



Richardson-Gordan inequality from simple connected changes

~~We found a connected change~~

The most general renormalization Hamiltonian is

$$H = \sum_{\lambda} \epsilon_{\lambda} S_{\lambda}^0 + \sum_{i \neq j} v_{i\lambda} S_i^{\dagger} S_{\lambda} + \sum_{i\lambda} \underbrace{w_{i\lambda} S_i^0 S_{\lambda}^0}_{\text{monopole terms}}$$

We found connected changes of the form.

$$\tau_i = \sum_{\lambda \neq j} \frac{\bar{S}_j \cdot \bar{S}_{\lambda}}{\epsilon_j - \epsilon_{\lambda}} = \sum_{\lambda \neq j} \frac{1}{2} \frac{(S_j^{\dagger} S_{\lambda} + S_{\lambda}^{\dagger} S_j) + S_{\lambda}^0 S_j^0}{\epsilon_j - \epsilon_{\lambda}}$$

in the ASA only a specific R-matrix.

Q: can we go more general?

$$\bar{S}_i \cdot \bar{S}_j |s; m_i\rangle = d_i(d_i+1) |s; m_i\rangle$$

$$R_i = \sum_{\alpha=1}^3 A_i^{\alpha} S_i^{\alpha} + \sum_{\lambda \neq i} \sum_{\alpha=1}^3 X_{i\lambda}^{\alpha} S_i^{\alpha} S_{\lambda}^{\alpha}$$

* eqy dependent!

$$= (A_i^x S_i^x + A_i^y S_i^y + A_i^z S_i^z) + \sum_{\lambda \neq i} (X_{i\lambda}^x S_i^x S_{\lambda}^x + X_{i\lambda}^y S_i^y S_{\lambda}^y + X_{i\lambda}^z S_i^z S_{\lambda}^z)$$

$$\text{notation: } (X_{i\lambda}^x, X_{i\lambda}^y, X_{i\lambda}^z) = (\underline{X}_{i\lambda}^x; \underline{X}_{i\lambda}^y, \underline{X}_{i\lambda}^z) = (X_{i\lambda}^x, Y_{i\lambda}, Z_{i\lambda})$$

We require herm:

$$\boxed{[R_i, R_j] = 0 \quad \forall i, j}$$

all matrices

$$X = \begin{pmatrix} \end{pmatrix}$$

we use $[S_i^{\alpha}, S_j^{\beta}] = i \delta_{ij} \epsilon^{\alpha\beta\gamma} S_i^{\gamma}$ as only ingredient

$$[R_i, R_j] = \left[\sum_{\alpha=1}^3 A_i^{\alpha} S_i^{\alpha} + \sum_{\lambda \neq i} \sum_{\alpha} X_{i\lambda}^{\alpha} S_i^{\alpha} S_{\lambda}^{\alpha}, \sum_{\beta} A_j^{\beta} S_j^{\beta} + \sum_{\lambda \neq j} \sum_{\beta} X_{j\lambda}^{\beta} S_j^{\beta} S_{\lambda}^{\beta} \right]$$

$$= \left[\sum_{\alpha=1}^3 A_i^{\alpha} S_i^{\alpha}, \sum_{\lambda \neq j} \sum_{\beta} X_{j\lambda}^{\beta} S_j^{\beta} S_{\lambda}^{\beta} \right] + \left[\sum_{\lambda \neq i} \sum_{\alpha} X_{i\lambda}^{\alpha} S_i^{\alpha} S_{\lambda}^{\alpha}, \sum_{\beta} A_j^{\beta} S_j^{\beta} \right]$$

$$+ \left[\sum_{\lambda \neq i} \sum_{\alpha} X_{i\lambda}^{\alpha} S_i^{\alpha} S_{\lambda}^{\alpha}, \sum_{\lambda \neq j} \sum_{\beta} X_{j\lambda}^{\beta} S_j^{\beta} S_{\lambda}^{\beta} \right]$$

$$= i \sum_{\rho \neq \gamma} \sum_{\alpha} \epsilon_{\rho \gamma \alpha} (A_i^\alpha X_{ji}^\rho + A_j^\alpha X_{ij}^\sigma) s_i^\delta s_j^\beta \quad (1)$$

$$+ i \sum_{\rho \neq \gamma} \sum_{\lambda \neq ij} \epsilon_{\rho \gamma \lambda} (-X_{i\lambda}^r X_{ji}^\alpha - X_{j\lambda}^r X_{ij}^\alpha + X_{i\lambda}^\alpha X_{j\lambda}^\alpha) s_i^\alpha s_j^\beta s_\lambda^\gamma \quad (2)$$

$$+ i \sum_{\rho \neq \gamma} \epsilon_{\rho \gamma \alpha} (X_{ij}^\alpha X_{ji}^\rho s_j^\alpha s_j^\beta s_i^\delta - X_{ji}^\alpha X_{ij}^\rho s_i^\alpha s_i^\beta s_j^\delta) \quad (3)$$

$$(3): \sum_{\rho \neq \gamma} \epsilon_{\rho \gamma \alpha} (X_{ij}^\alpha X_{ji}^\rho s_j^\alpha s_j^\beta s_i^\delta - X_{ji}^\alpha X_{ij}^\rho s_i^\alpha s_i^\beta s_j^\delta) \\ = \sum_{\rho \neq \gamma} \epsilon_{\rho \gamma \alpha} X_{ij}^\alpha X_{ji}^\rho (s_j^\alpha s_j^\beta s_i^\delta + s_i^\alpha s_i^\beta s_j^\delta)$$

= 0 sufficient (but also necessary condition)

$$\sum_{\rho \neq \gamma} \epsilon_{\rho \gamma \alpha} X_{ij}^\alpha X_{ji}^\rho = 0 \quad \forall \gamma$$

must be parallel.

$$\Rightarrow \vec{X}_{ij}^\alpha \times \vec{X}_{ji}^\alpha = \vec{0} \quad \forall ij$$

$$\Rightarrow \boxed{X_{ij}^\alpha = \sigma X_{ji}^\alpha}$$

flipping indices twice $i \rightarrow j$

$$\Rightarrow X_{ji}^\alpha = \sigma^2 X_{ji}^\alpha$$

$$\Rightarrow \sigma^2 = 1 \\ \Rightarrow \sigma = \pm 1$$

X_{ij}^α is anti/symmetric

proof: (also (5.2.a))!

$$\begin{aligned} & X_{ij}^1 X_{ji}^2 (s_j^1 s_j^2 s_i^3 + s_i^1 s_i^2 s_j^3) \\ & + X_{ij}^2 X_{ji}^1 (s_j^2 s_j^1 s_i^3 + s_i^2 s_i^1 s_j^3) \\ & + X_{ij}^3 X_{ji}^3 (s_j^3 s_j^3 s_i^1 + s_i^3 s_i^3 s_j^1) \\ & - X_{ij}^1 X_{ji}^3 (s_j^1 s_j^3 s_i^1 + s_i^1 s_i^3 s_j^1) \\ & - X_{ij}^2 X_{ji}^3 (s_j^2 s_j^3 s_i^1 + s_i^2 s_i^3 s_j^1) \\ & - X_{ij}^3 X_{ji}^2 (s_j^3 s_j^2 s_i^1 + s_i^3 s_i^2 s_j^1) \\ & = (X_{ij}^1 X_{ji}^2 - X_{ij}^2 X_{ji}^1) (s_j^1 s_j^2 s_i^3 + s_i^1 s_i^2 s_j^3) \\ & + (X_{ij}^2 X_{ji}^1 - X_{ij}^1 X_{ji}^2) (s_j^2 s_j^1 s_i^3 + s_i^2 s_i^1 s_j^3) \\ & + (X_{ij}^3 X_{ji}^3 - X_{ij}^3 X_{ji}^3) (s_j^3 s_j^3 s_i^1 + s_i^3 s_i^3 s_j^1) \\ & - X_{ij}^1 X_{ji}^3 (-i s_j^1 s_i^1 + i s_i^1 s_j^1) \\ & - X_{ij}^2 X_{ji}^3 (-i s_j^2 s_i^1 + i s_i^1 s_j^2) \\ & - X_{ij}^3 X_{ji}^2 (-i s_j^3 s_i^1 + i s_i^1 s_j^3) \end{aligned}$$

(e) make 2 copies of (e) : and change indices.

$$-X_{i\lambda}^r X_{ji}^\alpha - X_{j\lambda}^r X_{ij}^\alpha + X_{i\lambda}^\alpha X_{j\lambda}^\rho = 0 \quad (\forall j\lambda)$$

$$-X_{\lambda i}^\alpha X_{\lambda j}^\rho - X_{\lambda j}^\alpha X_{\lambda i}^\rho + X_{\lambda i}^\rho X_{\lambda j}^\alpha = 0$$

$$-X_{ij}^\alpha X_{\lambda i}^\rho - X_{\lambda j}^\alpha X_{i\lambda}^\rho + X_{ij}^\rho X_{\lambda j}^\alpha = 0$$

$$\Rightarrow \begin{pmatrix} -1 & -1 & 1 \\ \sigma & -\sigma^2 & -\sigma \\ -\sigma^2 & +\sigma & -\sigma \end{pmatrix} \begin{pmatrix} X_{i\lambda}^r X_{ji}^\beta \\ X_{j\lambda}^r X_{ij}^\alpha \\ X_{i\lambda}^\alpha X_{j\lambda}^\rho \end{pmatrix} = 0. \quad \text{we want solutions } \neq 0$$

diag, $i \rightarrow \lambda$
 $\lambda \rightarrow j, j \rightarrow \lambda$

$$\Rightarrow \det \begin{vmatrix} -1 & -1 & 1 \\ \sigma & -\sigma^2 - \sigma & \\ -\sigma^2 & \sigma & -\sigma \end{vmatrix} = 0$$

$$\Rightarrow \sigma^2(\sigma+1)^2 = 0$$

$\sigma \neq 0 \wedge \sigma = -1$

$$\Rightarrow \boxed{X_{ij}^\alpha = -X_{ji}^\alpha} \quad \text{anti-symmetric}$$

$$\begin{aligned} & \kappa_3 | \kappa_2 + \kappa_3 \\ & \begin{vmatrix} -1 & -1 & 1 \\ \sigma & -\sigma^2 - \sigma & \\ -\sigma^2 & \sigma & -\sigma \end{vmatrix} = \begin{vmatrix} -1 & -1 & 0 \\ \sigma & -\sigma^2 - \sigma(\sigma+1) & \\ -\sigma^2 & \sigma & 0 \end{vmatrix} \\ & = -\sigma(\sigma+1) \begin{vmatrix} -1 & -1 \\ -\sigma^2 & \sigma \end{vmatrix} \\ & = -\sigma(\sigma+1) (-\sigma - \sigma^2) \\ & = \sigma^2(\sigma+1)^2 \end{aligned}$$

not also. $-X_{ji}^\alpha X_{il}^\beta - X_{ij}^\alpha X_{jl}^\beta - X_{il}^\alpha X_{lj}^\beta = 0 \quad \forall j, l$

$$\Rightarrow X_{ij}^\alpha X_{jl}^\beta + X_{jl}^\beta X_{li}^\alpha + X_{li}^\beta X_{ij}^\alpha = 0$$

equations are cyclic.

$$\boxed{X_{ij}^1 X_{jk}^2 + X_{jk}^2 X_{ki}^3 + X_{ki}^3 X_{ij}^1 = 0}$$

YANG-DARTER-GAUSSIAN equations.

(1) what about the Cartan algebra

$$\Rightarrow \sum_\alpha \epsilon_{\alpha\beta\gamma} (A_i^\alpha X_{ji}^\beta + A_j^\alpha X_{ij}^\gamma) = 0 \quad \forall i, j$$

(*) $\bar{A}_i = \bar{0}$:: trivial solutions (cfu ABA)

(*) suppose we want $A_i^3 \neq 0$ because $A_i \sim A_i^3 s_i^2 + \dots = \frac{1}{2} A_i m_i + \dots$

$$\Rightarrow \begin{aligned} \epsilon_{312} (A_i^3 X_{ji}^1 + A_j^3 X_{ij}^2) &= 0 \\ \epsilon_{321} (A_i^3 X_{ji}^2 + A_j^3 X_{ij}^1) &= 0 \end{aligned}$$

sp generator for i, k level

$$\Rightarrow \begin{pmatrix} X_{ji}^1 & X_{ij}^2 \\ X_{ji}^2 & X_{ij}^1 \end{pmatrix} \begin{pmatrix} A_i^3 \\ A_j^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

\rightarrow non-zero solutions when $X_{ji}^1 X_{ij}^1 - X_{ji}^2 X_{ij}^2 = 0$

$$\Rightarrow (X_{j^1}^1)^2 - (X_{j^2}^1)^2 = 0$$

$$\Rightarrow X_{ij}^1 = \pm X_{ij}^2$$

$$\Rightarrow -A_i^3 X_{ij}^1 + A_j^3 X_{ij}^2 = 0$$

$$A_i^3 = \pm A_j^3 \quad (\text{choose } \pm \text{ sign.})$$

$A_i^{(3)} = A_j^{(3)}$ independent of i (choose $A_i^{(3)} = 1$ because A_i is defined up to a scale).

XXZ.

$$R_i = S_i^0 + \sum_{\lambda \neq i} X_{i\lambda}^1 S_i^{\lambda} S_{\lambda}^{\lambda} + X_{i\lambda}^2 S_i^{\lambda} S_{\lambda}^{\lambda} + X_{i\lambda}^3 S_i^{\lambda} S_{\lambda}^{\lambda}$$

$$A_i = S_i^0 +$$

cell: $\begin{cases} X_{i\lambda}^1 = X_{i\lambda}^2 = X_{i\lambda} \\ X_{i\lambda}^3 = Z_{i\lambda} \end{cases}$

$$\Rightarrow R_i = S_i^0 + \sum_{\lambda \neq i} X_{i\lambda} \frac{1}{2} (S_i^{\lambda} S_{\lambda}^{\lambda} + S_{\lambda}^{\lambda} S_i^{\lambda}) + Z_{i\lambda} S_i^{\lambda} S_{\lambda}^{\lambda}$$

XXZ model

$$\left\{ \begin{array}{l} \text{with } X_{ij} + X_{ji} = 0 = Z_{ij} + Z_{ji} \\ X_{ij} X_{j\lambda} = X_{i\lambda} (Z_{ij} + Z_{j\lambda}) \end{array} \right.$$

hyperbolic
-integrable
(rational)

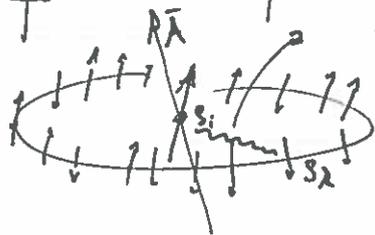
special case. XXX: $X=Z \rightarrow X_{ij} + X_{ji} = 0$
 $X_{ij} X_{j\lambda} = X_{i\lambda} (X_{ij} + X_{j\lambda})$

$$R_i = S_i^0 + \sum_{\lambda \neq i} X_{i\lambda} \left[\frac{1}{2} (S_i^{\lambda} S_{\lambda}^{\lambda} + S_{\lambda}^{\lambda} S_i^{\lambda}) + S_i^{\lambda} S_{\lambda}^{\lambda} \right]$$

$$A_i = S_i^0 + \sum_{\lambda \neq i} X_{i\lambda} \frac{S_i^{\lambda} S_{\lambda}^{\lambda}}{\dots}$$

XXX model

link with central spin model (quaternion stats) $X_{i\lambda} S_i^{\lambda} S_{\lambda}^{\lambda}$



X72 if $\vec{A} = \vec{0}$: most general R_i (elliptic).

$$R_i = \sum_{\lambda \neq i} X_{i\lambda} s_i^2 s_\lambda^2 + Y_{i\lambda} s_i^2 s_\lambda^2 + Z_{i\lambda} s_i^2 s_\lambda^2 \rightsquigarrow \text{does not cause total magnetization}$$

$$[R_i: \sum_{\lambda} s_\lambda^2] \neq 0$$

some special properties of the TQGE

$$(*) \quad \boxed{X_{ij}^2 - Z_{ij}^2 = \Gamma} \quad \forall ij$$

~~proof~~ $X_{ij} X_{j\lambda} = X_{i\lambda} (Z_{ij} + Z_{j\lambda}) \quad | \quad \times (Z_{ij} - Z_{j\lambda})$

$$X_{ij} X_{j\lambda} Z_{ij} - X_{ij} X_{j\lambda} Z_{j\lambda} = X_{i\lambda} (Z_{ij}^2 - Z_{j\lambda}^2) \quad \text{we want an extra } X_{j\lambda} \text{ in (1) and } X_{j\lambda} \text{ in (2)}$$

$$- X_{ij} X_{j\lambda} Z_{ij} + X_{j\lambda} X_{ji} Z_{j\lambda} = X_{i\lambda} (Z_{ij}^2 - Z_{j\lambda}^2)$$

$$- X_{ij} [X_{j\lambda} X_{ij} - X_{j\lambda} Z_{i\lambda}] + X_{j\lambda} [X_{j\lambda} X_{ji} - X_{ji} Z_{i\lambda}] = X_{i\lambda} (Z_{ij}^2 - Z_{j\lambda}^2)$$

$$+ X_{ij}^2 X_{i\lambda} + X_{j\lambda}^2 X_{i\lambda} - X_{ij} X_{j\lambda} Z_{i\lambda} = X_{i\lambda} (Z_{ij}^2 - Z_{j\lambda}^2)$$

$$(X_{ij}^2 - X_{j\lambda}^2) X_{i\lambda} = X_{i\lambda} (Z_{ij}^2 - Z_{j\lambda}^2)$$

$$\Rightarrow (X_{ij}^2 - Z_{ij}^2) = (X_{j\lambda}^2 - Z_{j\lambda}^2) \quad \text{independent of } i, \lambda$$

$$\Rightarrow (X_{ij}^2 - Z_{ij}^2) = (X_{\lambda j}^2 - Z_{\lambda j}^2) = (X_{j\lambda}^2 - Z_{j\lambda}^2) = (X_{\lambda\lambda}^2 - Z_{\lambda\lambda}^2) \quad \forall i, j, \lambda$$

$$\Rightarrow X_{ij}^2 - Z_{ij}^2 = \Gamma \quad \square$$

$$(*) \quad \boxed{Z_{ij} Z_{j\lambda} = Z_{i\lambda} (Z_{ij} + Z_{j\lambda}) + \Gamma}$$

~~proof~~ $X_{ij} X_{j\lambda} = X_{i\lambda} (Z_{ij} + Z_{j\lambda}) \quad | \quad \times (Z_{ji} + Z_{i\lambda})$
 i abs. as intermediate for l.h.s

$$X_{ij} X_{j\lambda} (Z_{ji} + Z_{i\lambda}) = X_{i\lambda} (Z_{ij} + Z_{j\lambda}) (-Z_{ij} + Z_{i\lambda})$$

$$X_{ij} X_{ji} X_{i\lambda} = X_{i\lambda} [-Z_{ij}^2 + Z_{ij} Z_{i\lambda} - Z_{j\lambda} Z_{ij} + Z_{j\lambda} Z_{i\lambda}]$$

~~$Z_{ij} Z_{jk} = Z_{ik} (Z_{ij} + Z_{jk}) + X_{ij}^2 - Z_{ij}^2$~~

$$Z_{ij} Z_{jk} = Z_{ik} (Z_{ij} + Z_{jk}) + \Gamma$$

(*) $Z_{ij} X_{jk} Z_{ki} = X_{ij} Z_{jk} X_{ki} + \Gamma X_{jk}$

(*) etc.

Q: what are the eigenstates?

remember. $\mathcal{H} = \otimes_i \mathcal{H}_i = \otimes_i (\mathcal{R}^{S_{2i}})$

allow for an arbitrary degeneracy: $d_i = \frac{1}{2} S_{2i} - \frac{1}{2} \nu_i$

$$S_i |d_i, m_i\rangle = d_i (d_i + 1) |d_i, m_i\rangle$$

$$S_i^0 |d_i, m_i\rangle = m_i |d_i, m_i\rangle$$

$$d_i = \frac{1}{2} S_{2i} - \frac{1}{2} \nu_i$$

$$m_i = \frac{1}{2} M_i - \frac{1}{2} \nu_i$$

pair-wisdom: $S_i |d_i, -d_i\rangle = 0 \Rightarrow S_i^{(u)} |d_i, -d_i\rangle = -d_i |d_i, -d_i\rangle$

Construct from a (generalised) Jordan algebra (GGA)

$$S_\alpha^\dagger = \sum_{\lambda=1}^L X_{\lambda\alpha} S_\lambda^\dagger, \quad S_\alpha = \sum_{\lambda=1}^L X_{\lambda\alpha} S_\lambda, \quad S_\alpha^0 = \sum_{\lambda=1}^L Z_{\lambda\alpha} S_\lambda^0$$

$\alpha = 1 \dots N$:: as of yet free parameters lower saying 7GGE!

$$X = \left(\begin{array}{c|c} \left(\begin{array}{c} L \times L \\ i, j \end{array} \right) & i_\alpha \\ \hline \left(\begin{array}{c} N \times L \\ \alpha_i \end{array} \right) & d_\alpha \end{array} \right)$$

→ it is as if the matrices X and Z are augmented

note: $S_\alpha \rightarrow S_\lambda$ is not invertible (not a mapping) ($N \leq L$)

The GGA is an infinite D algebra

$$\begin{aligned} [S_\alpha^\dagger; S_\beta] &= 2X_{\alpha\beta} (S_\alpha^0 - S_\beta^0) \quad (1) \\ [S_\alpha^\dagger; S_\beta^\dagger] &= X_{\alpha\beta} S_\alpha^\dagger - 2Z_{\alpha\beta} S_\beta^\dagger \quad (2) \\ [S_\alpha^0; S_\beta] &= -(X_{\alpha\beta} S_\alpha - Z_{\alpha\beta} S_\beta) \end{aligned}$$

proof:

$$\begin{aligned} (1): [S_\alpha^\dagger; S_\beta] &= \sum_j X_{\alpha j} X_{j\beta} [S_i^\dagger; S_j] \\ &= \sum_j X_{\alpha j} X_{j\beta} 2\delta_{ij} S_i^0 \\ &= 2 \sum_i X_{i\alpha} X_{i\beta} S_i^0 \\ &= 2 \sum_i X_{\alpha i} X_{i\beta} S_i^0 \\ &= 2 \sum_i X_{\alpha\beta} (Z_{\alpha i} + Z_{i\beta}) S_i^0 \\ &= 2X_{\alpha\beta} (\sum_i Z_{i\alpha} S_i^0 - \sum_i Z_{i\beta} S_i^0) \\ &\stackrel{!}{=} 2X_{\alpha\beta} (S_\alpha^0 - S_\beta^0) \end{aligned}$$

(2): $[S_\alpha^\dagger; S_\beta^\dagger] = \sum_j X_{\alpha j} X_{j\beta} [S_i^\dagger; S_j^\dagger]$ (*) unital

$$\begin{aligned} &= \sum_j X_{\alpha j} X_{j\beta} \delta_{ij} S_j^\dagger \\ &= \sum_i X_{i\alpha} X_{i\beta} S_i^\dagger \\ &= \sum_i (X_{i\alpha} X_{\alpha\beta} - X_{i\beta} Z_{\alpha\alpha}) S_i^\dagger \\ &= X_{\alpha\beta} S_\alpha^\dagger - Z_{\alpha\alpha} S_\alpha^\dagger \\ &= X_{\alpha\beta} S_\alpha^\dagger - Z_{\alpha\beta} S_\beta^\dagger \end{aligned}$$

(*) note: $[S_\alpha^\dagger; S_\alpha] = \sum_j X_{\alpha j} X_{j\alpha} [S_i^\dagger; S_j]$

$$= 2 \sum_i X_{i\alpha}^2 S_i^0 \neq f(S_\alpha) \quad \left. \vphantom{\sum_i} \right\} \text{not well defined}$$

Assume a product state of generalized creation operators. (cf. B(\mu) operators from ABS)

$$|u\rangle = \prod_{\alpha=1}^N S_\alpha^\dagger |b\rangle$$

$$\begin{aligned} R_i |u\rangle &= R_i \prod_{\alpha=1}^N S_\alpha^\dagger |b\rangle = \left(\prod_{\alpha=1}^N S_\alpha^\dagger \right) R_i |b\rangle + [R_i; \prod_{\alpha=1}^N S_\alpha^\dagger] |b\rangle \\ &= \left(\prod_{\alpha=1}^N S_\alpha^\dagger \right) R_i |b\rangle + \sum_{\alpha=1}^N \frac{1}{N} (S_\alpha^\dagger) [R_i; S_\alpha^\dagger] \left(\prod_{\gamma=\alpha+1}^N S_\gamma^\dagger \right) |b\rangle \\ &= \left(\prod_{\alpha=1}^N S_\alpha^\dagger \right) R_i |b\rangle + \sum_{\alpha=1}^N \frac{1}{N} (S_\alpha^\dagger) \left\{ \left(\prod_{\gamma=\alpha+1}^N S_\gamma^\dagger \right) [R_i; S_\alpha^\dagger] + [R_i; S_\alpha^\dagger] \left(\prod_{\gamma=\alpha+1}^N S_\gamma^\dagger \right) \right\} |b\rangle \end{aligned}$$

$$\begin{aligned}
 R_i | \psi \rangle &= \left(\frac{N}{n} s_\alpha^+ \right) R_i | \psi \rangle + \sum_{\alpha=1}^N \left(\frac{N}{n} s_\alpha^+ \right) [R_i : s_\alpha^+] | \psi \rangle \\
 &+ \sum_{\alpha=1}^N \left(\frac{\alpha-1}{n} s_\alpha^+ \right) \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^N \left(\frac{\beta-1}{n} s_\beta^+ \right) \underbrace{[[R_i : s_\alpha^+] : s_\beta^+]}_{\text{commutes with } s_\alpha^+} \left(\frac{N}{n} s_\beta^+ \right) | \psi \rangle. \\
 &= \left(\frac{N}{n} s_\alpha^+ \right) R_i | \psi \rangle + \sum_{\alpha=1}^N \left(\frac{N}{n} s_\alpha^+ \right) [R_i : s_\alpha^+] | \psi \rangle \\
 &+ \sum_{\alpha=1}^N \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^N \left(\frac{N}{n} s_\alpha^+ \right) \left(\frac{N}{n} s_\beta^+ \right) [[R_i : s_\alpha^+] : s_\beta^+] | \psi \rangle.
 \end{aligned}$$

calculate commutators, and act on $|\psi\rangle$

we're here specified in wegen 7 BGE yeluid maar μ de $R(\mu) = E(\mu)$.

$$R_i = s_i^0 + \kappa \sum_{\lambda \neq i} \left[\frac{1}{2} X_{i\lambda} (s_i^+ s_\lambda^+ + s_\lambda^+ s_i^+) + Z_{i\lambda} s_i^0 s_\lambda^0 \right] \quad \kappa \text{ male factor.}$$

$$\begin{aligned}
 \underline{\text{we:}} \quad [s_i^0 : s_\alpha^+] &= \sum_{\lambda} X_{\lambda\alpha} [s_i^0 : s_\lambda^+] = X_{i\alpha} s_i^+ \\
 [s_i^+ : s_\alpha^+] &= \sum_{\lambda} X_{\lambda\alpha} [s_i^+ : s_\lambda^+] = -2 X_{i\alpha} s_i^0 \\
 [s_i^+ : s_\alpha^+] &= 0
 \end{aligned}$$

$$\begin{aligned}
 [R_i : s_\alpha^+] &= [s_i^0 + \kappa \sum_{\lambda \neq i} \frac{1}{2} X_{i\lambda} (s_i^+ s_\lambda^+ + s_\lambda^+ s_i^+) + Z_{i\lambda} s_i^0 s_\lambda^0 : s_\alpha^+] \\
 &= [s_i^0 : s_\alpha^+] + \kappa \sum_{\lambda \neq i} \left\{ \frac{1}{2} X_{i\lambda} (s_i^+ [s_\lambda^+ : s_\alpha^+] + s_\lambda^+ [s_i^+ : s_\alpha^+]) + Z_{i\lambda} (s_i^0 [s_\lambda^+ : s_\alpha^+] + [s_i^0 : s_\alpha^+] s_\lambda^0) \right\} \\
 &= X_{i\alpha} s_i^+ + \kappa \sum_{\lambda \neq i} \left\{ \frac{1}{2} X_{i\lambda} (s_i^+ (-2) X_{\lambda\alpha} s_\lambda^0 + s_\lambda^+ (-1) X_{i\alpha} s_i^0) + Z_{i\lambda} (s_i^0 X_{\lambda\alpha} s_\lambda^+ + X_{i\alpha} s_i^+ s_\lambda^0) \right\} \\
 &= X_{i\alpha} s_i^+ + \kappa \sum_{\lambda \neq i} \left\{ \underbrace{(Z_{i\lambda} X_{i\alpha} - X_{i\lambda} X_{\lambda\alpha})}_{\text{YBGE}} s_i^+ s_\lambda^0 + \underbrace{(Z_{i\lambda} X_{\lambda\alpha} - X_{i\lambda} X_{i\alpha})}_{\text{YBGE}} s_\lambda^+ s_i^0 \right\} \\
 &= X_{i\alpha} s_i^+ + \kappa \sum_{\lambda \neq i} \left\{ \underbrace{(Z_{i\lambda} X_{i\alpha} - X_{i\alpha} (Z_{i\lambda} + Z_{\lambda\alpha}))}_{\text{YBGE}} s_i^+ s_\lambda^0 + \underbrace{(Z_{i\lambda} X_{\lambda\alpha} + X_{\lambda\alpha} (Z_{i\lambda} + Z_{i\alpha}))}_{\text{YBGE}} s_\lambda^+ s_i^0 \right\} \\
 &= X_{i\alpha} s_i^+ + \kappa \sum_{\lambda \neq i} \left\{ -X_{i\alpha} Z_{\lambda\alpha} s_i^+ s_\lambda^0 + X_{\lambda\alpha} Z_{i\alpha} s_\lambda^+ s_i^0 \right\} \\
 &= X_{i\alpha} s_i^+ + \kappa \left\{ -X_{i\alpha} s_i^+ (s_\alpha^0 - Z_{i\alpha} s_i^0) + Z_{i\alpha} (s_\alpha^+ - X_{i\alpha} s_i^+) s_i^0 \right\} \\
 &= X_{i\alpha} s_i^+ [1 - \kappa s_\alpha^0] + \kappa Z_{i\alpha} s_\alpha^+ s_i^0
 \end{aligned}$$

↳ make sure s_i^0 are left

$$\begin{aligned}
 [R_i: s_\alpha^+, s_\beta^+] &= [X_{i\alpha} s_i^+ (1 - \kappa s_\alpha^0) + \kappa Z_{i\alpha} s_\alpha^+ s_i^0 : s_\beta^+] \\
 &= -\kappa X_{i\alpha} s_i^+ [s_\alpha^0 : s_\beta^+] + \kappa Z_{i\alpha} s_\alpha^+ [s_i^0 : s_\beta^+] \\
 &= -\kappa X_{i\alpha} s_i^+ (\underbrace{X_{\alpha\gamma} s_\alpha^+ - Z_{\alpha\gamma} s_\gamma^+}_{\text{GGA}}) + \kappa Z_{i\alpha} s_\alpha^+ X_{i\gamma} s_\gamma^+ \\
 &= -\kappa s_i^+ [(X_{i\alpha} X_{\alpha\gamma} - X_{i\gamma} Z_{i\alpha}) s_\alpha^+ - X_{i\alpha} Z_{\alpha\gamma} s_\gamma^+] \\
 &= -\kappa s_i^+ [(X_{i\gamma} (Z_{i\alpha} + Z_{\alpha\gamma}) - X_{i\gamma} Z_{i\alpha}) s_\alpha^+ - X_{i\alpha} Z_{\alpha\gamma} s_\gamma^+] \\
 &= -\kappa s_i^+ [X_{i\gamma} Z_{\alpha\gamma} s_\alpha^+ - X_{i\alpha} Z_{\alpha\gamma} s_\gamma^+] \\
 &= -\kappa Z_{\alpha\gamma} s_i^+ [X_{i\gamma} s_\alpha^+ - X_{i\alpha} s_\gamma^+].
 \end{aligned}$$

$$\begin{aligned}
 [R_i: s_\alpha^+] &= X_{i\alpha} s_i^+ [1 - \kappa s_\alpha^0] + \kappa Z_{i\alpha} s_\alpha^+ s_i^0 \\
 [R_i: s_\alpha^+ : s_\beta^+] &= -\kappa Z_{\alpha\gamma} s_i^+ [X_{i\gamma} s_\alpha^+ - X_{i\alpha} s_\gamma^+]
 \end{aligned}$$

indeed,
 $[R_i: s_\alpha^+ : s_\beta^+ : s_\gamma^+] = 0$
 (only 2-body interactions).

action on the vacuum $|0\rangle$. (define $s_\alpha^0 |0\rangle = -d_\alpha |0\rangle = -\sum_{\beta \neq \alpha} Z_{\alpha\beta} d_\beta |0\rangle$)

$$\begin{aligned}
 R_i |0\rangle &= -d_i |0\rangle + \kappa \sum_{\beta \neq i} Z_{i\beta} (-d_i) (-d_\beta) |0\rangle = d_i [-1 + \kappa \sum_{\beta \neq i} Z_{i\beta}] |0\rangle \\
 [R_i: s_\alpha^+] |0\rangle &= (X_{i\alpha} s_i^+ [1 + \kappa d_\alpha] - \kappa Z_{i\alpha} s_\alpha^+ d_i) |0\rangle \\
 [R_i: s_\alpha^+ : s_\beta^+] |0\rangle &= -\kappa Z_{\alpha\gamma} s_i^+ [X_{i\gamma} s_\alpha^+ - X_{i\alpha} s_\gamma^+] |0\rangle.
 \end{aligned}$$

plugging that in $R_i |0\rangle$ (rearranging, etc: proof r.s.a)

$$\begin{aligned}
 R_i \left(\sum_{\alpha=1}^N s_\alpha^+ \right) |0\rangle &= d_i \left[-1 + \kappa \sum_{\beta \neq i} Z_{i\beta} d_\beta - \kappa \sum_{\alpha=1}^N Z_{i\alpha} \right] \left(\sum_{\alpha=1}^N s_\alpha^+ \right) |0\rangle \\
 &\quad + \sum_{\alpha=1}^N X_{i\alpha} \left[1 + \kappa d_\alpha + \kappa \sum_{\beta \neq \alpha} Z_{\alpha\beta} \right] \left(s_i^+ \sum_{\beta \neq \alpha} s_\beta^+ \right) |0\rangle
 \end{aligned}$$

eigenvalues: $\epsilon_i = d_i \left[-1 + \kappa \sum_{\beta \neq i} Z_{i\beta} d_\beta - \kappa \sum_{\alpha=1}^N Z_{i\alpha} \right]$

\neq $1 + \kappa \sum_{\beta} Z_{\alpha\beta} d_\beta + \kappa \sum_{\beta \neq \alpha} Z_{\alpha\beta} = 0 \quad \forall \alpha = 1 \dots N$
 RICHARDSON-GAUDIN equations

We have come very far without specifying $X_{ij}, Z_{ij}, X_{i\lambda}, Z_{i\lambda}$.

\Rightarrow REALISATIONS of YDGE.

$$\text{YDGE: } \begin{cases} X_{ij} + X_{ji} = 0, & Z_{ij} + Z_{ji} = 0 \\ X_{ij} X_{j\lambda} \neq X_{i\lambda} (Z_{ij} + Z_{j\lambda}) \end{cases} \quad \forall i, j, \lambda. \quad X_{ij}^2 - Z_{ij}^2 = \Gamma$$

Take an auxiliary level : 0 : $\boxed{Z_{0i} = \gamma_i} \Rightarrow X_{0i} = \sqrt{\Gamma + \gamma_i^2}$

$\Rightarrow X_{i0} X_{0\lambda} = X_{i\lambda} (Z_{i0} + Z_{0\lambda})$

$\Rightarrow X_{i\lambda} = \frac{X_{i0} X_{0\lambda}}{Z_{i0} + Z_{0\lambda}} = \frac{-\sqrt{\Gamma + \gamma_i^2} \sqrt{\Gamma + \gamma_\lambda^2}}{-\gamma_i + \gamma_\lambda} = \frac{(\Gamma + \gamma_i^2)(\Gamma + \gamma_\lambda^2)}{\gamma_\lambda^2 - \gamma_i^2}$

better: $Z_{ij} Z_{j\lambda} = Z_{i\lambda} (Z_{ij} + Z_{j\lambda}) + \Gamma$

substitute $j=0$: $Z_{i0} Z_{0\lambda} = Z_{i\lambda} (Z_{i0} + Z_{0\lambda}) + \Gamma$

$\Rightarrow \boxed{Z_{i\lambda} = \frac{Z_{i0} Z_{0\lambda} - \Gamma}{Z_{i0} + Z_{0\lambda}} = \frac{-\gamma_i \gamma_\lambda - \Gamma}{-\gamma_i + \gamma_\lambda} = \frac{\Gamma + \gamma_i \gamma_\lambda}{\gamma_i - \gamma_\lambda}}$

note: (*) $Z_{i\lambda} = -Z_{\lambda i}$ $X_{i\lambda} = -X_{\lambda i}$

(*) **only** \equiv free parameters (logical, because we have all \perp covered changes).

Special case: (*) $\Gamma=0$: $X \times X$.

$X_{i\lambda} = \frac{\gamma_i \gamma_\lambda}{\gamma_i - \gamma_\lambda} = Z_{i\lambda} \Rightarrow X_{i\lambda} = Z_{i\lambda} = \frac{1}{\frac{1}{\gamma_\lambda} - \frac{1}{\gamma_i}}$ let $\varepsilon_\lambda = -\frac{1}{\gamma_\lambda}$

$\Rightarrow \boxed{X_{i\lambda} = Z_{i\lambda} = \frac{1}{\varepsilon_i - \varepsilon_\lambda}}$ rational model

(*) $P=1$: XXZ

$$X_{i2} = \frac{\sqrt{(1+\eta_i^2)(1+\eta_j^2)}}{\eta_i - \eta_j} ; Z_{i2} = \frac{1 + \eta_i \eta_j}{\eta_i - \eta_j}$$

oder $\eta_i = -\cot \theta_i$

$$X_{i2} = \frac{1}{\sinh(\theta_i - \theta_j)} ; Z_{i2} = \coth(\theta_i - \theta_j)$$

hyperbolic model

$$\begin{aligned} Z_{i2} &= - \frac{1 + \frac{\omega \theta_i \omega \theta_j}{\mu \theta_i \mu \theta_j}}{\frac{\omega \theta_i}{\mu \theta_i} - \frac{\omega \theta_j}{\mu \theta_j}} \\ &= - \frac{\mu \theta_j \omega \theta_i + \omega \theta_i \omega \theta_j}{\mu \theta_j \omega \theta_i - \omega \theta_j \mu \theta_i} \\ &= \frac{\omega(\theta_i - \theta_j)}{\mu \theta_i - \mu \theta_j} \\ X_{i2} &= \sqrt{1 + \frac{\omega^2 \theta_i \theta_j}{\mu^2 \theta_i \theta_j}} = \frac{1}{\mu(\theta_i - \theta_j)} \end{aligned}$$

(*) $P=-1$: XXZ

$$X_{i2} = \frac{\sqrt{(-1+\eta_i^2)(-1+\eta_j^2)}}{\eta_i - \eta_j} ; Z_{i2} = \frac{-1 + \eta_i \eta_j}{\eta_i - \eta_j}$$

oder $\eta_i =$

$$X_{i2} = \frac{1}{\sinh(\theta_i - \theta_j)} ; Z_{i2} = \coth(\theta_i - \theta_j)$$

hyperbolic model

$$\begin{aligned} X_{i2} &= \frac{\sqrt{\frac{\cosh^2 \theta_i}{\mu^2 \theta_i^2} - 1} \sqrt{\frac{\cosh^2 \theta_j}{\mu^2 \theta_j^2} - 1}}{\frac{\cosh^2 \theta_i}{\mu^2 \theta_i^2} \mu \theta_j - \cosh^2 \theta_j \mu \theta_i} \\ &= - \frac{\frac{1}{\mu \theta_i} \frac{1}{\mu \theta_j}}{\frac{\cosh^2 \theta_i - \theta_i^2}{\mu \theta_i^2} - \frac{\cosh^2 \theta_j - \theta_j^2}{\mu \theta_j^2}} \\ &= \frac{1}{\sinh(\theta_i - \theta_j)} \\ Z_{i2} &= \frac{1 + \mu^2 \theta_i \theta_j \cosh \theta_i \cosh \theta_j}{\mu \theta_i \theta_j} = \coth(\theta_i - \theta_j) \end{aligned}$$

also

$$\begin{aligned} Z_{i2} \cdot \frac{e^{\theta_i + \theta_j}}{e^{\theta_i + \theta_j}} &= \frac{e^{\theta_i + \theta_j}}{e^{\theta_i + \theta_j}} \cdot \frac{e^{\theta_i - \theta_j} + e^{-\theta_i + \theta_j}}{e^{\theta_i - \theta_j} - e^{-\theta_i + \theta_j}} = \frac{e^{2\theta_i} + e^{2\theta_j}}{e^{2\theta_i} - e^{2\theta_j}} \\ X_{i2} \cdot \frac{e^{\theta_i + \theta_j}}{e^{\theta_i + \theta_j}} &= \frac{e^{\theta_i + \theta_j}}{e^{\theta_i + \theta_j}} \cdot \frac{2}{e^{\theta_i - \theta_j} - e^{-\theta_i + \theta_j}} = 2 \frac{e^{\theta_i} e^{\theta_j}}{e^{2\theta_i} - e^{2\theta_j}} \end{aligned}$$

oder $e^{\theta_i} = c_i$

$$\begin{aligned} Z_{i2} &= \frac{c_i^2 + c_j^2}{c_i^2 - c_j^2} \\ X_{i2} &= \frac{2c_i c_j}{c_i^2 - c_j^2} \end{aligned}$$

→ factorisierbares Modell

proof of $\sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} X_j^\alpha X_i^\beta = 0 \quad \forall \gamma$

$$\begin{aligned}
 &\Rightarrow \sum_{\alpha\beta\gamma} X_j^\alpha X_i^\beta (s_j^\alpha s_j^\beta s_i^\gamma + s_i^\alpha s_i^\beta s_j^\gamma) \epsilon_{\alpha\beta\gamma} \\
 &= \frac{1}{2} \sum_{\alpha\beta\gamma} X_j^\alpha X_i^\beta (s_j^\alpha s_j^\beta s_i^\gamma + s_i^\alpha s_i^\beta s_j^\gamma) \\
 &\quad + \frac{1}{2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} X_j^\alpha X_i^\beta (s_j^\alpha s_j^\beta s_i^\gamma + s_i^\alpha s_i^\beta s_j^\gamma) \\
 &= \frac{1}{2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} X_j^\alpha X_i^\beta (s_j^\alpha s_j^\beta s_i^\gamma + s_i^\alpha s_i^\beta s_j^\gamma) \\
 &\quad - \frac{1}{2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} X_j^\alpha X_i^\beta ((s_j^\alpha s_j^\beta + i \sum_{\mu \neq \alpha\beta} \epsilon_{\alpha\beta\mu} s_j^\mu) s_i^\gamma + (s_i^\alpha s_i^\beta + i \sum_{\mu \neq \alpha\beta} \epsilon_{\alpha\beta\mu} s_i^\mu) s_j^\gamma) \\
 &= \frac{1}{2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} (X_j^\alpha X_i^\beta - X_i^\alpha X_j^\beta) (s_j^\alpha s_j^\beta s_i^\gamma + s_i^\alpha s_i^\beta s_j^\gamma) \\
 &\quad - \frac{1}{2} i \sum_{\alpha\beta\gamma} \sum_{\mu} \epsilon_{\alpha\beta\gamma} \epsilon_{\alpha\beta\mu} X_j^\alpha X_i^\beta s_j^\mu s_i^\gamma (\epsilon_{\alpha\beta\mu} s_j^\mu s_i^\gamma + \epsilon_{\beta\alpha\mu} s_i^\mu s_j^\gamma) \\
 &= \frac{1}{2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} (X_j^\alpha X_i^\beta - X_i^\alpha X_j^\beta) (s_j^\alpha s_j^\beta s_i^\gamma + s_i^\alpha s_i^\beta s_j^\gamma) \\
 &\quad - \frac{1}{2} i \sum_{\alpha\beta} \sum_{\gamma\mu} \epsilon_{\alpha\beta\gamma} \epsilon_{\alpha\beta\mu} X_j^\alpha X_i^\beta s_j^\mu s_i^\gamma \\
 &\quad + \frac{1}{2} i \sum_{\alpha\beta} \sum_{\gamma\mu} \epsilon_{\alpha\beta\gamma} \epsilon_{\alpha\beta\mu} X_j^\alpha X_i^\beta s_i^\mu s_j^\gamma \\
 &= \frac{1}{2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} \underbrace{\left(\sum_{\mu\nu} \epsilon_{\mu\nu\gamma} X_j^\mu X_i^\nu \right)}_{=0} (s_i^\alpha s_j^\beta s_i^\gamma + s_j^\alpha s_i^\beta s_j^\gamma).
 \end{aligned}$$

$$\begin{aligned}
 r_i(w) &= \left(\frac{N}{n} s_a^+ \right) r_i(w) + \sum_{\alpha=1}^N \left(\frac{N}{n} s_p^+ \right) [r_i: s_a^+] w \\
 &+ \sum_{\alpha=1}^N \sum_{\gamma=\alpha+1}^N \left(\frac{N}{n} s_p^+ \right) [r_i: s_a^+] s_\gamma^+ w \\
 &= \left(\frac{N}{n} s_a^+ \right) d_i [-1 + \alpha \sum_{\beta \neq i} z_{i\beta} d_\beta] w + \sum_{\alpha=1}^N \left(\frac{N}{n} s_p^+ \right) [X_{i\alpha} s_i^+ (1 + \alpha d_\alpha) - \alpha z_{i\alpha} s_a^+ d_i] w \\
 &+ \sum_{\alpha=1}^N \sum_{\gamma=\alpha+1}^N \left(\frac{N}{n} s_p^+ \right) (-\alpha) z_{\alpha\gamma} s_i^+ [X_{i\gamma} s_a^+ - X_{i\alpha} s_\gamma^+] w. \quad \text{can be incorporated in } \frac{N}{n} s_p^+ \\
 &= \left(\frac{N}{n} s_a^+ \right) d_i [-1 + \alpha \sum_{\beta \neq i} z_{i\beta} d_\beta] w + \sum_{\alpha=1}^N \left(\frac{N}{n} s_p^+ \right) (-\alpha) z_{i\alpha} d_i w \\
 &+ \sum_{\alpha=1}^N \left(\frac{N}{n} s_p^+ \right) X_{i\alpha} s_i^+ (1 + \alpha d_\alpha) w \\
 &- \alpha \sum_{\alpha=1}^N \sum_{\gamma=\alpha+1}^N \left(\frac{N}{n} s_p^+ \right) z_{\alpha\gamma} s_i^+ X_{i\gamma} s_a^+ w \\
 &+ \alpha \sum_{\alpha=1}^N \sum_{\gamma=\alpha+1}^N \left(\frac{N}{n} s_p^+ \right) z_{\alpha\gamma} X_{i\alpha} s_i^+ s_\gamma^+ w \\
 &= d_i \left[-1 + \alpha \sum_{\beta \neq i} z_{i\beta} d_\beta - \alpha \sum_{\alpha=1}^N z_{i\alpha} \right] \left(\frac{N}{n} s_a^+ \right) w \\
 &+ \sum_{\alpha=1}^N \left(\frac{N}{n} s_p^+ \right) X_{i\alpha} s_i^+ (1 + \alpha d_\alpha) w - \alpha \sum_{\alpha=1}^N \sum_{\gamma=\alpha+1}^N \left(\frac{N}{n} s_p^+ \right) s_i^+ z_{\alpha\gamma} X_{i\gamma} w \\
 &+ \alpha \sum_{\alpha=1}^N \sum_{\gamma=\alpha+1}^N \left(\frac{N}{n} s_p^+ \right) s_i^+ X_{i\alpha} z_{\alpha\gamma} w \\
 &= r_i(w) + \sum_{\alpha=1}^N \left(\frac{N}{n} s_p^+ \right) X_{i\alpha} s_i^+ (1 + \alpha d_\alpha) w - \alpha \sum_{\gamma=1}^N \sum_{\alpha=1}^{\gamma-1} \left(\frac{N}{n} s_p^+ \right) s_i^+ z_{\alpha\gamma} X_{i\gamma} w \\
 &+ \alpha \sum_{\alpha=1}^N \sum_{\gamma=\alpha+1}^N \left(\frac{N}{n} s_p^+ \right) s_i^+ X_{i\alpha} z_{\alpha\gamma} w \\
 &= r_i(w) + \sum_{\alpha=1}^N \left(s_i^+ \frac{N}{n} s_p^+ \right) X_{i\alpha} (1 + \alpha d_\alpha) w - \alpha \sum_{\alpha=1}^N \sum_{\gamma=\alpha+1}^N \left(\frac{N}{n} s_p^+ \right) s_i^+ z_{\alpha\gamma} X_{i\alpha} w \\
 &+ \alpha \sum_{\alpha=1}^N \sum_{\gamma=\alpha+1}^N \left(\frac{N}{n} s_p^+ \right) X_{i\alpha} z_{\alpha\gamma} w \\
 &= r_i(w) + \sum_{\alpha=1}^N \left(s_i^+ \frac{N}{n} s_p^+ \right) X_{i\alpha} (1 + \alpha d_\alpha) w \\
 &+ \alpha \sum_{\alpha=1}^N \left(s_i^+ \frac{N}{n} s_p^+ \right) \sum_{\gamma \neq \alpha} X_{i\alpha} z_{\alpha\gamma} w. \\
 &= r_i(w) + \sum_{\alpha=1}^N \left(s_i^+ \frac{N}{n} s_p^+ \right) X_{i\alpha} \left[1 + \alpha d_\alpha + \alpha \sum_{\gamma \neq \alpha} z_{\alpha\gamma} \right] w.
 \end{aligned}$$

