

FACTORISABLE MODEL (XX)

$$(*) X_{ik} = \frac{\varepsilon_i \varepsilon_k}{\varepsilon_i^2 - x_k^2} \quad Z_{ik} = \frac{1}{2} \frac{\varepsilon_i^2 + \varepsilon_k^2}{\varepsilon_i^2 - x_k^2} \quad P = -\frac{1}{4}$$

$$(*) R_i = S_i^0 + \lambda \sum_{k \neq i} \frac{\varepsilon_i \varepsilon_k}{\varepsilon_i^2 - x_k^2} \left[\frac{1}{2} (S_i^+ S_k^- + S_k^+ S_i^-) + \frac{1}{2} \frac{\varepsilon_i^2 + \varepsilon_k^2}{\varepsilon_i^2 - x_k^2} S_i^0 S_k^0 \right]$$

$$= S_i^0 + \lambda \sum_{k \neq i} \left\{ \frac{\varepsilon_i \varepsilon_k}{\varepsilon_i^2 - x_k^2} (S_i^+ S_k^-) + \frac{1}{2} \underbrace{\frac{\varepsilon_i^2 + \varepsilon_k^2}{\varepsilon_i^2 - x_k^2} S_i^0 S_k^0}_{\text{ANISOTROPY}} \right\}$$

$$(*) X_{ik} = \frac{\varepsilon_i x_k}{\varepsilon_i^2 - x_k^2}$$

$$\text{GOAL: } \begin{cases} [S_\alpha^1 : S_\beta^+] = 2 \frac{x_\alpha x_\beta}{x_\alpha^2 - x_\beta^2} (S_\alpha^0 - S_\beta^0) \\ [S_\alpha^0 : S_\beta^+] = \frac{1}{2} \frac{2x_\alpha x_\beta S_\alpha^+ - (x_\alpha^2 + x_\beta^2) S_\beta^+}{x_\alpha^2 - x_\beta^2} \end{cases}$$

$$(*) q_i = d_i \left[-1 + \lambda \sum_{k \neq i} \frac{1}{2} \frac{\varepsilon_i^2 + \varepsilon_k^2}{\varepsilon_i^2 - x_k^2} S_k - \lambda \sum_{k=1}^N \frac{1}{2} \frac{\varepsilon_i^2 + x_k^2}{\varepsilon_i^2 - x_k^2} \right]$$

$$(*) |q\rangle = \sum_{\alpha=1}^N S_\alpha^+ |0\rangle = \sum_{\alpha=1}^N \left(\sum_{k=1}^L \frac{\varepsilon_i x_k}{\varepsilon_i^2 - x_k^2} S_i^+ \right) |0\rangle = \left(\prod_{\alpha=1}^N x_\alpha \right) \sum_{\alpha=1}^N \frac{\varepsilon_i S_\alpha^+}{\varepsilon_i^2 - x_\alpha^2} S_i^+$$

$$(*) R \underline{\text{eq}}: 1 + \lambda \sum_{k \neq i} \frac{1}{2} \frac{\varepsilon_i^2 + x_k^2}{\varepsilon_i^2 - x_k^2} S_k \stackrel{\text{normalize}}{\underset{\text{sa.}}{\sim}} \frac{1}{2} \frac{x_\rho^2 + x_\sigma^2}{x_\rho^2 - x_\sigma^2} \sim$$

One can derive the factorisable model

$$\sum_{i=1}^L \varepsilon_i^2 R_i = \underbrace{\left[1 + \frac{1}{2} \lambda \sum_k S_k^2 - \frac{1}{2} \lambda \right]}_{\gamma = \text{ab.}} \sum_{i=1}^L \varepsilon_i^2 S_i^0 + \frac{1}{2} \lambda \sum_k \frac{\varepsilon_i \varepsilon_k}{\varepsilon_i^2 - x_k^2} S_i^+ S_k^- - \frac{\lambda}{2} \sum_i \varepsilon_i^2 C_i$$

$$\gamma = \text{ab.}$$

$$\text{because } \langle \sum_i S_i^0 \rangle = N - \sum_k x_k = N - 1$$

$$\text{Ansatz: } = \underbrace{\gamma \sum_i \varepsilon_i^2 S_i^0}_{\text{1 part}} + \underbrace{\gamma \sum_k \varepsilon_i \varepsilon_k S_i^+ S_k^-}_{\text{2 parts}} - \underbrace{\gamma \sum_i \varepsilon_i^2 C_i}_{\text{3 part}}$$

Proof (always the same game)

$$\begin{aligned}
 \sum_i \varepsilon_i^2 R_i &= \sum_i \varepsilon_i^2 \left[S_i^0 + \frac{\chi}{2} \sum_i \frac{\varepsilon_i^2 \varepsilon_{ik}}{\varepsilon_i^2 - \varepsilon_k^2} \frac{1}{2} (S_i^+ S_k^- + S_k^+ S_i^-) + \frac{1}{2} \frac{\varepsilon_i^2 + \varepsilon_k^2}{\varepsilon_i^2 - \varepsilon_k^2} \varepsilon_i^2 S_k^0 \right] \\
 &= \sum_i \varepsilon_i^2 S_i^0 + \frac{\chi}{2} \sum_i \sum_k \frac{\varepsilon_i^2 \varepsilon_{ik}}{\varepsilon_i^2 - \varepsilon_k^2} S_i^+ S_k^- + \frac{\chi}{2} \sum_i \sum_k \frac{\varepsilon_i^2 \varepsilon_{ik}}{\varepsilon_i^2 - \varepsilon_k^2} S_k^+ S_i^- \\
 &\quad + \frac{\chi}{2} \sum_i \sum_k \frac{\varepsilon_i^2 (\varepsilon_i^2 + \varepsilon_k^2)}{\varepsilon_i^2 - \varepsilon_k^2} S_i^0 S_k^0 + \frac{\chi}{4} \sum_i \sum_k \frac{\varepsilon_i^2 \varepsilon_{ik}}{\varepsilon_i^2 - \varepsilon_k^2} S_i^0 S_k^0 \\
 &= \sum_i \varepsilon_i^2 S_i^0 + \frac{\chi}{2} \sum_i \sum_k \frac{\varepsilon_i^2 \varepsilon_{ik}}{\varepsilon_i^2 - \varepsilon_k^2} S_i^+ S_k^- + \frac{\chi}{2} \sum_{k \neq i} \sum_{i \neq k} \frac{\varepsilon_i^2 \varepsilon_{ik}}{\varepsilon_i^2 - \varepsilon_k^2} S_i^+ S_k^- \\
 &\quad + \frac{\chi}{4} \sum_i \sum_k \frac{\varepsilon_i^2 (\varepsilon_i^2 + \varepsilon_k^2)}{\varepsilon_i^2 - \varepsilon_k^2} S_i^0 S_k^0 + \frac{\chi}{4} \sum_{k \neq i} \sum_{i \neq k} \frac{\varepsilon_i^2 (\varepsilon_i^2 + \varepsilon_k^2)}{\varepsilon_i^2 - \varepsilon_k^2} S_i^0 S_k^0 \\
 &\rightarrow \sum_i \varepsilon_i^2 S_i^0 + \frac{\chi}{2} \sum_i \sum_k \frac{(\varepsilon_i^2 + \varepsilon_k^2)}{\varepsilon_i^2 - \varepsilon_k^2} \varepsilon_{ik} S_i^+ S_k^- + \frac{\chi}{4} \sum_{k \neq i} \sum_{i \neq k} \frac{(\varepsilon_i^2 + \varepsilon_k^2)(\varepsilon_i^2 + \varepsilon_k^2)}{\varepsilon_i^2 - \varepsilon_k^2} S_i^0 S_k^0 \\
 &= \sum_i (\varepsilon_i^2 S_i^0) + \frac{\chi}{2} \sum_{ik} \varepsilon_{ik} S_i^+ S_k^- - \frac{\chi}{2} \sum_{ik} \varepsilon_i^2 S_i^+ S_k^- \\
 &\quad + \frac{\chi}{4} \sum_{ik} (\varepsilon_i^2 + \varepsilon_k^2) S_i^0 S_k^0 - \frac{\chi}{4} \sum_i (\varepsilon_i^2 + \varepsilon_i^2) (S_i^0)^2 \\
 &= \sum_i (\varepsilon_i^2 S_i^0) + \frac{\chi}{2} \sum_{ik} \varepsilon_{ik} S_i^+ S_k^- + \frac{\chi}{4} ((\sum_i \varepsilon_i^2 S_i^0) \sum_k S_k^0 + (\sum_i S_i^0) (\sum_k \varepsilon_k^2 S_k^0)) \\
 &= - \frac{\chi}{2} \sum_i \varepsilon_i^2 [S_i^+ S_i^- + (S_i^0)^2] \stackrel{=: S^0}{=} \sum_i (\varepsilon_i^2 S_i^0 + S_i^0 (S_{i-1}^0))
 \end{aligned}$$

$$\begin{aligned}
 &\rightarrow \sum_i (\varepsilon_i^2 S_i^0) + \frac{\chi}{2} \sum_{ik} \varepsilon_{ik} S_i^+ S_k^- + \frac{\chi}{2} S_{n+1}^0 \sum_i \varepsilon_i^2 S_i^0 - \frac{\chi}{2} \sum_i \varepsilon_i^2 [(\varepsilon_i^2 + S_i^0)] \\
 &= \left[1 + \frac{\chi}{2} S^0 - \frac{\chi}{2} \right] \sum_i \varepsilon_i^2 S_i^0 + \frac{\chi}{2} \sum_{ik} \varepsilon_{ik} S_i^+ S_k^- - \frac{\chi}{2} \sum_i \varepsilon_i^2 \varepsilon_i^2
 \end{aligned}$$

~~Diagrammatic calculation.~~

Energy of palindromic Hamiltonian

$$H_{\text{part}} = \sum_i \varepsilon_i^2 R_i + g \sum_i \varepsilon_i^2 C_i$$

$$\rightarrow E_{\text{part}} = g \left(\sum_k \alpha_k^2 - \sum_i \varepsilon_i^2 d_i \right)$$

$$\begin{aligned}
 \text{Proof: } \langle E_{\text{part}} \rangle &= \sum_i \varepsilon_i^2 R_i + \frac{\chi}{2} \sum_i \varepsilon_i^2 d_i (d_{i+1}) \quad \text{note: if } \underline{g} \neq 0 \text{ - spinless core,} \\
 &= \sum_i \varepsilon_i^2 d_i \left[-1 + \chi \sum_{k \neq i} \frac{1}{2} \frac{\varepsilon_i^2 + \varepsilon_k^2}{\varepsilon_i^2 - \varepsilon_k^2} d_k - \chi \sum_{k \neq i} \frac{1}{2} \frac{\varepsilon_i^2 + \varepsilon_k^2}{\varepsilon_i^2 - \varepsilon_k^2} \right] + \frac{\chi}{2} \sum_i \varepsilon_i^2 d_i d_{i+1} \quad \text{because one of } \alpha_k = 0. \\
 &\quad \text{see later}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_i \varepsilon_i^2 d_i + \frac{\gamma}{4} \sum_{i \neq j} \sum_{\alpha} \frac{\varepsilon_i^2 (\varepsilon_j^2 + \varepsilon_\alpha^2)}{\varepsilon_i^2 - \varepsilon_\alpha^2} d_i d_k + \frac{\gamma}{2} \sum_{k=1}^N \sum_{i \neq k} \frac{d_i^2 (\varepsilon_i^2 + \varepsilon_\alpha^2)}{\varepsilon_k^2 - \varepsilon_\alpha^2} \\
&\quad - \frac{\gamma}{2} \sum_i \sum_{\alpha} \frac{(\varepsilon_i^2 - \varepsilon_\alpha^2)(\varepsilon_i^2 + \varepsilon_\alpha^2)}{(\varepsilon_i^2 - \varepsilon_\alpha^2)} d_i + \frac{\gamma}{2} \sum_i \varepsilon_i^2 d_i (d_i + 1) \\
&= \left(\frac{\gamma}{2} - 1 \right) \sum_i \varepsilon_i^2 d_i + \frac{\gamma}{4} \sum_{i \neq j} \sum_{\alpha} \frac{\varepsilon_i^2 \varepsilon_k^2}{\varepsilon_i^2 - \varepsilon_\alpha^2} (\varepsilon_i^2 + \varepsilon_\alpha^2) d_i d_k \\
&\quad - \frac{\gamma}{2} \sum_i \sum_{\alpha} \frac{(\varepsilon_i^2 - \varepsilon_\alpha^2)(\varepsilon_i^2 + \varepsilon_\alpha^2)}{(\varepsilon_i^2 - \varepsilon_\alpha^2)} d_i - \frac{\gamma}{2} \sum_i \sum_{\alpha} \varepsilon_\alpha^2 \frac{\varepsilon_i^2 + \varepsilon_\alpha^2}{\varepsilon_i^2 - \varepsilon_\alpha^2} d_i + \frac{\gamma}{2} \sum_i \varepsilon_i^2 d_i \\
&= \left(\frac{\gamma}{2} - 1 \right) \sum_i \varepsilon_i^2 d_i + \frac{\gamma}{4} \sum_{i \neq j} (\varepsilon_i^2 + \varepsilon_\lambda^2) d_i d_k - \frac{\gamma}{2} \sum_{i \neq j} \frac{\alpha \text{ or } R \text{ key}}{(\varepsilon_i^2 + \varepsilon_\lambda^2) d_i^2} \\
&\quad - \frac{\gamma}{2} \sum_i \sum_{\alpha} (\varepsilon_i^2 + \varepsilon_\alpha^2) d_i - \sum_{\alpha} \varepsilon_\alpha^2 \left[\frac{\gamma}{2} \sum_i \frac{\varepsilon_i^2 + \varepsilon_\alpha^2}{\varepsilon_i^2 - \varepsilon_\alpha^2} d_i \right] + \frac{\gamma}{2} \sum_i \varepsilon_i^2 d_i \\
&= \left(\frac{\gamma}{2} - 1 \right) \sum_i \varepsilon_i^2 d_i + 2 \frac{\gamma}{4} d \sum_i \varepsilon_i^2 d_i \\
&\quad - \frac{\gamma}{2} \left(\sum_{i \neq j} \sum_{\alpha} \varepsilon_i^2 d_i \frac{\varepsilon_k^2}{\varepsilon_i^2 - \varepsilon_\alpha^2} + \sum_{i \neq j} \sum_{\alpha} \varepsilon_\alpha^2 \right) - \sum_{\alpha} \varepsilon_\alpha^2 \left[\sum_{\beta \neq \alpha} \frac{\gamma}{2} \frac{\varepsilon_\beta^2 + \varepsilon_\alpha^2}{\varepsilon_\beta^2 - \varepsilon_\alpha^2} - 1 \right] \\
&= \left(\frac{\gamma}{2} - 1 + \frac{\gamma}{2} d \right) \sum_i \varepsilon_i^2 d_i - \frac{\gamma}{2} N \sum_i \varepsilon_i^2 d_i - \frac{\gamma}{2} d \sum_{\alpha} \varepsilon_\alpha^2 \\
&\quad - \frac{\gamma}{2} \sum_{\beta \neq \alpha} \varepsilon_\alpha^2 \frac{\varepsilon_\beta^2 + \varepsilon_\alpha^2}{\varepsilon_\beta^2 - \varepsilon_\alpha^2} + \sum_{\alpha} \varepsilon_\alpha^2 \\
&= \left[-1 - \frac{\gamma}{2}(N-d) + \frac{\gamma}{2} \right] \sum_i \varepsilon_i^2 d_i + (1 - \frac{\gamma}{2} d) \sum_{\alpha} \varepsilon_\alpha^2 \\
&\quad - \frac{\gamma}{2} \sum_{\alpha} \sum_{\beta \neq \alpha} \frac{\varepsilon_\alpha^2 (\varepsilon_\beta^2 + \varepsilon_\alpha^2)}{\varepsilon_\beta^2 - \varepsilon_\alpha^2} - \frac{\gamma}{2} \sum_{\alpha} \sum_{\beta \neq \alpha} \frac{\varepsilon_\beta^2 (\varepsilon_\beta^2 + \varepsilon_\alpha^2)}{\varepsilon_\alpha^2 - \varepsilon_\beta^2} \\
&= -\gamma \sum_i \varepsilon_i^2 d_i + (1 - \frac{\gamma}{2} d) \sum_{\alpha} \varepsilon_\alpha^2 - \frac{\gamma}{2} \sum_{\alpha} \sum_{\beta \neq \alpha} \frac{(\varepsilon_\alpha^2 - \varepsilon_\beta^2)}{\varepsilon_\beta^2 - \varepsilon_\alpha^2} (\varepsilon_\beta^2 + \varepsilon_\alpha^2) \\
&= -\gamma \sum_i \varepsilon_i^2 d_i + (1 - \frac{\gamma}{2} d) \sum_{\alpha} \varepsilon_\alpha^2 + \frac{\gamma}{2} \sum_{\alpha} \sum_{\beta \neq \alpha} (\varepsilon_\beta^2 + \varepsilon_\alpha^2) \\
&= -\gamma \sum_i \varepsilon_i^2 d_i + (1 - \frac{\gamma}{2} d) (\sum_{\alpha} \varepsilon_\alpha^2) + 2 \frac{\gamma}{2} (N-1) (\sum_{\alpha} \varepsilon_\alpha^2) \\
&= -\gamma \sum_i \varepsilon_i^2 d_i + (1 + \frac{\gamma}{2}(N-d) - \frac{\gamma}{2}) \sum_{\alpha} \varepsilon_\alpha^2 \\
&\therefore \gamma \left(\sum_{\alpha} \varepsilon_\alpha^2 - \sum_i \varepsilon_i^2 d_i \right).
\end{aligned}$$

PHASE DIAGRAM from BCS theory. IBADUZ PRB73, 180501(1) (2009)

$$H = \sum_i \varepsilon_i^2 s_i^0 + g \sum_{ij} \varepsilon_i \varepsilon_j s_i^+ s_j^-$$

$$\hat{\Delta}_i^+ = \sum_k g \varepsilon_i \varepsilon_k s_k^+ = g \varepsilon_i \sum_k \varepsilon_k s_k^+ := \varepsilon_i \hat{\Delta}^+ \quad \left| \begin{array}{l} (\text{in singlet: } j=1) \\ \Rightarrow \text{difference in self-interaction} \end{array} \right.$$

$\Rightarrow \Delta_i \sim \varepsilon_i$ (gap is proportional to ε_i).

$$(\text{BCS}) \approx e^{-\frac{\mu}{kT} \frac{\partial \ln S_R^+}{\partial \mu}}$$

we obtain the GHP equations.

$$\text{with } \Delta_i = \langle \hat{\Delta}_i \rangle = \varepsilon_i \langle \hat{\Delta} \rangle = \varepsilon_i \Delta$$

$$\underbrace{\text{GHP equations}}_{\text{L}} \left\{ \begin{array}{l} \lambda \frac{\varepsilon_k^2}{\sqrt{(\varepsilon_k^2 - \mu)^2 + \varepsilon_k^2 (2\Delta)^2}} = -\frac{1}{g} \\ \text{gn. } \sum_k \frac{1}{\sqrt{(\varepsilon_k^2 - \mu)^2 + \varepsilon_k^2 (2\Delta)^2}} = 2N - L - \frac{1}{g}. \quad (\text{number conserv.}) \end{array} \right.$$

\Rightarrow The ground-state energy is given by

$$E_0 = \sum_{k \neq 0} \left(\frac{\varepsilon_k^2}{2} \right) \left[1 - \frac{(\varepsilon_k^2 - \mu) + 2\Delta^2}{\sqrt{(\varepsilon_k^2 - \mu)^2 + \varepsilon_k^2 (2\Delta)^2}} \right]$$

Excitation energy (pair breaking)

$$E(\varepsilon_k) = \frac{1}{2} \sqrt{(\varepsilon_k^2 - \mu)^2 + \varepsilon_k^2 (2\Delta)^2}$$

How to obtain? $s_R = e^{-\mu/kT}$.

$$H_0 = \sum_i w_i^2 \left(\frac{\varepsilon_i^2}{2} \right) + \sum_{ik} G_{ik} s_i s_k u_i u_k.$$

take $G_{ik} > 0$ for the derivation

$$\Delta_i = \sum_k G_{ik} s_k u_k$$

$$s_i^2 = \frac{1}{2} \left[1 - \frac{(\varepsilon_i^2 - \mu)}{\sqrt{(\varepsilon_i^2 - \mu)^2 + \Delta_i^2}} \right]; u_i^2 = 1 - s_i^2.$$

$$\rightarrow \text{GHP eq.: } \Delta_i = +\frac{1}{2} \sum_k \frac{G_{ik} \Delta_k}{\sqrt{(\varepsilon_i^2 - \mu)^2 + \Delta_k^2}}; \quad \epsilon N = \sum_k w_k^2$$

are: $G_i h = \frac{1}{2} \int \varepsilon_i \varepsilon_R \sin u_R$

$$2\pi \times 1 \Delta_i = \sum_k \lg | \varepsilon_i \varepsilon_R \sin u_R | = \varepsilon_i \lg \underbrace{\sum_k \varepsilon_R \sin u_R}_{\Delta :=} = \varepsilon_i \Delta$$

$$(*) \sigma_i^2 = \frac{1}{2} \left[1 - \frac{(\varepsilon_i^2 - \gamma_M)}{\sqrt{(\varepsilon_i^2 - \gamma_M)^2 + \varepsilon_i^2 (\Delta)^2}} \right]$$

$$(*) H_0 = \sum_i \left(\frac{\varepsilon_i^2}{2} \right) \left[1 - \frac{(\varepsilon_i^2 - \gamma_M)}{\sqrt{(\varepsilon_i^2 - \gamma_M)^2 + \varepsilon_i^2 (\Delta)^2}} \right] + \sum_k g \varepsilon_i \varepsilon_R \sin u_R \sin u_R$$

$$(*) = \sum_i \frac{\varepsilon_i^2}{2} \left[1 - \frac{(\varepsilon_i^2 - \gamma_M)}{\sqrt{(\varepsilon_i^2 - \gamma_M)^2 + \varepsilon_i^2 (\Delta)^2}} \right] + \frac{\Delta^2}{1g}$$

(*) GAP eq.

$$\hat{\varepsilon} \cdot \Delta = \frac{1}{2} \sum_k \frac{\lg | \varepsilon_i \varepsilon_R \varepsilon_R \Delta |}{\sqrt{(\frac{\varepsilon_i^2}{2} - \gamma_M)^2 + \varepsilon_i^2 \Delta^2}}$$

$$\Rightarrow \frac{1}{1g} = \sum_k \frac{\varepsilon_R}{\sqrt{(\varepsilon_R^2 - \gamma_M)^2 + \alpha^2 (\Delta)^2}} \quad v(1)$$

(*) number continuing:

$$2N = \sum_k \left[1 - \frac{\varepsilon_R^2 - \gamma_M}{\sqrt{(\varepsilon_R^2 - \gamma_M)^2 + (2\Delta)^2 \varepsilon_R^2}} \right]$$

$$2N \leftarrow L - \sum_k \frac{\varepsilon_R}{\sqrt{(\varepsilon_R^2 - \gamma_M)^2 + (2\Delta)^2 \varepsilon_R^2}} + \gamma_M \sum_k \frac{1}{\sqrt{(\varepsilon_R^2 - \gamma_M)^2 + (2\Delta)^2 \varepsilon_R^2}}$$

~~$\Rightarrow L$~~ ch. 6.1 p. 1

$$2N - L = - \frac{1}{1g} + \gamma_M \sum_k \frac{1}{\sqrt{(\varepsilon_R^2 - \gamma_M)^2 + (2\Delta)^2 \varepsilon_R^2}}$$

$$\Rightarrow (\gamma_M) \sum_k \frac{1}{\sqrt{(\varepsilon_R^2 - \gamma_M)^2 + (2\Delta)^2 \varepsilon_R^2}} = 2N - L + \frac{1}{1g}. \quad v(1).$$

(*) emissive spectrum:

$$E(\varepsilon_R) = \sqrt{(\frac{\varepsilon_R^2}{2} - \gamma_M)^2 + \varepsilon_R^2 \Delta^2} = \frac{1}{2} \sqrt{(\varepsilon_R^2 - \gamma_M)^2 + \varepsilon_R^2 (\Delta)^2}$$

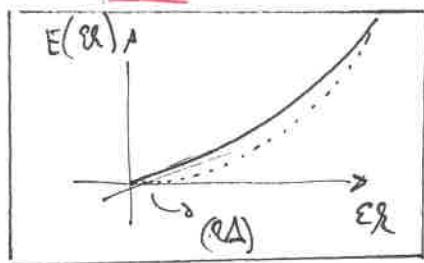
2 important "points"

(2.6)

(1) Read GREEN: $\mu \gg \rightarrow 2N-L-\frac{1}{g} = 0$ TERIBLOC because it does not depend on ε_L ! $\sim \frac{\partial E}{\partial L}$

$$\rightarrow E(\varepsilon_L) = \frac{1}{2} \sqrt{(\varepsilon_L - \mu)^2 + \varepsilon_L^2(2A)^2}$$

$E(\varepsilon_L) = \varepsilon_L \sqrt{\varepsilon_L^2 + (2A)^2} \rightarrow$ important when $\varepsilon_L \ll \mu$
 ↳ GAPLESS when $\varepsilon_L \approx \mu$ cp satellite joining.



define: $\alpha = \frac{N}{L}$ $G = gL$

$$= 2\alpha - 1 - \frac{1}{G} = 0$$

$$\rightarrow \alpha = \frac{1}{2} \left(1 + \frac{1}{G} \right)$$

3rd order LPT.

(2) Moss READS $\mu = \frac{\Delta^2}{2}$

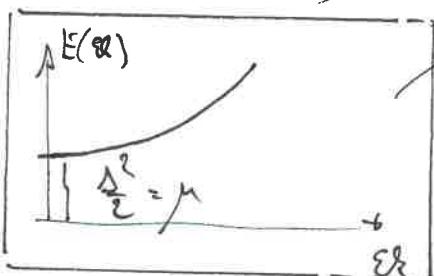
$$\Rightarrow (*) \quad E(\varepsilon_L) = \frac{1}{2} \sqrt{(\varepsilon_L - \mu)^2 + \varepsilon_L^2(2A)^2}$$

$$= \frac{1}{2} \sqrt{(\varepsilon_L^2 - \Delta^2)^2 + \varepsilon_L^2(2A)^2}$$

$$= \frac{1}{2} \sqrt{\varepsilon_L^4 - 2\Delta^2\varepsilon_L^2 + \Delta^4 + 4\Delta^2\varepsilon_L^2}$$

$$= \frac{1}{2} \sqrt{(\varepsilon_L + \Delta^2)^2}$$

$$E(\varepsilon_L) = \frac{1}{2} (\varepsilon_L^2 + \Delta^2)$$



almost like a free spectrum when $B\Delta \sim \mu$

(*) Energy: $E_0 = \sum_{n=1} \left(\frac{\varepsilon_L^n}{2} \right) \left[1 - \frac{\varepsilon_L^n - \Delta^2 + 2\Delta^2}{\varepsilon_L^n + \Delta^2} \right] = 0$

⇒ The energy is always > 0 , independent of ε_L (**) (2.6)

→ Topological? plug solution in GPE eq.

$$\begin{aligned} \text{u} \left\{ \begin{aligned} \sum_{\lambda} \frac{\varepsilon_{\lambda}^l}{\varepsilon_{\lambda}^l + \Delta^2} &= -\frac{1}{g} \quad (*) \\ 2\mu \sum_{\lambda} \frac{1}{\varepsilon_{\lambda}^l + \Delta^2} &= 2N - L - \frac{1}{g} \end{aligned} \right. \end{aligned}$$

$$(*) \rightarrow \sum_{\lambda} \frac{\varepsilon_{\lambda}^l + (\Delta^2 - \Delta^2)}{\varepsilon_{\lambda}^l + \Delta^2} = -\frac{1}{g}$$

$$L - \Delta^2 \sum_{\lambda} \frac{1}{\varepsilon_{\lambda}^l + \Delta^2} = -\frac{1}{g} \quad \Delta^2 = 2\mu$$

$$L - 2\mu \sum_{\lambda} \frac{1}{\varepsilon_{\lambda}^l + \Delta^2} = -\frac{1}{g}$$

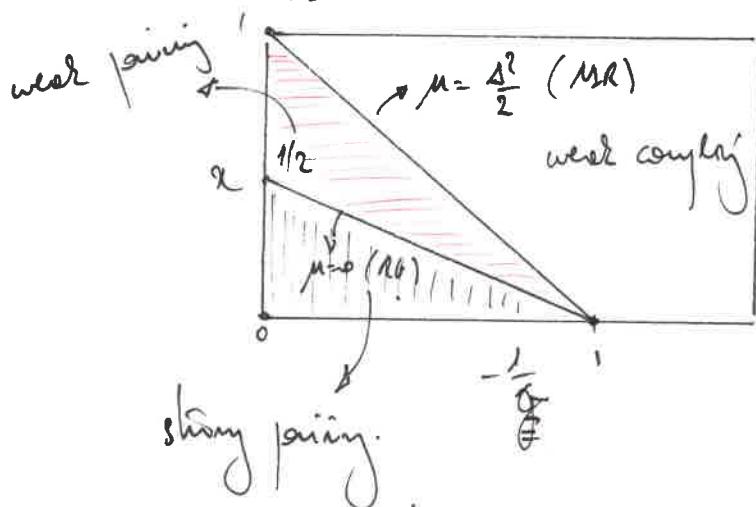
$$L - (2N - L - \frac{1}{g}) = -\frac{1}{g}$$

$$\rightarrow \varepsilon L - 2N + \frac{2}{g} = 0 \rightarrow$$

→ Topological!

$$x = 1 + \frac{1}{6}$$

PHASE DIAGRAM



(*) How does the MR state look like?

$$\langle BCS \rangle = e^{\sum_{\lambda} \frac{\partial \ln S_{\lambda}^+}{\partial \mu} \omega_{\lambda}}$$

$$\text{For MR: } \frac{\partial \ln S_{\lambda}^+}{\partial \mu} = \frac{1 - \frac{\varepsilon_{\lambda}^l - \Delta^2}{\varepsilon_{\lambda}^l + \Delta^2}}{1 + \frac{\varepsilon_{\lambda}^l - \Delta^2}{\varepsilon_{\lambda}^l + \Delta^2}} = \frac{\varepsilon_{\lambda}^l + \Delta^2 - \varepsilon_{\lambda}^l - \Delta^2}{\varepsilon_{\lambda}^l + \Delta^2 + \varepsilon_{\lambda}^l - \Delta^2} = \left(\frac{\Delta}{\varepsilon_{\lambda}}\right)^2$$

$$\rightarrow \langle BCS \rangle = e^{\sum_{\lambda} \frac{\Delta}{\varepsilon_{\lambda}} S_{\lambda}^+ \omega_{\lambda}}$$

Compare this with: $\lim_{\Delta \rightarrow 0} (S_{\lambda}^+) = \lim_{\Delta \rightarrow 0} \sum_{\lambda} \frac{c_{\lambda} n_{\lambda}}{\varepsilon_{\lambda}^l - \mu_{\lambda}} S_{\lambda}^+ \approx \sum_{\lambda} \frac{S_{\lambda}^+}{\varepsilon_{\lambda}^l / \Delta}$

\Rightarrow Q: Is the state with $x_{\alpha=0}$ a special EXACT state?

Investigate:

$$|\psi\rangle = (S_0^+)^p \left(\prod_{\alpha=1}^q S_\alpha^+ \right) |0\rangle$$

$$\text{with } S_0^+ = \lim_{\alpha \rightarrow 0} \left(\frac{1}{N_\alpha} S_\alpha^+ \right)$$

$$p+q=N$$

Same game all over again.

$$H(S_0^+)^p \left(\prod_{\alpha=1}^q S_\alpha^+ \right) |0\rangle = (S_0^+)^p H \left(\prod_{\alpha=1}^q S_\alpha^+ \right) |0\rangle + (S_0^+)^{p-1} S_{\alpha=0}^+ \left(\prod_{\alpha=1}^{q-1} S_\alpha^+ \right) \underbrace{H[-2g - (p-1) + \frac{1}{g} + 2d] |0\rangle}_{=0}$$

original RG for the
non-condensed eq.

YES, provided $-2g - (p-1) + \frac{1}{g} + 2d = 0$ Independent of ϵ_L

Special cases:

$$d = \sum d_\alpha = \frac{L}{2}$$

$$(1) \quad p=1; q=N-1 \quad \Rightarrow \quad -2(N-1) - 0 + \frac{1}{g} + \frac{2L}{8} = 0$$

$$-2\frac{N}{L} + \frac{g}{L} + \frac{1}{gL} + 1 = 0$$

$$\Rightarrow 2\alpha = 1 + \frac{1}{g} + \frac{g}{L}$$

$$\alpha = \frac{1}{2} \left(1 + \frac{1}{g} + \frac{g}{L} \right)$$

limit:

$$\underset{L \rightarrow \infty}{\lim} \boxed{\alpha = \frac{1}{2} \left(1 + \frac{1}{g} \right)} \quad \text{Read-Green!}$$

The read green state is a single $x_{\alpha=0}$ state.

$$(2) \quad p=N, q=0$$

$$-(N-1) + \frac{1}{g} + L = 0$$

$$-\left(\frac{N}{L} - \frac{1}{L}\right) + \frac{1}{g} + 1 = 0$$

$$\Rightarrow \alpha = 1 + \frac{1}{g} + \frac{1}{L}$$

$$\lim_{L \rightarrow \infty} : \boxed{x = 1 + \frac{1}{6}} \quad \text{More - Read}$$

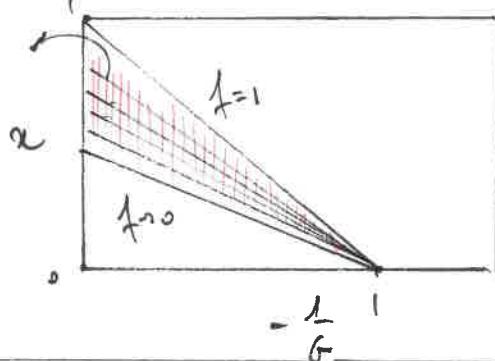
(3) more general ~~sol~~

def: $\frac{\rho}{N} = f$ condensation fraction

$$x = \frac{1}{(e-f)} \left(1 + \frac{1}{6} + \frac{1}{L} \right)$$

$$\Rightarrow \lim_{L \rightarrow \infty} x = \frac{1}{e-f} \left(1 + \frac{1}{6} \right)$$

continuum of states



Importance: (*) PBCS state is an exact state of some Integrable model.

(*) experimentally: $E_R = \text{Lat. Ry} \rightarrow E^2 = g^2$

$$H = \sum_k g^2 n_k + g \sum_l (\text{Lat. Ry}) (a_{k+l}^\dagger - i g_j) S_k^+ S_{k+l}^-$$

p + p wave pairings \rightarrow possible topological states

(*) Stefan's $\sqrt{(E_i - \mu)}$ fit !!