

overlaps:

assume a general pairing state

$$|\Psi(G)\rangle = \frac{N}{L!} \left( \sum_{i=1}^L g_{xi} s_i^+ \right) |0\rangle$$

genuine state (chemistry).

genuine pairing  $G = \begin{pmatrix} G_{11} & \cdots & G_{1N} & | & G_{1,N+1} & \cdots & G_{1L} \\ \vdots & & \vdots & | & \vdots & & \vdots \\ G_{N1} & \cdots & G_{NN} & | & G_{N,N+1} & \cdots & G_{NL} \end{pmatrix} \quad \left\{ \begin{array}{l} N \\ L-N \end{array} \right.$

Q1: what is the overlap with an SD? (no degeneracy,  $\Rightarrow$  feasible).

$$\langle j_1 j_2 \dots j_N | = S_{j_1}^+ S_{j_2}^- \dots S_{j_N}^+ |0\rangle \quad \text{---}$$

There is no ordering: ~~---~~

$$\langle j_1 j_2 | = \langle j_2 j_1 |$$

N2:  $\langle j_1 | \Psi(G) \rangle = ?$

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~~---~~  
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$$\begin{aligned} |\Psi(G)\rangle &= \sum_j g_{xj} S_j^+ |0\rangle \\ &= \sum_{j \in L} g_{xj} |j\rangle \Rightarrow \langle j_1 | \Psi(G) \rangle = \sum_j g_{xj} \langle j | j \rangle = g_{xj_1} \end{aligned}$$

N3:  $\langle \Psi(G) | = \left( \sum_{j \in L} g_{xj} S_j^+ \right) \left( \sum_i g_{xi} S_i^+ \right) |0\rangle$

$$= \sum_j g_{xj} g_{xi} \underbrace{S_j^+ S_i^+}_{|0\rangle} = \sum_j g_{xj} g_{xi} \langle j | i \rangle$$

$$\begin{aligned} \rightarrow \langle j_1 j_2 | \Psi(G) \rangle &= \sum_j g_{xj} g_{xi} \underbrace{\langle j_1 j_2 | j i \rangle}_{(*)} \\ &\quad (\delta_{j_1 j} \delta_{j_2 i} + \delta_{j_1 i} \delta_{j_2 j}) \end{aligned}$$

$$= g_{xj_1} g_{xi} j_2 + g_{xj_2} g_{xi} j_1$$

$$= \begin{vmatrix} g_{xj_1} & g_{xj_2} \\ g_{xj_2} & g_{xj_1} \end{vmatrix} \quad \text{permanent}$$

$$\underline{N}: \quad \langle j_1, j_2, \dots, j_n | \psi(\sigma) \rangle = \left| \begin{array}{cccc} G_{1,j_1} & G_{1,j_2} & \dots & G_{1,j_n} \\ G_{2,j_1} & G_{2,j_2} & & : \\ : & : & & \\ G_{n,j_1} & G_{n,j_2} & \dots & G_{n,j_n} \end{array} \right| +$$

analogously: because you keep all permutations of  $(j_1, j_2, \dots, j_n)$

General formula for a PERMANT.

$$\boxed{\text{per}(G) = \sum_{\sigma} \prod_{i=1}^n G_{i,\sigma(i)}} \quad \text{LAPLACE}$$

Note: (\*) switching 2 columns/rows  $\Rightarrow \text{per}(A) \rightarrow \text{per}(B)$ .

(\*\*)  $\hookrightarrow$  no  $\text{per}(A) \neq 0$  when  $G_{i,i} = G_{j,j} \forall i$ .  
two equal rows/columns  
 $\hookrightarrow$  no Gaussian Elimination!!! :-c

$\hookrightarrow$  only Laplace formula is known (horribly, owing)  
[brunnerissement safety  $n!$  here to that! ]

[complexity issues: if you find a general algorithm  
to compute  $\text{per}(A)$  in Polynomial time -

$$\#P = FP \text{ stronger than } P = NP$$

$\rightarrow$  no way José ...

you have to stress it

examples: (1) Projected BCS: (AGP)

$$\langle \psi(G) \rangle = \left( \sum_{j_1}^l G_{j_1} S_j^+ \right)^n |0\rangle$$

$$G = \begin{pmatrix} G_1 & G_2 & G_N & G_{N+1} & \dots & G_n \\ G_1 & G_2 & G_N & & & \\ : & : & : & & & \\ G_N & G_2 & G_N & & & G_n \end{pmatrix}$$

$\rightsquigarrow$  per

$$\begin{aligned} \langle j_1, j_2, \dots, j_N | G \rangle &= \text{per} \begin{pmatrix} G_{j_1} & G_{j_2} & \dots & G_{j_N} \\ & G_{j_1} & & G_{j_N} \\ & & \ddots & \\ & & & G_{j_N} \end{pmatrix} \\ &= \left( \prod_{k=1}^N G_{j_k} \right) \text{per} \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & & & \\ 1 & 1 & \dots & 1 \end{pmatrix} = \\ &= N! \left( \prod_{k=1}^N G_{j_k} \right) \end{aligned}$$

### (1) Generalized Valence Bond (APer)

$$G = \left( \begin{array}{c|c|c|c} ( ) & ( ) & \vdots & ( ) \\ ( ) & ( ) & \vdots & ( ) \\ \hline \end{array} \right)$$

### (2) Richardson-Greenlin:

$$\boxed{G_{ia} = X_{ia}} \quad \text{with } \begin{cases} X_{ij} X_{ji} = X_{ii} (z_j + z_{ji}), \\ X_{ij} = -X_{ji}. \end{cases}$$

How to get an expression?

Muir's theorem: (Singer El. Jour. Combinatorics 13, 73 (2004))

$$\boxed{\det(A) \cdot \text{per}(B) = \sum_{\sigma} \det(A * B_{\sigma})}$$

$\rightarrow *$   $*$   $\therefore$  Hadamard product:  $(A * B)_{ij} = A_{ij} B_{ij}$

$\rightarrow (B_{\sigma})_{ij} = B_{\sigma(i)j} \rightarrow$  you ~~swap~~ permute the columns according to  $\sigma(j)$

Proof: Muir

$$\det A \operatorname{per} B = \sum_{\sigma} (-1)^{\sigma} \frac{n}{n} A_{i(\sigma(i))} \sum_{\tau} (-1)^{\tau} \frac{n}{n} B_{j(\tau(j))}$$

$$= \sum_{\sigma} \sum_{\tau} (-1)^{\sigma} \frac{n}{n} A_{i(\sigma(i))} B_{j(\tau(i))}$$

but  $\tau = g \circ \sigma$  ("g after  $\sigma$ ")

$$\begin{aligned} &= \sum_g \sum_{\sigma} (-1)^{\sigma} \frac{n}{n} A_{i(\sigma(i))} B_{i(g(\sigma(i)))} \\ &= \sum_g \sum_{\sigma} (-1)^{\sigma} \frac{n}{n} A_{i(\sigma(i))} (B_g)_{i(\sigma(i))} \\ &= \sum_g \sum_{\sigma} (-1)^{\sigma} \frac{n}{n} (A * B_g)_{i(\sigma(i))} \\ &= \sum_g \det(A * B_g) \end{aligned}$$

Through the proof: Muir is basically a different way to write the LAPLACE expansion

However: Borchardt Theorem (generalised) ~~(classy PRB)~~

$$\boxed{\det(X) \operatorname{per}(X) = \det(X * X)}.$$

$\hookrightarrow$  no summation: wow!

Proof: Muir:  $\det X \operatorname{per} X = \sum_{\sigma} \det(X * X_{\sigma})$

- consider each  $\sigma \in S_n$  (not  $A$ ).

- identify a cycle (each permutation has at least one).

$\Rightarrow$  assume this cycle to have e.g.:  $\sigma = (135)(24)$   
length  $m \leq n$   $\Rightarrow \sigma^m(i) = i = \overbrace{\sigma \circ \sigma \circ \dots \circ \sigma}^m$

claim: as the  $m$  columns  $(X * X_\sigma)_{j \in \sigma^k(i)}$  are linear dependent with weights  $(X_{\sigma^k(i)} \sigma^{k+1}(i))^{-1}$

$$\begin{aligned}
 & \cancel{\sum_{\lambda=0}^{m-1}} \underbrace{X_{j \sigma^k(i)} X_{j \sigma(\sigma^k(i))}}_{X_{\sigma^k(i)} \sigma^{k+1}(i)} \\
 & = - \sum_{\lambda=0}^{m-1} \frac{X_{\sigma^k(i) j} X_{j \sigma^{k+1}(i)}}{X_{\sigma^k(i)} \sigma^{k+1}(i)} \quad \text{YBGE} \\
 & = - \sum_{\lambda=0}^{m-1} \frac{X_{\sigma^k(i) \sigma^{k+1}(i)} (Z_{\sigma^k(i) j} + Z_{j \sigma^{k+1}(i)})}{X_{\sigma^k(i)} \sigma^{k+1}(i)} \\
 & = - (Z_{ij} + Z_{j \sigma^m(i)}) \quad (\text{other cancel exactly}) \\
 & = - (Z_{ij} + Z_{ji}) = 0. \quad Z_{ij} + Z_{ji} = 0
 \end{aligned}$$

$\Rightarrow \det(X * X_\sigma) = 0 \quad \text{if } \sigma \neq \text{id}.$

$\Rightarrow \det(X) \operatorname{per}(X) = \det(X * X)$

example:  $XX^\top$ :  $X_{i\alpha} = \frac{1}{\varepsilon_i - x_\alpha} \Rightarrow (X * X)_{i\alpha} = \frac{1}{(\varepsilon_i - x_\alpha)^2}$

$$\begin{aligned}
 & \Rightarrow \operatorname{per} \left( \begin{array}{ccc} \frac{1}{\varepsilon_1 - x_1} & \frac{1}{\varepsilon_1 - x_2} & \cdots & \frac{1}{\varepsilon_1 - x_N} \\ \vdots & & & \\ \frac{1}{\varepsilon_N - x_1} & \cdots & \frac{1}{\varepsilon_N - x_N} \end{array} \right) = \frac{\det \left( \begin{array}{ccc} \frac{1}{(\varepsilon_1 - x_1)^2} & \cdots & \frac{1}{(\varepsilon_1 - x_N)^2} \\ \vdots & & \\ \frac{1}{(\varepsilon_N - x_1)^2} & \cdots & \frac{1}{(\varepsilon_N - x_N)^2} \end{array} \right)}{\det \left( \begin{array}{ccc} \frac{1}{\varepsilon_1 - x_1} & \cdots & \frac{1}{\varepsilon_1 - x_N} \\ \vdots & & \\ \frac{1}{\varepsilon_N - x_1} & \cdots & \frac{1}{\varepsilon_N - x_N} \end{array} \right)}
 \end{aligned}$$

Consequently  $\exists$  closed form  
in context of Lagrange interpolation

$$\det\left(\frac{1}{\epsilon_\alpha - \epsilon_\beta}\right) = \frac{\prod_{i=1}^N (\epsilon_i - \epsilon_\alpha) \prod_{\alpha < \beta} (\alpha_\alpha - \eta_\beta)}{\prod_{i=1}^N \prod_{\alpha < \beta} (\epsilon_\alpha - \alpha_\alpha)}$$

degeneracies? : not nice, not instructive, terrible.

d<sub>2</sub>: what about overlaps? NORMS!

From the ABA : (in the "classical" limit  $\epsilon_\alpha \approx \alpha$ )

Hermitian ( $\delta(\alpha)$ ) [GAUDIN, RICHARDSON JMP 6, 1034 (1965)]

NORMS:  $\langle u(x_\alpha) | u(x_\beta) \rangle = \det G$  (Gaudin matrix)

$$G_{\alpha\beta} = \begin{cases} \sum_{i=1}^N \frac{1}{(\alpha_\alpha - \epsilon_i)^2} - 2 \sum_{\beta \neq \alpha} \frac{1}{(\alpha_\alpha - \eta_\beta)^2} & \Rightarrow \alpha \neq \beta \\ \frac{g}{(\alpha_\alpha - \eta_\beta)^2} & \alpha \neq \beta \end{cases}$$

OVERLAP:  $\langle u(y_\alpha) | u(x_\alpha) \rangle = \frac{\prod_{\alpha < \beta} (y_\alpha - y_\beta)}{\prod_{\alpha < \beta} (\alpha_\alpha - \eta_\beta) \prod_{\alpha < \beta} (y_\beta - y_\alpha)} \det(J)$   
(SLAVNOV)

$$J_{\alpha\beta} = \frac{x_\beta - y_\beta}{\alpha_\alpha - y_\beta} \left( \sum_i \frac{1}{(\alpha_\alpha - \epsilon_i)(y_\beta - \epsilon_i)} - 2 \sum_{\gamma \neq \alpha} \frac{1}{(\alpha_\alpha - \eta_\gamma)(y_\beta - \eta_\gamma)} \right)$$

only one! of  $|u(x_\alpha)\rangle$  or  $|u(y_\beta)\rangle$  need to be on shell!!

(not very beautiful)

other ways?: DUAL STATES

## DUAL STATES:

$$|\tilde{q}\rangle = \prod_{\alpha=1}^N (S_\alpha^+) |0\rangle \rightarrow |\tilde{q}\rangle \stackrel{?}{=} \prod_{\alpha=1}^{L-N} S_\alpha^- |\tilde{\alpha}\rangle$$

thus now the algebra:

remember:  $s(i)$  rotates

$$\begin{pmatrix} A_i^+ \\ A_i^0 \\ A_i^- \end{pmatrix} = \begin{pmatrix} u_i^2 & 2u_i s_i & -s_i^2 \\ -u_i s_i^* & u_i^2 - u_i^2 & -u_i^* s_i \\ -s_i^{*2} & 2u_i^* s_i^* & u_i^{*2} \end{pmatrix} \begin{pmatrix} S_i^+ \\ S_i^0 \\ S_i^- \end{pmatrix}$$

take  $u_i = 0$ ,  $s_i = 1$  ( $\alpha$   $s_i = i$ ).  
doesn't matter

$$\Rightarrow \begin{pmatrix} A_i^+ \\ A_i^0 \\ A_i^- \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} S_i^+ \\ S_i^0 \\ S_i^- \end{pmatrix}$$

check:  
 $A_i^0 |0\rangle = -S_i^0 |0\rangle$   
 $= -d_i |0\rangle$

$$\begin{array}{l} \sim A_i^+ = -S_i \\ A_i^0 = -S_i^0 \\ S_i^- = -S_i^+ \end{array} \left\{ \begin{array}{l} \text{lowest weight becomes highest weight etc} \\ A_i^+ |\tilde{\alpha}\rangle = 0 \Rightarrow -S_i^+ |\tilde{\alpha}\rangle = 0 \end{array} \right.$$

$$\Rightarrow R_i = S_i^0 + \alpha \sum_{\tilde{\alpha} \neq i} \frac{1}{2} \chi_{i\tilde{\alpha}} (S_i^+ S_{\tilde{\alpha}} + S_{\tilde{\alpha}}^+ S_i) + 2i \chi_{i\tilde{i}} S_i^0 S_{\tilde{i}} \Rightarrow |\tilde{\alpha}\rangle = \text{filled.}$$

$$= -A_i^0 + \alpha \sum_{\tilde{\alpha} \neq i} \frac{1}{2} \chi_{i\tilde{\alpha}} (A_i^+ A_{\tilde{\alpha}} + A_{\tilde{\alpha}} A_i^+) + 2i \chi_{i\tilde{i}} A_i^0 A_{\tilde{i}}^0$$

$$= - [ A_i^0 - \alpha \sum_{\tilde{\alpha} \neq i} \frac{1}{2} \chi_{i\tilde{\alpha}} (A_i^+ A_{\tilde{\alpha}} + A_{\tilde{\alpha}} A_i^+) + 2i \chi_{i\tilde{i}} A_i^0 A_{\tilde{i}}^0 ].$$

$\sim |\tilde{q}\rangle = \frac{L-N}{N} (A_{\tilde{\alpha}}^+) |0\rangle = \frac{L-N}{N} A_{\tilde{\alpha}}^+ |0\rangle$

with ~~no~~ relationship

Is an eigenstate of  $R_i$ ?

There is a guaranteed dual state! (because we have converted it to a canonical form)

$$1 - \alpha \sum_{\tilde{\alpha}} 2g_{\tilde{\alpha}} d_{\tilde{\alpha}} + \alpha \sum_{\tilde{\alpha} \neq \tilde{\beta}} 2\tilde{\alpha} \tilde{\beta} = 0. \quad \forall \tilde{\alpha} = 1 \dots L-N$$

$\Rightarrow$  Eigenvalues are

$$\eta_i = -d_i \left[ -1 + \chi \sum_{\alpha \neq i} z_{i\alpha} d_\alpha + \chi \sum_{\alpha=1}^{L-N} z_{i\bar{\alpha}} \right]$$

$$\eta_i = d_i \left[ 1 + \chi \sum_{\alpha \neq i} z_{i\alpha} d_\alpha - \chi \sum_{\alpha=1}^N z_{i\bar{\alpha}} \right]$$

anyone with

$$\eta_i = d_i \left[ -1 + \chi \sum_{\alpha \neq i} z_{i\alpha} d_\alpha - \chi \sum_{\alpha=1}^N z_{i\bar{\alpha}} \right]$$

note that

$$\eta_i = \eta_i : \quad \eta_i \left[ 1 + \chi \sum_{\alpha \neq i} z_{i\alpha} d_\alpha - \chi \sum_{\alpha=1}^{L-N} z_{i\bar{\alpha}} \right]$$

$$= d_i \left[ -1 + \chi \sum_{\alpha \neq i} z_{i\alpha} d_\alpha - \chi \sum_{\alpha=1}^N z_{i\bar{\alpha}} \right]$$

$$\Rightarrow \chi \left( \sum_{\alpha=1}^N z_{i\alpha} - \sum_{\alpha=1}^{L-N} z_{i\bar{\alpha}} \right) + 2 = 0. \quad \text{See later}$$

$$\Rightarrow \left| \tilde{4} \right\rangle = \prod_{\alpha=1}^{L-N} A_\alpha^+ | \tilde{4} \rangle = \cancel{\left( \prod_{\alpha=1}^{L-N} S_\alpha^+ \right)} | \tilde{4} \rangle.$$

$$\left| 4 \right\rangle = \prod_{\alpha=1}^N S_\alpha^+ | 0 \rangle$$

$$\Rightarrow \langle \tilde{4} | \tilde{4} \rangle \langle 4 | 4 \rangle$$

Let  $|4_N\rangle = N |4\rangle$  so  $\langle 4_N | 4_N \rangle = 1$

$\langle \tilde{4}_N | \tilde{4}_N \rangle = \tilde{N} | \tilde{4} \rangle$  "  $\langle \tilde{4}_N | 4_N \rangle = 1$

$$\left| 4_N \right\rangle \equiv | \tilde{4}_N \rangle$$

$$\begin{aligned} 1 &= \langle \tilde{4}_N | \tilde{4}_N \rangle = N \cdot \tilde{N} \langle \tilde{4} | \tilde{4} \rangle \\ &= N \tilde{N} \langle \tilde{4} | \left( \prod_{\alpha=1}^{L-N} S_\alpha^+ \right)^+ \left( \prod_{\alpha=1}^N S_\alpha^+ \right) | 0 \rangle \\ &= N \tilde{N} \underbrace{\langle \tilde{4} | \prod_{\alpha=1}^{L-N} S_\alpha^+}_{\tilde{N}} \underbrace{\prod_{\alpha=1}^N S_\alpha^+}_{N} | 0 \rangle \end{aligned}$$

$$|\tilde{4}\rangle = |j_1, j_2, \dots, j_L\rangle \quad \text{everything filled SD}$$

$$\frac{L-N}{\prod_{\alpha=1}^{L-N} S_\alpha^+} \frac{N}{\prod_{\alpha=1}^N S_\alpha^+} | 0 \rangle : \text{General with } .$$

$$\delta x = \begin{pmatrix} x_{d1}, \dots, x_{dN}, x_{1\bar{N}}, \dots, x_{1\bar{L-N}} \\ x_{d2} \\ x_d \\ \vdots \\ x_L \end{pmatrix}$$

(8.9)

$$G = \begin{pmatrix} X_{\alpha_1 i_1} & X_{\alpha_1 i_2} & \dots & X_{\alpha_1 i_L} \\ X_{\alpha_2 i_1} & X_{\alpha_2 i_2} & & \vdots \\ \vdots & \vdots & & \vdots \\ X_{\alpha_n i_1} & & & \\ X_{\alpha_n i_1} & & & \\ \vdots & & & \\ X_{\alpha_{L+1} i_1} & & & X_{\alpha_{L+1} i_L} \end{pmatrix}$$

$$\cancel{\det G} \Rightarrow \langle \tilde{\psi} | \psi \rangle = \text{per } G = \frac{\det(G \times G)}{\det G}$$

$$\Rightarrow L = \tilde{N} N \frac{\det G \times G}{\det G}$$

or, we know  $\tilde{N} N$ ,  $\rightarrow$  how about separately?

$$|\tilde{\psi}_N\rangle = |\psi_N\rangle$$

$$\rightarrow \tilde{N} |\tilde{\psi}\rangle = N |\psi\rangle \quad \text{take in product with any SD}$$

$$\tilde{N} \langle \text{SD} | \tilde{\psi} \rangle = N \langle \text{SD} | \psi \rangle$$

$$\Rightarrow \frac{\tilde{N}}{N} = \frac{\langle \text{SD} | \psi \rangle}{\langle \text{SD} | \tilde{\psi} \rangle} \sim \text{ratio of determinants.}$$

note: orthogonal when

note: SLAVON equivalent:  $\langle \{x_\alpha\} | \{y_\alpha\} \rangle$

\* assume only  $\{x_\alpha\}$  is coming from RG (on shell)

$\rightarrow$  make the dual of  $|\{x_\alpha\}\rangle$

$\rightarrow$  promote  $|\tilde{x}_\alpha\rangle$  (and  $|y_\alpha\rangle$ )

$$\rightarrow \langle \{x_\alpha\} | \{y_\alpha\} \rangle = \frac{1}{N} \langle \tilde{x}_\alpha | \{x_\alpha\} | \{y_\alpha\} \rangle$$

$$= \frac{1}{N} \langle \tilde{x}_{\alpha \beta} | \{y_\alpha\} \rangle$$

(8.10)

$$\begin{aligned}
 &= \frac{\tilde{N}}{N} \langle s_i^+ | s_\alpha^+ \rangle \\
 &= \frac{\tilde{N}}{N} \det_{\alpha=1}^{\tilde{N}} (s_\alpha^+ | \overline{s}_\alpha^+) (\overline{s}_\alpha^+ | \overline{s}_\alpha^+) \text{ b} \rangle \\
 &\stackrel{!}{=} \frac{\tilde{N}}{N} \frac{\det_{\alpha=1}^{\tilde{N}} (s_\alpha^+ | s_\alpha^+)}{\det_{\alpha=1}^{\tilde{N}} (s_\alpha^+ | s_\alpha^+)}
 \end{aligned}$$

Q3: What about correlation wef?

$$\langle s_i^+ \rangle = ?$$

$$\begin{aligned}
 \langle s_i^+ | \psi \rangle &= \langle s_i^+ | \frac{N}{\tilde{N}} \sum_{\alpha=1}^{\tilde{N}} s_\alpha^+ | \text{b} \rangle \\
 &= \left( \left[ \langle s_i^+ | \frac{N}{\tilde{N}} s_\alpha^+ \right] \frac{N}{\tilde{N}} s_\alpha^+ | s_i^+ \rangle \right) \text{b} \rangle \\
 &= \sum_{\alpha=1}^{\tilde{N}} \left( \frac{N}{\tilde{N}} s_\alpha^+ \right) [\langle s_i^+ | s_\alpha^+ ] \frac{N}{\tilde{N}} s_\alpha^+ | \text{b} \rangle \\
 &= \sum_{\alpha=1}^{\tilde{N}} \left( \frac{N}{\tilde{N}} s_\alpha^+ \right) (X_{i\alpha} s_i^+) \frac{N}{\tilde{N}} s_\alpha^+ | \text{b} \rangle - d_i | \psi \rangle \\
 &= \sum_{\alpha=1}^{\tilde{N}} X_{i\alpha} s_i^+ \frac{N}{\tilde{N}} s_\alpha^+ | \text{b} \rangle - d_i | \psi \rangle \\
 \Rightarrow \langle \psi | s_i^+ | \psi \rangle &= \underbrace{\sum_{\alpha=1}^{\tilde{N}} \langle \psi | X_{i\alpha} s_i^+ \frac{N}{\tilde{N}} s_\alpha^+ | \text{b} \rangle}_{\text{sum of genuine s. overlaps}} - d_i \langle \psi | \psi \rangle.
 \end{aligned}$$

$\langle s_i^+ s_j^+ \rangle$  - analogously.