	2 (even)	direct
glide	$r\hat{r}r'$	3 (odd)	opposite

(b) *The Limiting Case.*
Suppose two lines m, $\hat{m}$ intersect in a far distant point A at a very small angle $\theta \approx 0°$:

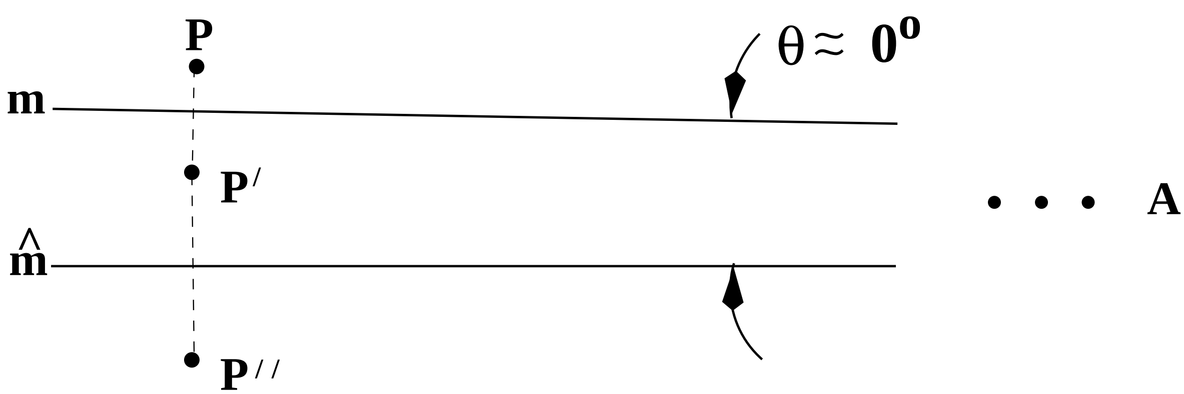


Figure 58: Parallelism in a limiting sense.

Then m and $\hat{m}$ are nearly parallel. Moreover, the rotation $s = r\hat{r}$ through $\alpha = 2\theta \approx 0°$ acts essentially like a translation on a typical point P. Taking A to be further and further away, we thus find it convenient to say that:

(i) * Parallel lines intersect at an infinitely distant point.

(ii) ** A translation acts like a rotation through 0° about an infinitely distant centre.

It should be emphasized that for now this is just an intuitive way of thinking: we don't say 'infinity' exists; and distinct parallel lines, of course, don't intersect. However, in *projective geometry* these ideas can be developed in a precise and rigorous fashion: see [5], for example.

(c) *A Final Example.* By using 3 identical pocket mirrors you can study this example visually. Some very interesting patterns result.
Mirrors m_1, m_2, m_3 enclose an equilateral triangle of side length 4 units. Determine the isometry

$$q = r_1 r_2 r_3.$$

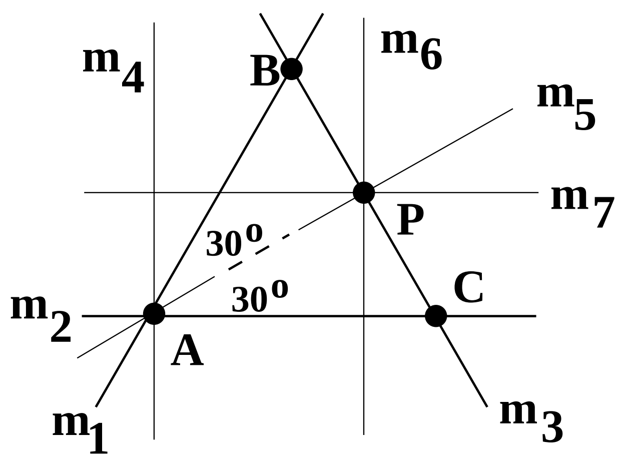


Figure 59: Three mirrors enclosing an equilateral triangle.

Solution:

(i) Let P be the midpoint of BC and let AP be mirror m_5. The crucial point is that this makes m_5 *perpendicular* to m_3.

(ii) Write $r_1 r_2 = r_4 r_5$. We thus require that the angle from m_4 to m_5 should equal the angle from m_1 to m_2, namely $-60°$. Hence, m_4 must be perpendicular to m_2 at A.

(iii) Write $r_5 r_3 = r_6 r_7$, where mirror m_6 passes through P and is perpendicular to m_2. Hence the angle from m_6 to m_7 equals the angle from m_5 to m_3, namely 90°; thus mirror m_7 is horizontal through P.

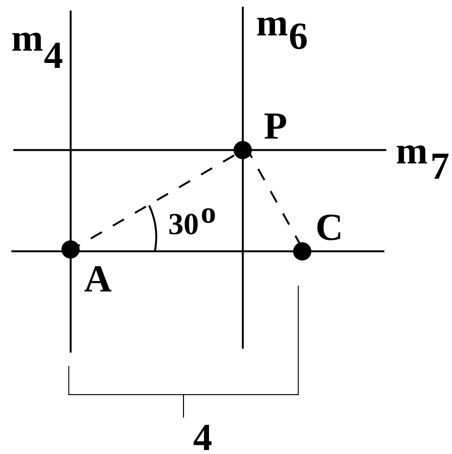


Figure 60: Rearranged reflections.

(iv) Thus

$$\begin{aligned} q &= (r_1r_2)r_3 &= (r_4r_5)r_3 \\ &= r_4(r_5r_3) &= r_4(r_6r_7) \\ &= tr_7 & . \end{aligned}$$

By trigonometry we find that

$$\begin{aligned} d &= (AP)\cos 30^\circ \\ &= (4\sin 60^\circ)\cos 30^\circ \\ &= 4\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = 3\ . \end{aligned}$$

Conclusion. $r_1r_2r_3$ is a glide with axis m_7, though $2\cdot 3 = 6$ units to the right.

19 The Classification Theorem for Isometries

Here we shall actually prove that our list of isometries in Section 15 is complete: in other words, no amount of tricky manoeuvring can produce any isometry other than a reflection, rotation, translation or glide.

First of all we need an easy theorem. We omit the proof, the idea of which comes from the proof of Theorem 4.1.

Theorem 19.1 *The perpendicular bisector of a line segment PP' is the locus of all points Q equidistant from P and P'.*

In other words, suppose M is the midpoint of PP' and m is the line perpendicular to PP' at M. Then Q lies on m if and only if $QP = QP'$.

Next we characterize the identity isometry.

Theorem 19.2 *Suppose an isometry q fixes* each *vertex of some $\triangle ABC$. Then $q = 1$.*

Proof. Looking for a contradiction, we suppose that $q : P \to P'$ where $P \neq P'$. On the other hand we are given that $q : A \to A' = A$. Since q is an isometry, we have

$$PA = P'A' = P'A.$$

Thus A is equidistant from P and P', so by Theorem 19.1 we conclude that A is on the perpendicular bisector m of line segment PP'. But similarly, B and C lie on m. Thus the three vertices of *triangle* $\triangle ABC$ lie on line m: this is a contradiction. In other words, q must fix every point P, so $q = 1$. //

Corollary 19.3 *Suppose $\triangle ABC \equiv \triangle DEF$ and each of two isometries*

$$u, v : \triangle ABC \to \triangle DEF$$

(so u and v each map $A \to D$, $B \to E$, $C \to F$). Then $u = v$.

Proof. The isometry $q = uv^{-1} : \triangle ABC \to \triangle ABC$. By Theorem 19.2, we have

$$q = uv^{-1} = 1\ .$$

Thus $u = v$. //

We have observed many times that an isometry u maps any triangle $\triangle ABC$ to some congruent triangle $\triangle DEF$. Conversely, if we are *given* two such congruent triangles, then we can find an isometry which does the job. This is the upshot of the next theorem:

Theorem 19.4 *Suppose* $\triangle ABC \equiv \triangle DEF$. *Then there exists an isometry* $q : \triangle ABC \to \triangle DEF$.

Proof. We construct q in at most three steps.

(a) If it happens that $A = D$, let $B = B'$ and $C = C'$ and go to step (b). Otherwise, $A \neq D$ and we let m_1 be the perpendicular bisector of segment AD. Thus the reflection r_1 in line m_1 maps

$$\triangle ABC \to \triangle DB'C'$$

and we proceed to the next step.

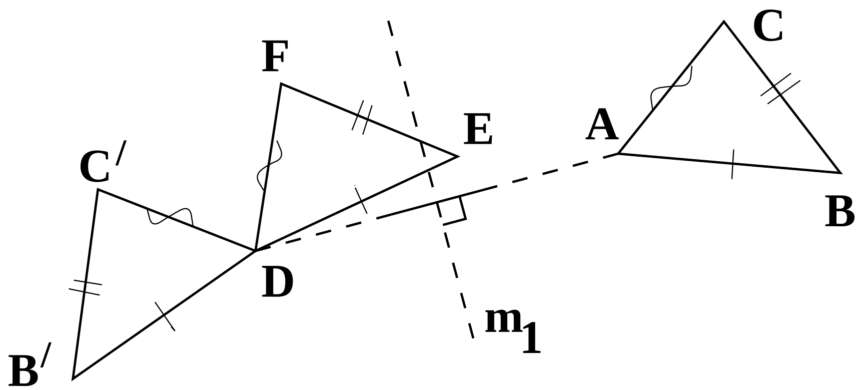


Figure 61: The construction of the first reflection.

(b) Our goal now is to map

$$\triangle DB'C' \to \triangle DEF.$$

Let m_2 be the line bisecting $\angle B'DE$, so that reflection r_2 in line m_2 maps $B' \to E$, $D \to D$, and $C' \to C''$, say. Thus

$$\triangle DB'C' \equiv \triangle DEC'' \equiv \triangle DEF.$$

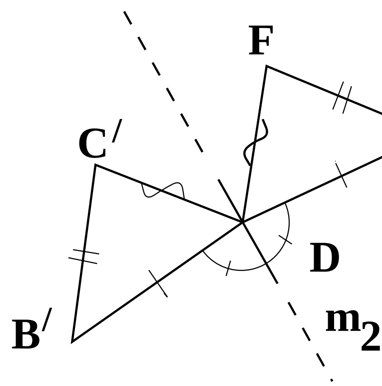


Figure 62: The construction of the second reflection.

Now we may proceed to the next and last step.

(c) Our final goal is to map

$$\triangle DEC'' \to \triangle DEF.$$

In this case, the two congruent triangles in question share two vertices, here D and E. Now depending on whether triangles $\triangle ABC$ and $\triangle DEF$ originally had the same or opposite orientation, we really have two cases here. (In the above Figures, the orientation happened to be initially the same.) As a consequence we can end with C'' and F either on the same or the opposite side of the line m_3 through D and E.

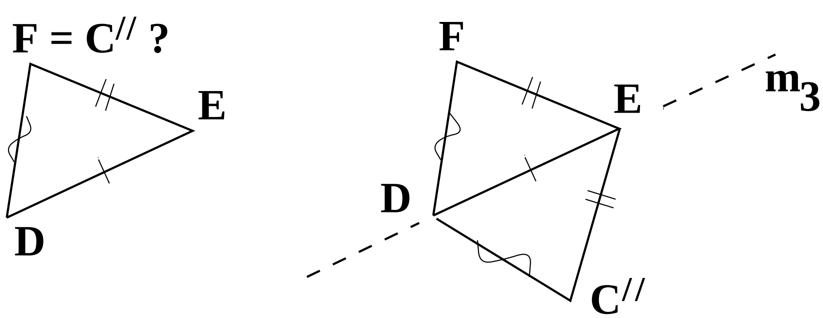


Figure 63: Two final possibilities.

If C'' lies on the same side as F, then since $DF = DC''$ and $\angle FDE = \angle C''DE$, it is easy to see that $F = C''$, and so

$$1 : \triangle DEC'' \to \triangle DEF \ .$$

On the other hand, if C'' and F lie on opposite sides of m_3, then the corresponding reflection r_3 must similarly map $C'' \to F$, and we are done.

In conclusion, we observe that by applying at most three reflections in succession, we are able to map

$$\triangle ABC \to \triangle DEF \qquad . \qquad //$$

In fact, Corollary 19.3 and Theorem 19.4 together imply that there is *exactly one* isometry mapping any given triangle to any other *congruent* triangle. Using the language of group theory, this can be summarized as

Corollary 19.5 *The Euclidean group* **Isom** *acts* sharply transitively *on the triangles in any complete class of mutually congruent triangles.*

Furthermore, the proof of Theorem 19.4 really provides a classification of all isometries.

Corollary 19.6 *Any isometry q is a product of at most three reflections. Any isometry is a reflection, rotation, translation or glide.*

Proof. The isometry q constructed in Theorem 19.4 was a product of at most three reflections. On the other hand, by Corollary 19.3, an arbitrary isometry must look like q anyway. Thus every possible isometry is a product of at most three reflections.

In the simplest case we may think of $q = 1$ as the product of *no* reflections, whereas a reflection $r = r_1$ is itself a 'product' of *one* reflection. We saw in Theorems 18.1 and 18.3 that the product of *two* reflections is either a rotation or translation.

This leaves the case of a product $q = r_1r_2r_3$ of *three* reflections. If the three mirrors are parallel or pass through a common point, then we may conclude as in the exercises of Sections 18.2 and 18.4 that $q = r$ reduces to a single reflection. Otherwise, the three mirrors enclose either a triangle or an infinite strip as in Figure 59 or Figure 57. Each of those figures was used to show that certain products of three reflections yield a glide. In fact, the reasoning is quite general: any product of three reflections in mirrors that are neither concurrent nor mutually parallel must produce a glide. //

This completes an exhaustive classification of the elements of the Euclidean group **Isom**.