

Eigenvalue - based method (2017) closely PRB 91, 155102, FRIGAULT

JPRA, 48520

$$R_i = S_i^P + \lambda \sum_{\alpha \neq i} \left\{ \frac{1}{2} X_{i\alpha} (S_i^{\frac{1}{2}} S_\alpha + S_\alpha^{\frac{1}{2}} S_i) + Z_{i\alpha} S_i^P S_\alpha^P \right\} \quad (2017)$$

$$\rightarrow R_i = \lambda_i \left[-1 + \lambda \sum_{\alpha \neq i} Z_{i\alpha} d_\alpha - \lambda \sum_{\alpha=1}^N Z_{i\alpha} \right]$$

with $1 + \lambda \sum_{\alpha \neq i} Z_{i\alpha} d_\alpha - \lambda \sum_{\alpha=1}^N Z_{i\alpha} = 0$ There are the only unknowns

→ solve for the rapidities first, plug in the eigenvalues later

Q: Can't we solve for the eigenvalues immediately?

$$\boxed{L_i = \sum_{\alpha=1}^N Z_{i\alpha}} \rightarrow R_i = \lambda_i \left[-1 + \lambda \sum_{\alpha \neq i} Z_{i\alpha} d_\alpha - \lambda L_i \right]$$

Start from: (quadratic form; we want to employ γ_{BGK})

$$\begin{aligned} L_i^2 &= \sum_{\alpha} \sum_{\beta} Z_{i\alpha} Z_{i\beta} \\ &= - \sum_{\alpha \neq \beta} Z_{i\alpha} Z_{i\beta} + \sum_{\alpha} Z_{i\alpha}^2 \\ &= - \sum_{\alpha \neq \beta} [Z_{\alpha\beta} (Z_{i\alpha} + Z_{i\beta}) + \Gamma] \\ &\quad + \sum_{\alpha} Z_{i\alpha}^2 \\ &= - N(N-1) \Gamma - \sum_{\alpha \neq \beta} Z_{\alpha\beta} Z_{i\beta} - \sum_{\alpha \neq \beta} Z_{\alpha\beta} Z_{i\alpha} + \sum_{\alpha} Z_{i\alpha}^2 \\ &= - N(N-1) \Gamma - 2 \sum_{\alpha=1}^N Z_{i\alpha} \sum_{\beta} Z_{\alpha\beta} + \sum_{\alpha} Z_{i\alpha}^2 \quad \text{exchange of dummy index} \end{aligned}$$

remember:
 $Z_{i\alpha} Z_{i\beta} \neq Z_{ij} (Z_{i\alpha} + Z_{i\beta}) + \Gamma$

up to now:

~~RG~~

everything is general as consequence of γ_{BGK}

→ Bring them on-shell!

$$= - N(N-1) \Gamma - 2 \sum_{\alpha=1}^N Z_{i\alpha} \left(-\frac{1}{\lambda} - \sum_{\beta} Z_{\alpha\beta} d_\beta \right) + \sum_{\alpha} Z_{i\alpha}^2$$

(3.2)

$$\begin{aligned}
 L_i^2 &= -N(N-1)P - \frac{2}{X} \sum_{\alpha=1}^N Z_{i\alpha}^2 + 2 \sum_{\alpha=1}^N \sum_{\beta=1}^L d_\beta Z_{i\alpha} Z_{\alpha\beta} + \sum_{\alpha} Z_{i\alpha}^2 \\
 &= -N(N-1)P - \frac{2}{X} L_i + 2 \sum_{\alpha=1}^N \sum_{\beta \neq i} d_\beta (Z_{i\beta} (Z_{i\alpha} + Z_{\alpha i}) + P) \\
 &\quad + 2 \sum_{\alpha=1}^N d_i Z_{i\alpha} Z_{\alpha i} + \sum_{\alpha} Z_{i\alpha}^2 \\
 &= -N(N-1)P - \frac{2}{X} L_i + 2 \sum_{\beta \neq i} d_\beta Z_{i\beta} \left(\sum_{\alpha=1}^N Z_{i\alpha} + \sum_{\alpha=1}^N Z_{\alpha i} \right) \\
 &\quad + 2P \sum_{\alpha} d_\alpha - 2d_i \sum_{\alpha=1}^N Z_{i\alpha}^2 + \sum_{\alpha} Z_{i\alpha}^2 \\
 &= [2N \sum_{\beta \neq i} d_\beta - N(N-1)]P - \frac{2}{X} L_i + 2 \sum_{\beta \neq i} d_\beta Z_{i\beta} (L_\beta - L_i) \\
 &\quad + (1-2d_i) \sum_{\alpha=1}^N Z_{i\alpha}^2 \\
 &= N \left[2 \sum_{\beta \neq i} d_\beta - N+1 \right] P - \frac{2}{X} L_i + 2 \sum_{\beta \neq i} d_\beta Z_{i\beta} (L_\beta - L_i) \\
 &\quad + (1-2d_i) \sum_{\alpha=1}^N Z_{i\alpha}^2
 \end{aligned}$$

still contains repulsive ...

but not for $d_i = 1/2$!! \rightarrow

In $d_i = 1/2$:

$$\boxed{L_i^2 = N \left[2 \sum_{\beta \neq i} \frac{1}{2} - N+1 \right] P - \frac{2}{X} L_i + 2 \sum_{\beta \neq i} \frac{1}{2} Z_{i\beta} (L_\beta - L_i)}$$

with σ

$$\boxed{L_i^2 = N (L-N)P - \frac{2}{X} L_i + \sum_{\beta \neq i} Z_{i\beta} (L_\beta - L_i)} \quad \forall i=1..L$$

Note: $XXX \rightarrow P_{20}$ independent of $N \dots \rightarrow$ we need something more to distinguish!

$$\begin{aligned}
 \rightarrow \sum_i L_i &= \sum_{i\alpha} Z_{i\alpha} = \sum_{\alpha} \left(\sum_{\beta \neq \alpha} \frac{1}{2} Z_{\beta\alpha} - \frac{1}{X} \right) = -\frac{2N}{X} \\
 \rightarrow \left(\binom{N}{2} \right) \sum_i L_i &= N \quad \text{symmetric sum over antisymmetric f-a}
 \end{aligned}$$

note:

(*) small g.

$$(\chi \lambda_i)^2 = \chi^2 N(L-N) \cap -2(\chi \lambda_i) + \chi \sum_{j \neq i} Z_R (\cancel{\lambda_j - \lambda_i})$$

$$\text{call } \chi \lambda_i = d_i$$

$$d_i^2 = \chi^2 N(L-N) \cap -2d_i + \chi \sum_{j \neq i} Z_R (d_j - d_i)$$

$$\text{for } \chi \rightarrow 0: \quad d_i^2 = -2d_i \quad \forall i$$

$$\hookrightarrow d_i(d_i + 2) = 0$$

$$\hookrightarrow \begin{cases} d_i = 0 \\ d_i = -2 \end{cases} \rightarrow \text{Q: what is the distribution?}$$

$$- \frac{\chi}{2} \sum_i \lambda_i = - \frac{1}{2} \left(\sum_i d_i \right) = N$$

→ we need $\sum d_i$ to be -2

→ the occupied?

verify for XXX:

$$d_i = \sum_{\alpha \geq 1} \frac{\chi}{\epsilon_i - \epsilon_\alpha} \quad \pi_\alpha \rightarrow \epsilon_j + \frac{\chi}{2} \quad (\text{rest of } L_{N=1}^{-2d_i+1} = -2)$$

* if ϵ_i = one of the occupied ϵ_j

$$d_i = \frac{\chi}{\epsilon_i - \epsilon_j - \frac{\chi}{2}} \rightarrow d_i = -2$$

* if ϵ_i = not one of occupied ϵ_j

$$d_i = \sum_{\alpha \geq 1} \frac{\chi}{\epsilon_i - \epsilon_\alpha - \frac{\chi}{2}} \xrightarrow{\pi_\alpha \rightarrow 0} 0$$

(*) degeneracies? constant higher $\lambda_i^{(n)} = \sum_{\alpha=1}^m z_{i\alpha}^n$ (9.4)

↳ are now $\sum_i z_{i\alpha}^2$ popping up

They where (with) for $n \geq 2$ when $d=1 \rightarrow m=2$
 $(N = -\chi \sum_{i=1}^n z_{i\alpha} d_i \lambda_i)$ $d=3/2 \rightarrow m=3$
 $m=2d$

(*) overlaps? Introduce an auxiliary level ϵ
 but technical ...

(continued)

$$\langle j_1 \dots j_n | \psi(x_\alpha) \rangle = \frac{\det J}{\prod_i x_{i\alpha}}$$

with $J_{ik} = \begin{cases} L_{ik} - \sum_{m \neq k} z_{i\alpha m} + z_{n\alpha k} \\ X_{i\alpha k} \end{cases}$ Closed in λ_i !!

now: more for $X_{i\alpha}$, and

an explicit realization

$$X_{i\alpha} = \frac{x_{n\alpha} x_{i\alpha}}{z_{n\alpha} - z_{i\alpha}}$$

$$z_{i\alpha} = \frac{n + z_{i\alpha} z_{n\alpha}}{z_{n\alpha} - z_{i\alpha}}$$

we know what x_α are ... (somehow)

(*) physical interpretation of λ_i ? $\Rightarrow d_i [-1 + \chi \sum_{\alpha} z_{i\alpha} d_\alpha - \chi \frac{\partial \lambda_i}{\partial x}]$

* Hellmann-Feynman Theorem

$$\begin{aligned} \langle \delta \frac{\partial}{\partial x} \rangle &= \langle R_i - \chi \frac{\partial R_i}{\partial x} \rangle = R_i - \chi \frac{\partial R_i}{\partial x} \\ &= R_i - \chi \left[d_i \left(\sum_{\alpha} z_{i\alpha} d_\alpha - \lambda_i \right) - \chi d_i \frac{\partial \lambda_i}{\partial x} \right] \\ &= R_i - \chi \left[\frac{R_i + d_i}{\chi} - \chi d_i \frac{\partial \lambda_i}{\partial x} \right] \\ &= R_i - \chi \left[R_i + d_i - \chi^2 d_i \frac{\partial \lambda_i}{\partial x} \right] \end{aligned}$$

$$\begin{aligned}\langle S_i^0 \rangle &= -d_i \left[1 - \alpha^2 \frac{\partial L_i}{\partial \alpha} \right] \\ &= -d_i \left[1 - \alpha^2 \frac{\partial}{\partial \alpha} \left(\frac{L_i}{\alpha} \right) \right] \\ &\quad \left(= -d_i \left[1 - \alpha \frac{\alpha^2 \frac{\partial L_i}{\partial \alpha} - d_i}{\alpha^2} \right] \right) \\ &\quad \left(= -d_i \left[1 + \lambda_i - \alpha \frac{\partial \lambda_i}{\partial \alpha} \right] \right)\end{aligned}$$

$$\Rightarrow S_i^0 = \frac{1}{2} M_i - \frac{d_i}{4} \rightarrow S_i^0 = \frac{1}{2} M_i - d_i$$

$$\Rightarrow \frac{1}{2} \langle M_i \rangle = d_i = -\lambda_i + d_i \alpha^2 \frac{\partial L_i}{\partial \alpha}$$

$$\langle M_i \rangle = 2 d_i \alpha^2 \frac{\partial L_i}{\partial \alpha}$$

(*) rapidities from λ_i ? tell λ .

$$P(\alpha) = \prod_{\alpha=1}^N (\alpha - \alpha_\alpha)$$

$$\frac{P'(\alpha)}{P(\alpha)} = \sum_{\alpha=1}^N \frac{1}{\alpha - \alpha_\alpha} \Rightarrow P'(\alpha) \text{ is } \#$$

$$\Rightarrow \frac{P'(\varepsilon_i)}{P(\varepsilon_i)} = \sum_{\alpha=1}^N \frac{1}{\varepsilon_i - \alpha_\alpha} = \lambda_i$$

$$\Rightarrow P'(\varepsilon_i) = \lambda_i P(\varepsilon_i)$$

$$\text{Def } P(x) = \sum_{m=0}^N p_m x^m \Rightarrow P'(x) = \sum_{m=0}^N m p_m x^{m-1}$$

$$\Rightarrow \sum_{m=0}^N m p_m \varepsilon_i^{m-1} = \lambda_i \sum_{m=0}^N p_m \varepsilon_i^m$$

$$\Rightarrow \sum_{m=0}^{N-1} \varepsilon_i^{m-1} (m - \lambda_i \varepsilon_i) p_m = 0 \quad \text{with } p_N = 0$$

$$\sum_{m=0}^{N-1} \varepsilon_i^{m-1} (m - \lambda_i \varepsilon_i) p_m = -\varepsilon_i^{N-1} (N - \lambda_i \varepsilon_i)$$

e.g. special (trial) case:

$$\underline{N=1}: \quad p(x) = (x - x_\alpha) \quad \rightarrow \quad p_0 = -x_\alpha$$

$$\lambda_i = \frac{1}{x - x_\alpha} + \varepsilon_i .$$

$$\sum_{m=0}^{\infty} \varepsilon_i^{m+1} (m - \lambda_i \varepsilon_i) p_m = -\varepsilon_i^0 (1 - \lambda_i \varepsilon_i)$$

$$-\lambda_i p_0 = -(1 - \lambda_i \varepsilon_i)$$

$$+ \frac{p_0}{\varepsilon_i - x_\alpha} = 1 - \left(1 - \frac{\varepsilon_i}{\varepsilon_i - x_\alpha} \right)$$

$$\begin{aligned} p_0 &= \frac{1}{\varepsilon_i - x_\alpha - \varepsilon_i} \\ p_0 &\stackrel{!}{=} -x_\alpha \end{aligned}$$

other algorithms exist!