

Intra messe: The quadratic algebra (Lie algebra).

Lie algebra $\mathfrak{g} = \{G_\alpha\}_{\alpha=1..L} \in \mathbb{R}$.

Properties: (1) It is linear: $\sum_{\alpha=1}^L c_\alpha G_\alpha \in \mathfrak{g}$.

(2) It loses under Lie bracket (commutation relations).

$$[G_\alpha, G_\beta] = \sum_{\gamma=1}^L c_{\alpha\beta}{}^\gamma G_\gamma$$

(3) It supports a family of linear operators

$$C_\mu =$$

→ representations clarify the total Hilbert space

(4) It is a linearization of a group

$$U(\vec{o}) = e^{i\vec{o} \cdot \vec{\sigma}} \approx 1 + i\vec{o} \cdot \vec{\sigma} \quad \xrightarrow{\vec{o} \cdot \vec{\sigma}} U(\vec{o}).$$

(5) Can be brought into Cartan-Weyl canonical form
 → defines a root-diagram (Cartan group).

~~examples~~ $SU(2) \cong SO(3)$ (6) Algebra in physics independent, representations

Examples (1) $SU(2) \cong SO(3)$ $L = \{L_x, L_y, L_z\} = \vec{\sigma} \times \vec{p}$ (angular momentum)

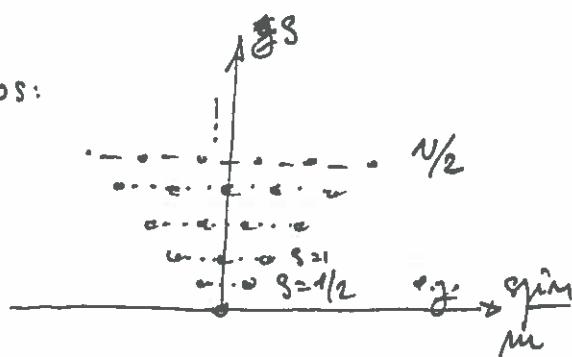
$$\hat{S} = \frac{1}{2} \{ \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z \} = \vec{\sigma} \quad (\text{spin})$$

$$[G_\alpha, G_\beta] = i \epsilon_{\alpha\beta}{}^\gamma G_\gamma$$

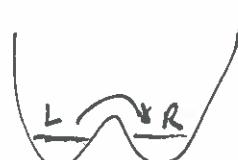
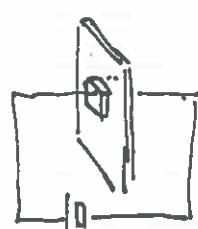
$$\text{Cartan: } G_2 = \vec{\sigma} \cdot \vec{\sigma} = \sigma_x^2 + \sigma_y^2 + \sigma_z^2.$$

$$\text{root diagram: } \leftarrow \overset{\sigma}{\longrightarrow} \rightarrow \overset{\sigma^+}{\longrightarrow} \dots$$

allowed irreps:



other su(2)'s (*1) Josephson junction (harmonic).



$$H = -t(d_R^\dagger d_L + d_L^\dagger d_R) + \sqrt{n_R n_L} d_R^\dagger d_L + \sqrt{n_R^2 n_L - d_L^\dagger d_R}$$

$$S^+ = d_R^\dagger d_L, \quad S = d_L^\dagger d_R, \quad S^0 = \frac{1}{2}(d_R^\dagger d_R - d_L^\dagger d_L)$$

$$S^x = \frac{1}{2}(d_R^\dagger d_L + d_L^\dagger d_R), \quad S^y = \frac{1}{2i}(d_R^\dagger d_L - d_L^\dagger d_R)$$

(*) Pairing, BCS.

(2) su(3) (*1) Gellmann's eightfold way.

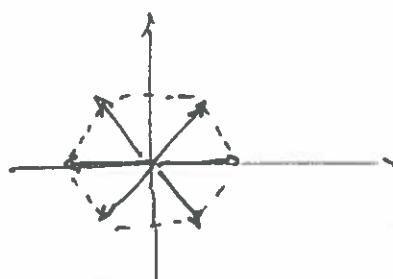
$$[d_1: d_2] = \begin{cases} 1 & d_1 \\ 1 & d_2 \end{cases}$$

(*) Elliot quasiparticle (shell model)

$$g = \underbrace{\vec{e} \times \vec{p}}_{SO(3)} \cup \vec{e} \times \vec{p} \cup \vec{e} \times \vec{p} = \vec{e} \times \vec{p}.$$

$$SO(3) \subset SU(3).$$

wot diagram:



(3) so(5): inverter pairing, quasiparticle collective model, ...

:

What is the algebra behind an \mathfrak{so} -problem problem?

$$H = \sum_j \langle :1T|j\rangle a_i^+ a_j + \frac{1}{2} \sum_{ijk} \langle ijk|V|ijk\rangle a_i^+ a_j a_k \quad \left. \begin{array}{l} \text{interactions.} \\ \text{why: they} \end{array} \right\}$$

$$\text{why: } G_{ij}^{\circ} = a_i^+ a_j \quad |j\rangle \quad i,j = 1..L \Rightarrow \# = L^2 \quad \left. \begin{array}{l} \text{why: } (a_i^+ a_j) \\ (+ \cancel{a_i^+ a_j}) \end{array} \right\}$$

$$\Rightarrow [G_{ij}^{\circ}; G_{kl}^{\circ}] = [a_i^+ a_j; a_k^+ a_l] = a_i^+ a_k \delta_{lj} - a_j^+ a_l \delta_{ik}$$

$$\text{here!} \quad \Rightarrow (G_{il}^{\circ} \delta_{lj} - G_{jl}^{\circ} \delta_{il}) \quad \Rightarrow U(L) \text{ unitary group}$$

are anti-commutation relations. $\{a_i^+ a_j\} = \delta_{ij}$

note: $i=j, j=l$

defines norm-constraining (unitary) rotations between orbitals (spins)

$$[G_{ii}^{\circ}; G_{jj}^{\circ}] = [a_i^+ a_i; a_j^+ a_j] (= [m_i; m_j]) = G_{ij}^{\circ} \delta_{ji} - G_{ji}^{\circ} \delta_{ij} \stackrel{!}{=} 0$$

Proof of $[G_{ij}^{\circ}; G_{kl}^{\circ}] = G_{il}^{\circ} \delta_{lj} - G_{jl}^{\circ} \delta_{il} =$

$$\begin{aligned} \Rightarrow [a_i^+ a_j; a_k^+ a_l] &= a_i^+ a_j a_k^+ a_l - a_k^+ a_l a_i^+ a_j \\ &= a_i^+ (\delta_{lk} - a_l^+ a_k) a_l - a_l^+ (\delta_{il} - a_i^+ a_l) a_j \\ &= a_i^+ a_l \delta_{jk} - a_i^+ a_k a_l - a_j^+ a_l \delta_{ik} + a_l^+ a_k a_j \\ &\stackrel{!}{=} a_i^+ a_l \delta_{jk} - a_j^+ a_l \delta_{ik} \end{aligned}$$

^{2nd way} $G_{ij}^{\pm} = a_i^{\pm} a_j^{\pm} \quad G_{ij} = (G_{ij}^{\pm})^{\pm} = a_j a_i \Rightarrow \# = \frac{L(L-1)}{2} + \frac{L(L)}{2}$

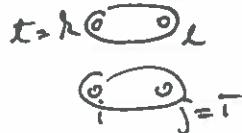
$$\Rightarrow [G_{ij}^{\pm}; G_{kl}^{\pm}] = \left(G_{il}^{\pm} - \frac{1}{2} \delta_{il} \right) \delta_{jk} + \left(G_{jl}^{\pm} - \frac{1}{2} \delta_{jl} \right) \delta_{ik} - \left(G_{ik}^{\pm} - \frac{1}{2} \delta_{ik} \right) \delta_{lj} - \left(G_{kj}^{\pm} - \frac{1}{2} \delta_{kj} \right) \delta_{il} \quad \left. \begin{array}{l} \text{Sp}(L) \supset U(L) \end{array} \right\}$$

$$[G_{ij}^{(\circ)}; G_{kl}^{\pm}] = G_{il}^{\pm} \delta_{jk} - G_{jl}^{\pm} \delta_{ik}$$

$$[G_{ij}^{(\circ)}; G_{kl}] = -(G_{li} \delta_{jk} - G_{ki} \delta_{jl})$$

(symmetric).
 $\# \in \frac{1}{2} (L(L-1) + L^2) = \frac{1}{2} L^2 = \frac{1}{2} (2L)(2L-1)$

take the special core



$$\left\{ \begin{array}{l} j = \bar{i} \quad \text{with } \bar{i} \text{ is "the special friend" of } i \text{ (and only } i) \\ \cancel{\text{---}} \\ \cancel{\text{---}} \\ L = \bar{L} \end{array} \right. \Rightarrow \begin{aligned} \delta_{jL} &= \delta_{iL} = \delta_{\bar{i}\bar{L}} \\ \delta_{iL} &= \delta_{i\bar{L}} = 0 = \delta_{\bar{i}L} \end{aligned}$$

$$\Rightarrow G_{i\bar{i}}^+ = d_i^+ a_{\bar{i}}^+ \quad G_{\bar{i}L} = \cancel{d_{\bar{i}}^+} a_L$$

$\frac{L}{2}$
 $i = \bar{i}$

$$\begin{aligned} [G_{i\bar{i}}^+; G_{\bar{i}L}] &= (G_{i\bar{i}}^{(0)} - \frac{1}{2} \delta_{iL}) \delta_{\bar{i}L} + (G_{\bar{i}L}^{(0)} - \frac{1}{2} \delta_{iL}) \delta_{iL} \\ &\quad - (G_{i\bar{i}}^{(0)} - \frac{1}{2} \delta_{i\bar{i}}) \cancel{\delta_{i\bar{i}}} - (G_{\bar{i}L}^{(0)} - \frac{1}{2} \delta_{i\bar{i}}) \cancel{\delta_{i\bar{i}}} \\ &= (G_{i\bar{i}}^{(0)} + G_{\bar{i}L}^{(0)} - 1) \delta_{iL} \end{aligned}$$

$$\begin{aligned} [G_{i\bar{i}}^{(0)} + G_{\bar{i}L}^{(0)}; G_{\bar{i}L}] &= G_{i\bar{i}}^+ \delta_{iL} - G_{i\bar{i}}^+ \cancel{\delta_{i\bar{i}}} \\ &\quad + G_{\bar{i}L}^+ \cancel{\delta_{i\bar{i}}} - G_{\bar{i}L}^+ \cancel{\delta_{i\bar{i}}} - \cancel{\delta_{iL}} \\ &\quad + (G_{i\bar{i}}^+ + G_{\bar{i}L}^+ - 1) \delta_{iL} = 2G_{i\bar{i}}^+ \delta_{iL}. \end{aligned}$$

$$[G_{i\bar{i}}^{(0)} + G_{\bar{i}L}^{(0)}; G_{\bar{i}L}] = -2 G_{i\bar{i}}^+ \delta_{iL}$$

$$\Rightarrow \text{Define: } \left\{ \begin{array}{l} S_i^{(0)} = \frac{1}{2}(G_{ii}^{(0)} + G_{\bar{i}\bar{i}}^{(0)} - 1) = \frac{1}{2}(d_i^+ a_i^+ + a_{\bar{i}}^+ a_{\bar{i}}^+ - 1) \\ S_i^+ = \cancel{d_i^+ a_i^+} = \cancel{G_{i\bar{i}}^+} \\ S_i^- = \cancel{a_{\bar{i}}^+ a_i^+} = \cancel{G_{\bar{i}L}^+} \end{array} \right.$$

$$\underline{S_i^{(0)}} \Rightarrow [S_i^{(0)}, S_L^+] = S_L^+ \delta_{iL}; [S_i^{(0)}; S_L^-] = -S_L^- \delta_{iL}; [S_L^+; S_i^-] = 2S_L^+ \delta_{iL}.$$

$\rightarrow S_i^{(0)}$; S_L^+

\hookrightarrow thanks to δ_{iL}

Significance: $\langle i | T | j \rangle = \cancel{\epsilon_i \delta_{ij}}$ (symmetry between ij and ff)

$$\begin{aligned} H &= \sum_{i=1}^{L/2} \cancel{\epsilon_i \delta_{ij}} (d_i^+ a_i^+ + a_{\bar{i}}^+ a_{\bar{i}}^+) + \cancel{\sum_{i \neq j} \epsilon_i \delta_{ij}} \\ &\quad + \frac{1}{2} \sum_{i \in L} \langle i | \psi | \bar{i} \rangle d_i^+ a_{\bar{i}}^+ a_{\bar{i}}^- a_L^- \end{aligned}$$

PAIRING HAMILTONIAN

$$+ \frac{1}{2} \sum_{i \in L} \langle i | \psi | \bar{i} \rangle d_i^+ a_{\bar{i}}^+ a_{\bar{i}}^- a_L^- + \frac{1}{2} \sum_{i \in L} \langle i | \psi | h \rangle d_i^+ a_h^-$$

Properties. (*) What are the quantum numbers?

→ We need the commutator.

$$C_{2i} = (S_i^x)^2 + (S_i^y)^2 + (S_i^z)^2 = (\hat{S}_i \cdot \vec{s}_i)$$

$$= \frac{1}{2} (S_i^+ S_i^- + S_i^- S_i^+) + (S_i^z)^2 = S_i^+ S_i^- + S_i^{(o)} (S_i^{(o)} - 1).$$

$$= \frac{1}{2} a_i^+ a_i^- a_i^- a_i^+ + \frac{1}{2} (M_i + M_i^- - 1) \frac{1}{2} (M_i + M_i^- - 3)$$

$$= \frac{1}{4} (4M_i M_i^- + (M_i + M_i^-)^2 - 4(M_i + M_i^-) + 3)$$

$$= \frac{1}{4} (4M_i M_i^- + M_i^2 + 2M_i M_i^- + M_i^-^2 - 4(M_i + M_i^-) + 3)$$

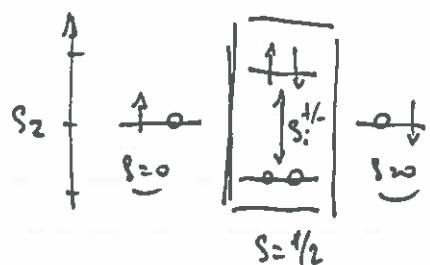
note that $M_i^2 = a_i^+ a_i^- a_i^+ a_i^- = a_i^+ (1 - a_i^+ a_i^-) a_i^- = M_i$

$$= \frac{1}{4} (6M_i M_i^- - 3M_i^2 - 3M_i^-^2 + 3)$$

$$= \frac{3}{4} (1 - (M_i - M_i^-)^2)$$

define: $|0\rangle = |0\rangle, |1_i\rangle = a_i^+ |0\rangle$ $\& |1_{ii}\rangle = a_i^+ a_i^- |0\rangle$
 $|0_i\rangle = a_i^+ |0\rangle$

	C_{2i}	$S_i^{(o)}$		S_i
$ 0\rangle$	$\frac{3}{4}$	$-\frac{1}{2}$	—	0
$ 1_i\rangle$	0	0	→ single	1
$ 1_{ii}\rangle$	0	0	→ single	1
$ 1_{li}\rangle$	$\frac{3}{4}$	$\frac{1}{2}$	—	0



$S = \frac{1}{2}$ doublet

contains a "broken pair"

introduce: singlet: $S_i = \# \text{ particles not coupled/joined}$



$$a_{j,m}^{\dagger} = (-)^{j+m} a_{j,-m} \quad a_{j,m} = (-)^{j+m} a_{j,-m}^{\dagger}$$

(*) degeneracies: e.g. spherical symmetry ($j(m)$)

$$\Rightarrow S_j^+ = \sum_{m=-1/2}^j a_{jm}^{\dagger} a_{j\bar{m}} = \frac{1}{2} [a_j^{\dagger} a_j]^{(+)^+} = -\frac{1}{2} (a_j^{\dagger} a_j^+)$$

$$S_j^- = \sum_{m=1/2}^j a_{j\bar{m}}^{\dagger} a_{jm} = \frac{1}{2} [a_j^{\dagger} a_j]^{(-)} = \frac{1}{2} (\tilde{a}_j \cdot \tilde{a}_j^+)$$

$$S_j^0 = \sum_{m=1/2}^j \frac{1}{2} (a_{jm}^{\dagger} a_{jm} + a_{j\bar{m}}^{\dagger} a_{j\bar{m}}) = \frac{1}{2} [a_j^{\dagger} a_j]^{(0)} - \frac{R}{4} j = \frac{1}{2} d_j^2 - \frac{R}{4} j$$

single occupancy ensures $\underline{\text{su}(2)}$ (because $SU(2)_M \otimes SU(2)_{M1}$)

quantum numbers? → take a state with a_j particles not coupled to zero
→ no pairs.

$$\Rightarrow S_j |m_j=j\rangle = 0 \quad \text{no pairs}$$

$$\langle \ell_{2j} | m_j=j\rangle = \frac{1}{2} [S_j^{\dagger} S_j + \frac{1}{2} S_j^0 (S_j - 1)] |m_j=j\rangle$$

other q numbers:

$$\overline{S_j^0} |m_j\rangle = \frac{1}{2} m_j - \frac{R}{4} j$$

$$\begin{aligned} &= \left(\frac{1}{2} \delta_j - \frac{R}{4}\right) \left(\frac{1}{2} \delta_j - \frac{R}{4} - 1\right) |m_j, \sigma_j\rangle \\ &= \left(\frac{R}{4} - \frac{1}{2} \delta_j\right) \left(\frac{R}{4} - \frac{1}{2} \delta_j + 1\right) |m_j=j\rangle. \end{aligned}$$

$$\Rightarrow \langle S_j \rangle = \frac{R}{4} j - \frac{1}{2} \delta_j \quad \text{e.g. } \ell_{22}: \quad S_j = \frac{1}{2} - \frac{\delta_j}{2} = \frac{1}{2} - \frac{\delta_2}{2}. \checkmark$$

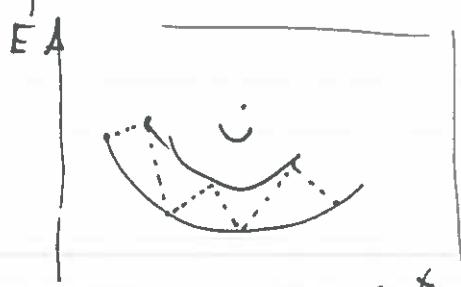
Pairing in a single shell.

$$H = \epsilon_j m_j + g S_j^+ S_j^- = \epsilon_j (S_j^0 + \frac{R}{2}) + g [\ell_{2j} - S_j^0 (S_j - 1)]$$

→ diagonal in $\text{SU}(2)_j$ basis $| \frac{R}{4} j - \frac{1}{2} \delta_j; \frac{R}{4} j - \frac{1}{2} \delta_j, \frac{1}{2} m_j - \frac{R}{4} j \rangle$

$$E_{m_j, \sigma_j} = 2\epsilon_j \left(\frac{1}{2} m_j - \frac{\delta_j}{2} + \frac{R}{4}\right) + g \left[\left(\frac{R}{4} - \frac{1}{2} \delta_j\right) \left(\frac{R}{4} - \frac{1}{2} \delta_j + 1\right) * \right. \\ \left. - \left(\frac{1}{2} m_j - \frac{\delta_j}{2}\right) \left(\frac{1}{2} m_j - \frac{\delta_j}{2} - 1\right) \right]$$

see plot therein ℓ_{22} .



(*) Tensor properties of $\{a_i^\dagger, a_i^\pm, u_i, d_i\}$

$$\text{for } a_i^\dagger: \quad \begin{cases} [S_i^0 : a_i^\dagger] = [\frac{1}{2}(a_i^\dagger a_i + a_i^\dagger a_T -) : a_i^\dagger] = \frac{1}{2} a_i^\dagger \\ [S_i^\pm : a_i^\dagger] = [a_i^\dagger a_T^\pm : a_i^\dagger] = 0 \\ [B_i : a_i^\dagger] = [a_T^\pm a_i^\dagger : a_i^\dagger] = a_T^\pm a_i^\dagger - a_i^\dagger a_T^\pm = a_T^\pm \end{cases}$$

$$a_T^\pm: \quad \begin{cases} [S_i^0 : a_T^\pm] = -\frac{1}{2} a_T^\pm \\ [S_i^\pm : a_T^\pm] = a_i^\pm \\ [B_i : a_T^\pm] = 0 \end{cases}$$

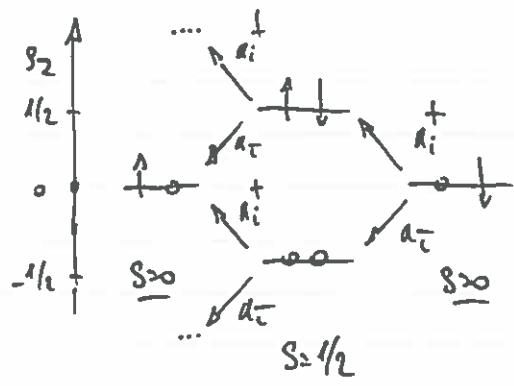
$su(2)$

definition: A tensor of \otimes multiplicity j \leftrightarrow is an object that transforms according to

$$\begin{cases} [S_i^0 : T_{jmn}] = m T_{jmn} \\ [S_i^\pm : T_{jmn}] = \sqrt{(j-m)(j+m+1)} T_{jm\pm 1} \\ [S : T_{jmn}] = \sqrt{(j+m)(j-m+1)} T_{jmn-1} \end{cases}$$

$$\Rightarrow \begin{cases} a_i^\dagger = T_{\frac{1}{2}\frac{1}{2}} \\ a_T^\pm = T_{\frac{1}{2}-\frac{1}{2}} \end{cases}$$

$\begin{pmatrix} a_i^\dagger \\ a_T^\pm \end{pmatrix}$ is a spinor in quark space.



+ dual picture for $\begin{pmatrix} u_i^\dagger \\ -u_i^- \end{pmatrix}$

(analogously for $\begin{pmatrix} d_i^\dagger \\ d_T^\pm \end{pmatrix}$)

(we need the phase for correct tensorial properties)

application: BCS - approximation

$$\begin{pmatrix} u_i^\dagger \\ u_i^- \end{pmatrix} \text{ transforms as a spinor:} \rightarrow \begin{pmatrix} c_i^\dagger \\ c_T \end{pmatrix} = D^{1/2}(\alpha\beta\gamma) \begin{pmatrix} a_i^\dagger \\ a_T^\pm \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_i^\dagger \\ c_T \end{pmatrix} = \begin{pmatrix} \cos \frac{\alpha}{2} e^{i\frac{\beta}{2}x} & -\sin \frac{\alpha}{2} e^{i\frac{\beta}{2}x} \\ \sin \frac{\alpha}{2} e^{-i\frac{\beta}{2}x} & \cos \frac{\alpha}{2} e^{-i\frac{\beta}{2}x} \end{pmatrix} \begin{pmatrix} u_i^\dagger \\ a_T^\pm \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_i^+ \\ c_i^- \end{pmatrix} = \underbrace{\begin{pmatrix} u & -v \\ v^* & u^* \end{pmatrix}}_{\text{BCS (Bogoliubov) transformation!}} \begin{pmatrix} a_i^+ \\ a_i^- \end{pmatrix}$$

BCS (Bogoliubov) transformation!

idea for the dual:

$$\begin{pmatrix} c_i^+ \\ -c_i^- \end{pmatrix} = \begin{pmatrix} u & -v \\ v^* & u^* \end{pmatrix} \begin{pmatrix} a_i^+ \\ -a_i^- \end{pmatrix}$$

Q: What about the algebra itself? A: $\{s_i^+, s_i^0, s_i^-\}$ transforms as a vector (s_i^0) in a generic spin space

$$\Rightarrow \text{define } \begin{pmatrix} A_i^+ \\ A_i^0 \\ A_i^- \end{pmatrix} = D^{(A)}(\alpha, \gamma) \begin{pmatrix} s_i^+ \\ s_i^0 \\ s_i^- \end{pmatrix} \quad \left| \text{note not completely correct...} \right.$$

$$\Rightarrow D^{(A)}(\alpha, \gamma) = [D^{1/2}(\alpha, \gamma) \alpha D^{1/2}(\alpha, \gamma)]^{A_0} \quad \left| \begin{array}{l} S_{+1} = -\frac{1}{2} S_+ \\ S_{-1} = \frac{1}{2} S_- \\ S_0 = S^0 \end{array} \right.$$

$$\Rightarrow \begin{pmatrix} A_i^+ \\ A_i^0 \\ A_i^- \end{pmatrix} = \begin{pmatrix} u_i^2 & 2u_i v_i & -v_i^2 \\ -u_i v_i^* & u_i^2 - v_i^2 & -u_i^2 v_i \\ -v_i^2 & 2u_i^* v_i & u_i^2 \end{pmatrix} \begin{pmatrix} s_i^+ \\ s_i^0 \\ s_i^- \end{pmatrix} \quad \left| \begin{array}{l} \text{note that } s_i^0 \in U(1) \\ \Rightarrow \text{we break } U(1) \text{ here} \end{array} \right.$$

$$H = \sum_i \epsilon_i (n_i + n_{i-}) + \sum_{ik} g_{ik} s_i^+ s_k^-$$

$$= \sum_i 2\epsilon_i s_i^0 + \sum_{ik} g_{ik} s_i^+ s_k^- + \sum_i (\epsilon_i)$$

$$\rightarrow H = \sum_i 2\epsilon_i [u_i^2 s_i^+ A_i^+ + (u_i^2 - v_i^2) A_i^0 + u_i v_i A_i^-] + \sum_i \epsilon_i + \sum_{ik} g_{ik} [u_i^2 s_i^+ - 2u_i v_i A_i^0 - v_i^2 A_i^-] [-v_i^2 A_i^+ - 2u_i^2 s_i^+ A_i^0 + u_i^2 A_i^-]$$

$|BCS\rangle$ is the state that is annihilated by A_i^- : $A_i^- |BCS\rangle = 0$ lowest weight
 $C_2(s) = C_2(A) \Rightarrow$ lowest weight keeps q-number $A_0 |BCS\rangle = -\frac{1}{2} |BCS\rangle$

$$\Rightarrow \langle \text{orbital} | BCS \rangle = \sum_i 2\epsilon_i (u_i^2 - v_i^2) \langle BCS | A_i^0 | \text{BCP} \rangle + \sum_i \epsilon_i$$

$$+ \sum_{ik} g_{ik} [4u_i^2 u_k v_i v_k^* \langle BCS | A_i^0 A_k^0 | \text{BCS} \rangle + v_i^2 v_k^2 \langle \text{orbital} | A_k^0 | \text{BCS} \rangle]$$

$$U_0 = \sum_{i=1}^L (2\varepsilon_i |\psi_i|^2 + \delta_{ii} |\psi_i|^4) + 4 \sum_{i,\lambda} \delta_{i\lambda} \psi_i^* u_\lambda^* v_\lambda$$