


RATIONAL MODEL XXX

$$(*) \quad X_i \lambda = Z_i \lambda = \frac{1}{\varepsilon_i - \varepsilon_\lambda} \quad P = 0$$

$$\begin{aligned} (*) \quad R_i &= S_i^0 + \lambda \sum_{j \neq i} \frac{1}{\varepsilon_j - \varepsilon_\lambda} [S_1^+ (S_i^- S_\lambda + S_\lambda^+ S_i^-) + S_i^0 S_\lambda^0] \\ &= S_i^0 + \lambda \sum_{j \neq i} \frac{S_i^- S_\lambda^+}{\varepsilon_i - \varepsilon_\lambda} \end{aligned}$$

$$(*) \quad X_i \alpha = \frac{1}{\varepsilon_i - \alpha} \quad \alpha_\alpha: \text{eigenvalues}$$

$$\text{GGK: } \left\{ \begin{array}{l} [S_\alpha^+ : S_\beta^-] = 2 \cdot \frac{S_\alpha^0 - S_\beta^0}{\alpha_\alpha - \alpha_\beta} \\ [S_\alpha^0 : S_\beta^+] = \frac{S_\alpha^+ - S_\beta^+}{\alpha_\alpha - \alpha_\beta} \\ [S_\alpha^0 : S_\beta^-] = - \frac{S_\alpha^- - S_\beta^-}{\alpha_\alpha - \alpha_\beta} \end{array} \right. \quad (\text{cpn ABA p. 3.13})$$

$$(*) \quad q_i = d_i \left[ -1 + \lambda \sum_{j \neq i} \frac{d_j}{\varepsilon_j - \varepsilon_\lambda} - \lambda \sum_{\alpha=1}^N \frac{1}{\varepsilon_i - \alpha} \right].$$

$$(*) \quad |q\rangle = \prod_{\alpha=1}^N S_\alpha^+ |0\rangle = \prod_{\alpha=1}^N \left( \sum_{i=1}^L \frac{S_i^+}{\varepsilon_i - \alpha} \right) |0\rangle.$$

$$(*) \quad \text{R.G. eq.:} \quad 1 + \lambda \sum_{\alpha} \frac{d_\alpha}{\varepsilon_\lambda - \alpha} + \lambda \sum_{\alpha \neq \lambda} \frac{1}{\varepsilon_\lambda - \alpha} = 0 \quad \forall \alpha = 1 \dots N.$$

Reduced BCS model:

$$\begin{aligned} H &= \sum_{i=1}^L \varepsilon_i R_i \\ &= \sum_{i=1}^L \varepsilon_i m_i + g \underbrace{\sum_{i,j} S_i^+ S_j^-}_{H_{BCS}} - g \sum_{i=1}^L \alpha_i + g \left( \sum_{i=1}^L S_i^0 \right) \left( \sum_{i=1}^L S_{i-1}^0 \right) - \sum_i \varepsilon_i \alpha_i \end{aligned}$$

$$\Rightarrow H_{BCS} = \sum_{i=1}^L \varepsilon_i R_i + g \sum_{i=1}^m d_i (d_{i+1}) - g (N-d) (N-d-1) + \sum_{i=1}^L \varepsilon_i \alpha_i$$

$$d = \sum_i d_i$$

Proof.:  $H = \sum_{i=1}^L 2\varepsilon_i R_i = \sum_{i=1}^L 2\varepsilon_i \left\{ \frac{1}{2} S_i^0 + \chi \sum_{k \neq i} \frac{1}{\varepsilon_i - \varepsilon_k} \left[ \frac{1}{2} (S_i^+ S_k + S_k^+ S_i) + S_i^0 S_k \right] \right\}$

$$= \sum_i 2\varepsilon_i S_i^0 + \chi \sum_{i=1}^L \sum_{k \neq i} \frac{2\varepsilon_i}{2\varepsilon_i - 2\varepsilon_k} (S_i^+ S_k + S_k^+ S_i) + \chi \sum_{i=1}^L \sum_{k \neq i} \frac{2\varepsilon_i}{\varepsilon_i - \varepsilon_k} S_i^0 S_k$$

$$= \sum_i 2\varepsilon_i S_i^0 + \chi \sum_{i=1}^L \sum_{k \neq i} \frac{2\varepsilon_i}{2\varepsilon_i - 2\varepsilon_k} S_i^+ S_k + \chi \sum_{i=1}^L \sum_{k \neq i} \frac{2\varepsilon_i}{2\varepsilon_i - \varepsilon_k} S_i^+ S_k$$

$$+ \cancel{\chi} \sum_{i=1}^L \sum_{k \neq i} \frac{\varepsilon_i}{\varepsilon_i - \varepsilon_k} S_i^0 S_k + \chi \sum_{k=1}^L \sum_{i \neq k} \frac{\varepsilon_k}{\varepsilon_k - \varepsilon_i} S_k^0 S_i$$

$$= \sum_i 2\varepsilon_i S_i^0 + \chi \sum_{i=1}^L \sum_{k \neq i} \frac{2\varepsilon_i - 2\varepsilon_k}{2\varepsilon_i - 2\varepsilon_k} S_i^+ S_k + \chi \sum_{i=1}^L \sum_{k \neq i} \frac{\varepsilon_i - \varepsilon_k}{\varepsilon_i - \varepsilon_k} S_i^0 S_k$$

$$= \sum_i 2\varepsilon_i S_i^0 + \chi \sum_{i=1}^L S_i^+ S_k + \chi \sum_{i=1}^L S_i^0 S_k - \chi \sum_i (S_i^+ S_i + S_i^0 S_i)$$

Defn.:  $C_{ii} = \frac{1}{2} (S_i^+ S_i + S_i^0 S_i) + (S_i^0)^2 = S_i^+ S_i + S_i^0 (S_i^0 - 1)$

$$= \sum_i 2\varepsilon_i S_i^0 + \chi \sum_{i=1}^L S_i^+ S_k + \chi \sum_{i=1}^L S_i^0 S_k - \chi \sum_i (C_{ii} + S_i^0)$$

$$= \sum_i 2\varepsilon_i S_i^0 + \chi \sum_{i=1}^L S_i^+ S_k + \chi (\sum_i S_i^0)^2 - \chi \sum_i C_{ii} - \chi (\sum_i S_i^0)$$

$$= \underbrace{\sum_i 2\varepsilon_i S_i^0}_{= \sum_i 2\varepsilon_i (M_i - \frac{1}{4} \sigma_i)} + \chi (\sum_i S_i^0) (\sum_i S_i^0 - 1) - \chi \sum_i C_{ii}$$

$$= \sum_i 2\varepsilon_i (M_i - \frac{1}{4} \sigma_i) = \sum_i 2\varepsilon_i M_i - \frac{1}{4} \sum_i 2\varepsilon_i \sigma_i$$

and  $\langle \sum_i S_i^0 \rangle = \langle \sum_i \left( \frac{1}{2} (M_i - \frac{1}{4} \sigma_i) \right) \rangle = \sum_i \frac{1}{2} (2M_i + \sigma_i) - \frac{1}{4} \sum_i \sigma_i$

$$= \sum_i M_i - \underbrace{\sum_i \frac{1}{4} \sigma_i}_{d_i} - \frac{1}{2} \sum_i \sigma_i$$

$$= N - d.$$

Step 2: Minimize

Energy:  $\sum_i 2\varepsilon_i Q_i = \sum_i 2\varepsilon_i M_i \left[ -1 + g \sum_{k \neq i} \frac{d_k}{\varepsilon_i - \varepsilon_k} - g \sum_{\alpha=1}^N \frac{1}{\varepsilon_i - \varepsilon_\alpha} \right]$ .

$$\sum_i 2\varepsilon_i Q_i = - \sum_i 2\varepsilon_i d_i + g(N-d)^2 - g \sum_i d_i^2 + \sum_\alpha 2x_\alpha - gN$$

Proof: are  $\mathbf{R}^2$  eq. at some point!

$$\begin{aligned}\sum_i 2\varepsilon_i \partial_{\varepsilon_i} &= \sum_i 2\varepsilon_i d_i \left[ -1 + g \sum_{\alpha \neq i} \frac{\partial h}{\partial \varepsilon_i - \varepsilon_\alpha} - g \sum_{\alpha=1}^N \frac{1}{\varepsilon_i - \varepsilon_\alpha} \right]. \\ &= - \sum_i 2\varepsilon_i d_i + 2g \underbrace{\sum_i \sum_{\alpha \neq i} \frac{\varepsilon_i d_i \partial h}{\varepsilon_i - \varepsilon_\alpha}}_{(*)} - 2g \underbrace{\sum_{\alpha \neq i} \frac{\varepsilon_i d_i}{\varepsilon_i - \varepsilon_\alpha}}_{\text{ALMOST ok for RO.}}\end{aligned}$$

$$(*) = g \sum_i \sum_{\alpha \neq i} \frac{\varepsilon_i d_i \partial h}{\varepsilon_i - \varepsilon_\alpha} + g \sum_{\alpha \neq i} \sum_{i \neq \alpha} \frac{\varepsilon_\alpha d_i \partial h}{\varepsilon_\alpha - \varepsilon_i}$$

$$= g \sum_i \sum_{\alpha \neq i} \left( \frac{\varepsilon_i - \varepsilon_\alpha}{\varepsilon_i - \varepsilon_\alpha} \right) d_i \partial h =$$

$$= g \sum_i \sum_{\alpha \neq i} d_i \partial h = g(d^2 - \sum_i d_i^2).$$

$$(**) = -2g \sum_i \sum_{\alpha=1}^N \frac{\varepsilon_i d_i}{\varepsilon_i - \varepsilon_\alpha} = -2g \sum_{\alpha=1}^N \sum_i \left( \frac{\varepsilon_i - \varepsilon_\alpha + \varepsilon_\alpha}{\varepsilon_i - \varepsilon_\alpha} \right) d_i$$

$$= -2g \sum_{\alpha=1}^N \sum_i d_i \left[ 1 + \frac{\varepsilon_\alpha}{\varepsilon_i - \varepsilon_\alpha} \right]$$

$$= -2g \sum_{\alpha=1}^N \left( d + \varepsilon_\alpha \sum_{i \in N} \frac{d_i}{\varepsilon_i - \varepsilon_\alpha} \right)$$

$$= -2g N d - 2g \sum_{\alpha=1}^N \varepsilon_\alpha \left( \sum_{i \in N} \frac{d_i}{\varepsilon_i - \varepsilon_\alpha} \right)$$

$$= -2g N d - 2g \sum_{\alpha=1}^N \varepsilon_\alpha \left[ -\frac{1}{g} - \sum_{\gamma \neq \alpha} \frac{1}{\varepsilon_\alpha - \varepsilon_\gamma} \right] = -\frac{1}{g} - \sum_{\alpha=1}^N \frac{1}{g \varepsilon_\alpha \varepsilon_\alpha - g y}$$

$$= -2g N d + \sum_{\alpha=1}^N 2\varepsilon_\alpha + 2g \sum_{\alpha=1}^N \sum_{\gamma \neq \alpha} \frac{\varepsilon_\alpha}{\varepsilon_\alpha - \varepsilon_\gamma}.$$

$$= -2g N d + \sum_{\alpha=1}^N (2\varepsilon_\alpha) + g \sum_{\alpha=1}^N \sum_{\gamma \neq \alpha} \frac{\varepsilon_\alpha}{\varepsilon_\alpha - \varepsilon_\gamma} + g \sum_{\gamma \neq \alpha} \sum_{\alpha=1}^N \frac{\varepsilon_\gamma}{\varepsilon_\gamma - \varepsilon_\alpha}$$

$$= -2g N d + \sum_{\alpha=1}^N 2\varepsilon_\alpha + g \sum_{\alpha=1}^N \sum_{\gamma \neq \alpha} \frac{\varepsilon_\alpha - \varepsilon_\gamma}{\varepsilon_\alpha - \varepsilon_\gamma}$$

$$= -2g N d + \sum_{\alpha=1}^N 2\varepsilon_\alpha + g N(N-1)$$

$$\sum_i 2\varepsilon_i \eta_i = -g \sum_i 2\varepsilon_i d_i + g(d^2 - \sum_i d_i^2) - 2g N d + \sum_{\alpha=1}^N 2\varepsilon_\alpha + g N(N-1)$$

$$= -\sum_i 2\varepsilon_i d_i + \sum_{\alpha=1}^N 2\varepsilon_\alpha + g(N-d)^2 - g \sum_i d_i^2 - g N$$

$$\begin{aligned}
 \Rightarrow E_{\text{ECS}} &= \sum_i 2\varepsilon_i \eta_i + g \sum_i d_i(d_i+1) - g(N-d)(N-d-1) + \sum_i \frac{\varepsilon_i \eta_i}{2} \\
 &= - \sum_i 2\varepsilon_i d_i + \sum_\alpha 2\eta_\alpha + g(N-d)^2 - g \cancel{\sum_i d_i^2} - g \cancel{\eta} \\
 &\quad + g \sum_i d_i(d_i+1) - g(N-d)^2 + g \cancel{\eta} + \sum_i 2\varepsilon_i \frac{\eta_i}{4} \\
 &= \sum_i 2\varepsilon_i \left( \frac{\eta_i}{4} - \frac{d_i}{4} + \frac{\eta_i}{2} \right) + \sum_\alpha 2\eta_\alpha \\
 \boxed{E_{\text{ECS.}} = \sum_{\substack{i \\ d_i \neq 1}} (\varepsilon_i \eta_i) + \sum_{i=1} \varepsilon_i \eta_i}
 \end{aligned}$$

- (1): Energy associated to the pairing: with each pair. or every  $\alpha$ .
- (2): single-particle energy of unpaired particles. They don't contribute to the ~~total~~ energy.

General behaviour: Stefan's rule (note:  $E_\alpha = 2\eta_\alpha$ ) PRC 66, 061203(R)  
 (2004)

- Note: (1)  $cc^\dagger$  pairs! or, because  $c + c^\dagger = 2\text{Re}(c)$
- (2) singular points! appear in  $\varepsilon_i$
- (3) Solving: start from  $g \rightarrow 0$  and convert adiabatically.

Het Heine STIEZTJES.

$$\text{At } g=0: |c\rangle = \prod_{i=1}^N (S_i^+)^{n_i} |0\rangle. \quad \cancel{\times} \cancel{\times} \cancel{\times} \cancel{\times} \text{oo}(\cdot).$$

on one single level  $i$ :

$$(S_i^+)^n |0\rangle \approx \lim_{\eta_\alpha \rightarrow \varepsilon_i} \prod_{\alpha=1}^N (S_\alpha^+)^{n_\alpha} |0\rangle = \lim_{\eta_\alpha \rightarrow \varepsilon_i} \prod_{\alpha=1}^N \sum_{k=1}^L \frac{S_\alpha^+}{\varepsilon_k - \eta_\alpha} |0\rangle$$

→ look for solutions in the neighbourhood of  $\varepsilon_i$ :

$$1 + g \sum_{\alpha=1}^L \frac{d\alpha}{\varepsilon_\alpha - \varepsilon_i} + g \sum_{\beta \neq i} \frac{1}{\varepsilon_\beta - \varepsilon_i} = 0 \quad \forall \alpha$$

$$\Rightarrow \boxed{\varepsilon_i \approx \varepsilon_i + g y_\alpha + O(g^2) + \dots}$$

$$\Rightarrow 1 + g \sum_{\alpha=1}^L \frac{d\alpha}{\varepsilon_\alpha - \varepsilon_i - g y_\alpha} + g \sum_{\beta \neq i} \frac{1}{\varepsilon_\beta - \varepsilon_i - g y_\beta} = 0 + O(g^1).$$

$$\Rightarrow 1 + g \sum_{\alpha=1}^L \frac{d\alpha}{\varepsilon_\alpha - \varepsilon_i - g y_\alpha} + g \frac{d_i}{-g y_\alpha} + g \sum_{\beta \neq i} \frac{1}{g(y_\alpha - y_\beta)} = 0 + O(g^1)$$

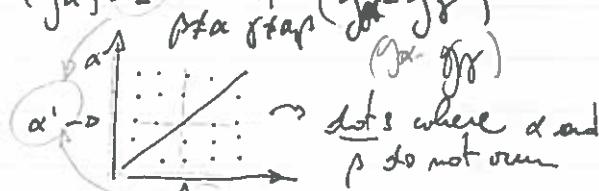
$$\Rightarrow 1 - \underbrace{\frac{d_i}{g y_\alpha}}_{\text{lhs}} + \underbrace{\sum_{\beta \neq i} \frac{1}{g y_\alpha - g y_\beta}}_{\text{rhs}} = 0 + O(g^1) \quad \forall \alpha$$

lhs how to solve this?

define:

$P(y) = \sum_{\alpha=1}^N (y - y_\alpha)$	$\Rightarrow P(y_\alpha) = 0$
$P'(y) = \sum_{\alpha=1}^N \frac{1}{y - y_\alpha} (y - y_\beta) \Rightarrow$	$P'(y_\alpha) = \sum_{\beta \neq \alpha} \frac{1}{y_\alpha - y_\beta} (y_\alpha - y_\beta)$
$P''(y) = \sum_{\alpha=1}^N \sum_{\beta \neq \alpha} \frac{1}{(y - y_\alpha)(y - y_\beta)} (y - y_\alpha)$	$P''(y_\alpha) = \sum_{\beta \neq \alpha} \frac{1}{y_\alpha - y_\beta} (y_\alpha - y_\beta)$

Multiplying with  $-y_\alpha P'(y_\alpha)$



$$\Rightarrow -y_\alpha P'(y_\alpha) + d_i P'(y_\alpha) = -y_\alpha \sum_{\beta \neq \alpha} \frac{1}{y_\alpha - y_\beta} (y_\alpha - y_\beta) = 0$$

$$\Rightarrow (d_i - y_\alpha) P'(y_\alpha) - y_\alpha \sum_{\beta \neq \alpha} \frac{1}{y_\alpha - y_\beta} (y_\alpha - y_\beta) = 0$$

$$\Rightarrow (d_i - y_\alpha) P'(y_\alpha) - y_\alpha \sum_{\beta \neq \alpha} \frac{1}{y_\alpha - y_\beta} P''(y_\alpha) = 0$$

call  $U(y) = (d_i - y) P'(y) - \frac{1}{2} y P''(y)$

$$\Rightarrow \begin{cases} U(y_\alpha) = 0 \quad \forall \alpha \\ U(y) \text{ is a polynomial of degree } N \end{cases} \Rightarrow U(y) = \sum_{n=0}^N U_n y^n$$

How to start?  $\Rightarrow$  compare highest coeff. of  $P(y) \cdot y^N$

$$\Rightarrow -y^N N y^{N-1} = N y^N \Rightarrow N = -N$$

scaling



$\Rightarrow$   $y_\alpha$  are the roots of the polynomial solutions of:

$$(d_i - y) P'(y) - \frac{1}{2} y P''(y) = -N P(y)$$

leb:  $z = -2y$

$$\Rightarrow z P''(z) + (-2d_i - z) P'(z) + N P(z) = 0$$

$\Rightarrow$  ASSOCIATED LEGENDRE POLYNOMIALS

$$L_N^\alpha \text{ mit } \alpha+1 = -2d_i$$

$\Rightarrow y_\alpha = -\frac{1}{2} z_\alpha$  mit  $z_\alpha$  roots of legendre

$$\Rightarrow x_\alpha = \varepsilon_i - \frac{1}{2} y^{2\alpha} + O(y^4)$$

note: very nice but not practical, as the roots are not given in closed form

How to obtain?

symbolic cases:

$$\begin{cases} L_0^\alpha = 1 & N=0 \dots \\ L_1^\alpha = -x + \alpha + 1 & N=1 \\ L_2^\alpha = \frac{x^2}{2} - (\alpha+2)x + \frac{(\alpha+2)(\alpha+1)}{2} & N=2 \\ \vdots \end{cases}$$

N=0: no roots

$$N=1: L_1^\alpha = -x + \alpha + 1 \Rightarrow x_1 = \alpha + 1 = -2d_i \in \mathbb{R}.$$

$$N=2: L_2^\alpha = \frac{x^2}{2} - (\alpha+2)x + \frac{(\alpha+2)(\alpha+1)}{2}$$

$$\Rightarrow x_{\pm} = \frac{\pm \sqrt{(\alpha+2)^2 - (\alpha+1)(\alpha+1) + \alpha+2}}{2 \cdot 1/2}$$

$$= (\alpha+2) \pm \sqrt{(\alpha+2)(\alpha+2 - \alpha - 1)}$$

$$x_{\pm} = \alpha + 2 \pm \sqrt{\alpha+2}$$

$$x_{\pm} = (1 - 2d_i) \pm \sqrt{1 - 2d_i}$$

we have: we need a  $S_{2,2}$  to accommodate  $N=2$  pairs

$$\Rightarrow d_i = \frac{\alpha_i}{\gamma} \rightarrow d_i > 1$$

$\alpha_i$  are always  $\text{cc}^*$

$N=...$  ? are recursion relation.

$$N L_N(x) = (2N-1+\alpha-\alpha) L_{N-1}(x) - (N-1+\alpha) L_{N-2}(x)$$

$$\begin{array}{c|ccc|c}
 1+\alpha-\alpha & \sqrt{1+\alpha} & & & \dots \\
 \hline
 \sqrt{1+\alpha} & 3+\alpha-\alpha & \sqrt{2+\alpha} & & \dots \\
 \dots & \sqrt{2+\alpha} & \dots & & \dots \\
 & & \dots & & \dots \\
 & & \sqrt{m-1+\alpha} & \frac{m-1+\alpha}{\sqrt{m}} & \dots \\
 & & \hline
 & & \frac{2m-1+\alpha-\alpha}{\sqrt{m}} & \dots & \dots
 \end{array} \Rightarrow \text{you can get them out when } (\alpha=0)$$

→ The roots are eigenvalues of the matrice complex symmetric matrix. for  $y \neq 0$ .

$$\sqrt{1+\alpha} = \sqrt{1-2d_i-1} + i\lambda.$$

generate fish!

(\*) Note: Stieltjes wanted to find the roots of the Laguerre polynomials, and "reduced" it to a set of algebraic equations that "only need to be solved now"

$$1 - \frac{d_i}{\gamma\alpha} + \sum \frac{1}{\gamma\alpha - \gamma_i - \gamma_j} = i\lambda.$$

Ramp up  $\gamma$  and solve adiabatically

→ you run into singular points.

- (\*)  $\sum (2\alpha_i) + \sum \epsilon_i \gamma_i$  has no singular behavior  $\rightarrow C_\infty$ .
- (\*) singularities for  $\alpha_i$  and  $\epsilon_i$ !
- (\*) Can we predict them?

(\*) Can we predict singular states?

$$\rightarrow |n\rangle = (S_j^+)^p \frac{1}{\alpha} S_\alpha^+ |0\rangle \quad p+\gamma = n.$$

SAME PROCEDURE

$$H(S_j^+)^p = (S_j^+)^p H + [H:(S_j^+)^p]$$

$$= (S_j^+)^p H + p(S_j^+)^{p-1} [H:S_j^+] + \frac{1}{2} p(p-1) [H:S_j^+] S_j^+.$$

$$\left\{ \begin{array}{l} [H:S_j^+] = 2c_j S_j^+ - \gamma S_j^+ S_j^+ \\ [[H:S_j^+]:S_j^+] = -\gamma S_j^+ S_j^+ \\ [[H:S_j^+]:S_\alpha^+] = -\gamma S_j^+ \frac{S_\alpha^+}{\gamma - \alpha} \end{array} \right. \quad S_j^+ = \sum_i S_i^+$$

$$\Rightarrow H(S_j^+)^p \frac{1}{\alpha} S_\alpha^+ |0\rangle = \left[ 2pc_j + \sum_{p=1}^p (2\alpha_p) + \sum_{i=1}^L \epsilon_i v_i \right] (S_j^+)^p \frac{1}{\alpha} S_\alpha^+ |0\rangle$$

$$+ \sum_{p=1}^p \left[ 1 + \gamma \sum_{i=1}^L \frac{\alpha_i}{\epsilon_i - \alpha_p} - \gamma \sum_{i \neq j} \frac{1}{\epsilon_i - \epsilon_j} - \gamma \frac{p}{\epsilon_j - \alpha_p} \right] (S_j^+)^p S_j^+$$

$$+ \gamma p \underbrace{[\text{adj} - (p-1)]}_{(n-p) S_j^+} (S_j^+)^{p-1} S_j^+ \frac{1}{\alpha} S_\alpha^+ |0\rangle$$

(\*) One can get a condensate  $(S_j^+)^p |0\rangle$  if

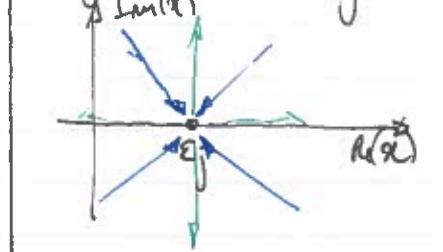
$$(1) \quad \gamma = 0 \quad \checkmark$$

$$(2) \quad \gamma = \frac{2\alpha_j}{\text{adj} - (p-1)} = 0 \quad \rightarrow \quad p = \text{adj} + 1 \quad (= \frac{n-j}{2} + 1)$$

| You try to squeeze one more pair than  
possible in a level!

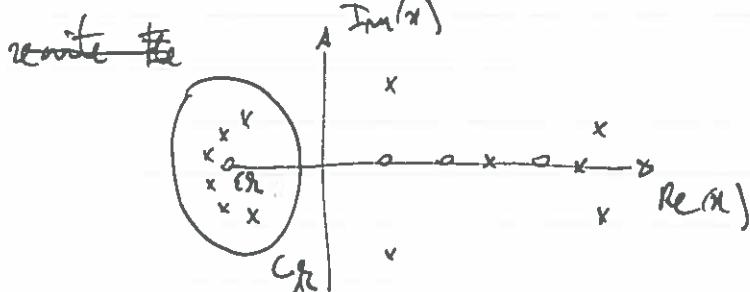
(\*) It can be shown that in the vicinity of a singular point.

$$\alpha_\alpha = \sum_j + 2\alpha := \underbrace{\epsilon_j + 2\omega^{n-j}}_{\text{Im}(\alpha)} \quad \text{imolecular} \quad (\omega^{n-j+1} = 1)$$



How to avoid the singular points?

METHOD 1: cluster method (Stefan). PNC 69, 061303(R) (2004)



separate cluster  $C_R$  from the others.

$$\frac{dk}{z_k - z_\alpha} = \sum_{\beta \in C_R} \sum_{p \neq \alpha} \frac{1}{(z_p - z_\alpha)} + F_R(z_\alpha) \quad \forall z_\alpha \in C_R.$$

→ merges up a Heine-Stieltjes!

$$F_R(z) := \frac{1}{z} + \sum_{j \neq R} \frac{d_j}{z_j - z} + \underbrace{- \sum_{p \in C_R} \frac{1}{z_p - z}}_{\text{effect of other residues}}$$

define:  $S_p = \sum_{\alpha \in C_R} (z_\alpha - z_\alpha)^p$   $p=1 \dots m_R$ . well-behaved

$$\Rightarrow \left( -2d_R + m_R - \frac{p}{2} \right) S_{p-1} + \frac{1}{2} \sum_{k=2}^R S_{p-1} S_{p+k} + \underbrace{\sum_{\alpha \in C_R} (z_\alpha - z_\alpha)^p F_R(z_\alpha)}_{\text{smooth}} = 0$$

solve  $S_p = z_\alpha$  (hand)

METHOD 2: Heine-Stieltjes on steroids. GUTW ch. PNC 86, 024313 (2012)

$$1 + g \sum_k \frac{dk}{z_k - z_\alpha} - g \sum_{p \neq \alpha} \frac{1}{(z_p - z_\alpha)} = 0$$

with: remember from  $\underline{j} \rightarrow \infty$ :

$$\sum_{\beta \neq \alpha} \frac{1}{(z_\beta - z_\alpha)^{p-1}} = \frac{P''(z_\alpha)}{2} \frac{P'(z_\alpha)}{P'(z_\alpha)}$$

$$1 + g \sum_k \frac{dk}{z_k - z_\alpha} + \frac{1}{2} g \frac{P''(z_\alpha)}{P'(z_\alpha)} = 0$$

$$\times P'(z_\alpha) \cdot \underline{A}(z_\alpha)$$

$$\Rightarrow \left( 1 + g \sum_k \frac{dk}{z_k - z_\alpha} \right) A(z_\alpha) P'(z_\alpha) + \frac{1}{2} g A(z_\alpha) P''(z_\alpha) = 0.$$

$B(z_\alpha) := \frac{A(z_\alpha) - \sum_j A(z_\alpha - \epsilon_j)}{\prod_j \epsilon_j}$

$B(z_\alpha) \text{ is polynomial in } z$

not necessarily 0!  
 $\left\{ \begin{array}{l} \epsilon_j = 0 \text{ and} \\ \sum_j = 1 \end{array} \right.$

$\Rightarrow$  note that  $x = x_\alpha$  is again a root of

$$Q(x) = B(x) P'(x) + \frac{1}{2} g Q(x) P''(x) \quad \text{degree: } N-1+L$$

$$\Rightarrow Q(x) = V(x) \cdot P(x)$$

some unknown polynomial of degree

$$\underbrace{N-1+L-N}_{Q(x)} = \underbrace{L-1}_{P(x)}$$

$\Rightarrow$  Van Vleck polynomial

$$\Rightarrow \boxed{B(x) P'(x) + \frac{1}{2} g A(x) P''(x) = V(x) P(x)}.$$

Inset:  $P(x) = \sum_{m=0}^N p_m x^m, \quad V(x) = \sum_{M=0}^{L-1} v_M x^M$ .

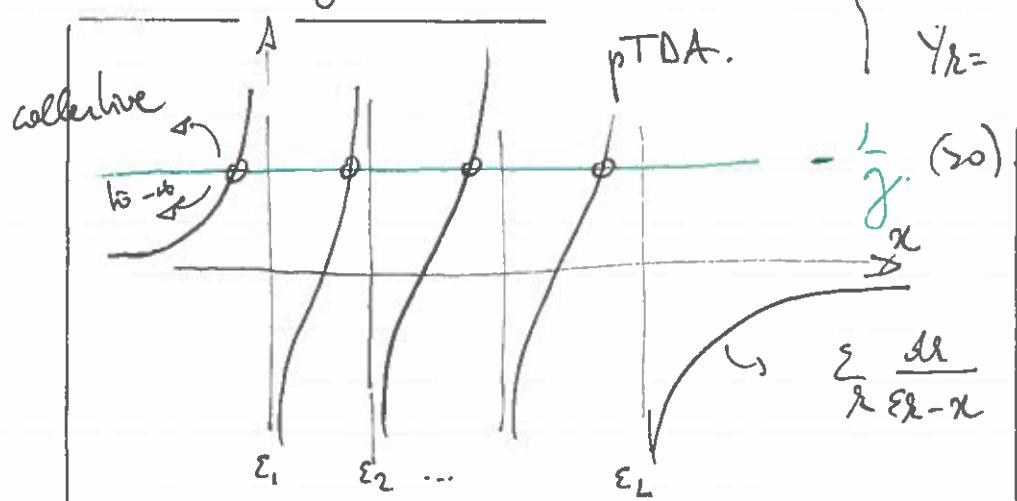
(\*) set of quadratic equations in  $p_m$  and  $v_m$

(\*) ~~invert to get~~ find roots of  $P(x)$  to get

~~rapidities~~  
 $\rightarrow$  only  ~~$p_{M-1}$~~  needs to be known to know the energy.

### METHOD 3: PSEUDO deformation SBS MRC 86, 044332 (2012)

$$\text{for } N=1: \quad 1 + g \sum_k \frac{ds}{\lambda} \frac{dx}{\lambda - x_\alpha} = 0 \quad \text{from } \left\{ H \sum_k Y_k \frac{1}{\lambda - \epsilon_k} = \sum_k g \delta(\lambda - \epsilon_k) \right\}$$



TDA: harmonic approximation:  $|q\rangle = (S_\alpha^+)^{\langle n \rangle} |0\rangle$

$\Rightarrow$   $\left\{ \begin{array}{l} \text{atomic configuration relations} \\ w(2) \end{array} \right.$

$$\left\{ \begin{array}{l} [b^{\pm}, b^{\pm}] = 0 \\ [S^{\pm}, S^{\pm}] = 0 \end{array} \right.$$

$$\left[ \begin{array}{l} b^{\pm}, b^{\pm} \\ S^{\pm}, S^{\pm} \end{array} \right] = \pm b^{\pm}$$

make a hybrid:

"HARD CORE" ATOMS

$$\left\{ \begin{array}{l} [b^{\pm}, b] = -i \\ [S^{\pm}, S] = 2S^0 \end{array} \right.$$

$$\left[ \begin{array}{l} b^{\pm}, b \\ S^{\pm}, S \end{array} \right] = \pm S^{\pm}$$

$$\left\{ \begin{array}{l} [S^0(S) : S^{\pm}(S)] = \pm S^{\pm}(S) \end{array} \right.$$

$$[S^{\pm}(S) : S(S)] = S S_i^{\pm}(S) + (S-1) \cdot i$$

PSEUDO DEFORMED ALGEBRA

$$\left\{ \begin{array}{l} g=1: \quad 3w(2) \\ g=0: \quad 2w(1) \end{array} \right.$$

$\rightarrow$  Turns a hard-core boson into a "real" boson.

$$H(S) \prod_{\alpha=1}^N S_{\alpha}^{+}(S) |0\rangle = E(S) \prod_{\alpha=1}^N S_{\alpha}^{+}(S) |0\rangle \quad \text{with } S_{\alpha}^{+}(S) = \sum_{k} \frac{S_{k\alpha}^{+}(S)}{e_k - \epsilon_{k\alpha}}$$

periodic:  $1 + g \neq \frac{g d(S)}{e_k - \epsilon_{k\alpha}} - g \sum_{\beta \neq \alpha} \frac{1}{\epsilon_{k\beta} - \epsilon_{k\alpha}} = 0$

$$g d(S) = \frac{1}{2} \partial g - S^{\frac{1}{2}} \zeta g$$

$$\left\{ \begin{array}{l} g=1: \quad RB \\ g=0: \quad TDA \end{array} \right.$$

unpaired particles  
are not felt

problem: start from  $S=0$  and re-imagine Pauli principle!

problem: # atomic configurations =  $\binom{L+N-1}{N}$  ?

$$\# w(2) \quad \# \binom{L+N-1}{N}$$

which one to pick?  $|k\rangle = b_1^{\pm} b_2^{\pm} b_3^{\pm} \dots b_N^{\pm} |0\rangle$

$$g=0: \quad |k\rangle = (b_{\text{collective}}^{\pm})^N |0\rangle$$

in between: switch at singular points!  
 $\rightarrow$  asymptotic

METHOD 4: Eigenvalue-based: later! section 3.