



The coordinate BETHE ANSATZ for the Heisenberg model.

$$\begin{aligned}
 H &= J \sum_{j=1}^N [(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + \Delta (S_j^z S_{j+1}^z - \frac{1}{4})] \\
 &= J \sum_{j=1}^N [S_j^x S_{j+1}^x + (\Delta - 1) S_j^z S_{j+1}^z - \frac{\Delta}{4}] \\
 &\quad \hookrightarrow \text{gives our anti/para up to higher moment} \\
 &\quad \left\{ \begin{array}{l} j > 0 : \text{anti para} \\ j < 0 : \text{para} \end{array} \right. \quad \begin{array}{l} \uparrow \text{anti} \\ \downarrow \text{para} \end{array} \quad \begin{array}{l} \text{up} \\ \text{down} \end{array} \\
 &= \text{Diagram of spins}
 \end{aligned}$$

su(2): spins

$$[S_j^\alpha : S_{j+1}^\beta] = i \epsilon_{ijk} S_j^k \Sigma_{kl} T \quad (\hbar=1)$$

$$\vec{S} = (S^x, S^y, S^z)$$

*1 Spectrum generating Algebra

independent copies of tensor products.

$$su(2)_1 \otimes su(2)_2 \otimes su(2)_3 \otimes \dots \otimes su(2)_L$$

*1 two rings that are important here

- (*) independent copies (tensor products $[S_j^\alpha : S_{j+1}^\beta] \neq 0 \forall j \neq k$).
- (*) they have a label: distinguishable
 \rightarrow triplet states as possible (opposite to fermions).

*1 su(2) algebra:

$$[S^\alpha : S^\beta] = i \epsilon_{ijk} S^k$$

$$\left\{ \begin{array}{l} [S_x : S_y] = i S_z \\ [S_y : S_z] = i S_x \\ [S_z : S_x] = i S_y \end{array} \right.$$

$$\text{Commutation operator: } [\vec{S} \cdot \vec{S} : S^\alpha] \neq 0 \quad \forall \alpha = x, y, z$$

choose $\alpha = z$: defines a basis for a representation $|SM\rangle$ $S = 0, \frac{1}{2}, 1, \dots$

$$\left\{ \begin{array}{l} S^z |SM\rangle = M |SM\rangle \\ \vec{S} \cdot \vec{S} |SM\rangle = S(S+1) |SM\rangle \end{array} \right.$$

Define raising/lowering operators:

$$S^\pm = S^x \pm i S^y$$

$$S^0 = S^z$$

$$\left| \begin{array}{l} \text{notation} \\ S^- = S \\ S^+ = (S)^+ \end{array} \right.$$

$$\begin{aligned} [S^0 \bar{S}^{\pm}] &= \pm S^{\pm} \\ [S^{\pm}, S^{\mp}] &= 2S^0 \end{aligned}$$

$$\Rightarrow S^{\pm}|SM\rangle = \sqrt{(S \mp M)(S \pm M + 1)} |SM \pm 1\rangle$$

Note: * S^z is constant. lowering and highest weight
 $M_z = -S$ $M_z = S$.

*1) special case $S=1/2$

$$\left\{ \begin{array}{l} |S=1/2, M_z=1/2\rangle = |1\rangle \\ |S=1/2, M_z=-1/2\rangle = |0\rangle. \end{array} \right.$$

$$\begin{array}{lll} (\ast) \quad \left\{ \begin{array}{l} S^0|1\rangle = \frac{1}{2}|1\rangle \\ S^+|1\rangle = 0 \\ S^-|1\rangle = |0\rangle \end{array} \right. & \begin{array}{l} S^0|0\rangle = -\frac{1}{2}|0\rangle \\ S^+|0\rangle = |1\rangle \\ S^-|0\rangle = 0 \end{array} & \begin{array}{l} S^0|1\rangle = \frac{1}{2}(1+1)|1\rangle = \frac{1}{2}|1\rangle \\ S^0|0\rangle = \frac{1}{2}(1-1)|0\rangle = 0 \end{array} \end{array}$$

*2) representation: Pauli matrices $\sigma_1, \sigma_2, \sigma_3$

$$\hat{S} = \frac{1}{2}\vec{\sigma} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S^0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (= (S^+)^*)$$

* Using this representation, it is straightforward to verify (\ast) .

* Matrix representation facilitates tensor products. \rightarrow each matrix is legge als ABS getrennt in blocks

$$\text{ex: } |1,1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1(1) \\ 0(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$S_1^- S_2^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0(0,1) & 0(0,1) \\ 1(0,1) & 0(0,1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_1^- S_2^+ |1,1\rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & (1) \\ 1 & (0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1,1\rangle$$

(*) reworking: $s_{\text{tot}}^x = \sum_{i=1}^L s_i^x \quad \forall i=1,2,\dots, L \quad (x,y,z)$

$$\Rightarrow [s_{\text{tot}}^x : s_{\text{tot}}^y] = \sum_{j=1}^L [s_j^x : s_j^y] = \sum_{j=1}^L i c_{qjy} \sum_{k=1}^L s_k^y \\ = i c_{qy} \sum_{k=1}^L s_k^y = i c_{qy} s_{\text{tot}}^y$$

\Rightarrow again an so(2)!

There is already one integral of motion. In the Heisenberg-W.

$$[H: s_{\text{tot}}^z] = \sum_{j=1}^L (s_j^x s_{j+1}^x + \sum_{k=1}^L s_k^y s_{k+1}^y) + A(s_j^z s_{j+1}^z - \frac{1}{4}) ; \\ = \sum_{j=1}^L (s_j^x s_{j+1}^x : s_{\text{tot}}^z) + [s_j^y s_{j+1}^y : s_{\text{tot}}^z] \\ = \sum_{j=1}^L (s_j^x [s_{j+1}^x : s_{\text{tot}}^z] + [s_j^x : s_{\text{tot}}^z] s_{j+1}^x \\ + s_j^y [s_{j+1}^y : s_{\text{tot}}^z] + [s_j^y : s_{\text{tot}}^z] s_{j+1}^y) \\ = \sum_{j=1}^L (s_j^x (-s_k^y \delta_{kj+1} - s_k^y \delta_{jk} s_{j+1}^x \\ + s_k^y \delta_{kj+1} + s_k^y \delta_{jk} s_{j+1}^x) \\ = i \sum_j (-s_j^x s_{j+1}^y - s_j^y s_{j+1}^x + s_j^y s_{j+1}^x + s_j^x s_{j+1}^y) \\ = 0$$

\Rightarrow Total M_{tot} is a good quantum number

Start with $M_{\text{tot}} = \frac{L}{2}$ all spins up. $|0\rangle = |1,1,\dots,1\rangle$

$$H|0\rangle = \sum_{j=1}^L [(s_j^x s_{j+1}^x + s_j^y s_{j+1}^y) + A(s_j^z s_{j+1}^z - \frac{1}{4})] |11\dots 1\rangle .$$

more convenient to go to raising/lowering representation.

$$s_j^x s_{j+1}^x + s_j^y s_{j+1}^y \\ - \frac{1}{4} (s_j^+ + s_j^-)(s_{j+1}^+ + s_{j+1}^-) \\ + \frac{1}{4} (s_j^z - s_{j+1}^z)(s_{j+1}^z - s_{j+1}^z)$$

$$\left| \begin{array}{l} s^+ = s_x + i s_y \\ s^- = s_x - i s_y \\ \Rightarrow \begin{cases} s_z^x = \frac{1}{2}(s_+^z + s_-^z) \\ s_z^y = \frac{1}{2i}(s_+^z - s_-^z) \end{cases} \end{array} \right.$$



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$$\begin{aligned}
 R(\lambda) &= S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ \\
 &= \frac{1}{2} \left[\cancel{S_j^+ S_{j+1}^-} + S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ + \cancel{S_j^- S_{j+1}^+} \right. \\
 &\quad \left. - S_j^+ S_{j+1}^+ + S_j^- S_{j+1}^- + S_j^- S_{j+1}^+ - S_j^+ S_{j+1}^- \right] \\
 &= \frac{1}{2} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+)
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow H(\lambda) &= J \sum_{j=1}^L \left[\frac{1}{2} (S_j^+ S_{j+1}^- + S_{j+1}^- S_j^+) + \Delta (S_j^+ S_{j+1}^- - \frac{1}{4}) \right] |\lambda\rangle \\
 &= J \sum_{j=1}^L \Delta \left(\frac{1}{2} \cdot \frac{1}{2} - \frac{1}{4} \right) |\lambda\rangle = \Delta |\lambda\rangle
 \end{aligned}$$

|\lambda\rangle \text{ is the "vector"}

$$\frac{M_{\text{tot}} = \frac{L}{2} - 1}{\text{one spin flip}} : L-\text{dim basis states } |\lambda\rangle = (1, 1, \dots, b_L, \dots, 1_L).$$

$$\begin{aligned}
 H(\lambda) &= J \sum_{j=1}^L \left[\frac{1}{2} (S_j^+ S_{j+1}^- + S_{j+1}^- S_j^+) + \Delta (S_j^+ S_{j+1}^- - \frac{1}{4}) \right] |\lambda\rangle \\
 &= J \left(\frac{1}{2} |\lambda+1\rangle + \frac{1}{2} |\lambda-1\rangle + \underbrace{\Delta \left(-\frac{1}{4} - \frac{1}{4} \right)}_{\text{for } j=2 \text{ and } j+1=L} |\lambda\rangle \right). \\
 &= J \frac{1}{2} ((\lambda+1) + (\lambda-1)) - J \Delta |\lambda\rangle.
 \end{aligned}$$

$$|1\rangle = \sum_{\lambda=1}^L c_\lambda |\lambda\rangle$$

note: for $|\lambda\rangle = |1\rangle : |\lambda-1\rangle = |0\rangle$
 $|\lambda\rangle = |L\rangle : |\lambda+1\rangle = |1\rangle$

$$H|1\rangle = E|1\rangle$$

$$\begin{aligned}
 H \sum_{\lambda} c_{\lambda} |\lambda\rangle &= \sum_{\lambda} \cancel{\sum_{j=1}^L} c_{\lambda} H(|\lambda\rangle) \\
 \text{projection on } \langle j | &= \sum_{\lambda} c_{\lambda} \left(\frac{1}{2} |\lambda+1\rangle + \frac{1}{2} |\lambda-1\rangle - \Delta |\lambda\rangle \right) = E \sum_{\lambda} c_{\lambda} |\lambda\rangle
 \end{aligned}$$



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$$\sum_{\lambda, \alpha} c_\alpha \left[J_{\frac{1}{2}}^{\pm} (\langle j | \lambda + \alpha \rangle + \langle j | \lambda - \alpha \rangle) - \Delta j \langle j | \lambda \rangle \right] = E \sum_\alpha c_\alpha g_j(\lambda).$$

$$\Rightarrow J_{\frac{1}{2}}^{\pm} (c_{j-1} + c_{j+1}) - \Delta j = E c_j$$

$$\Rightarrow \boxed{J_{\frac{1}{2}}^{\pm} (c_{j-1} + c_{j+1}) = (E + \Delta j) c_j}$$

note: $\begin{cases} j=1 & : j_{-1} = \Delta \\ j=L & : j_{L+1} = 1 \end{cases}$

$$\begin{aligned} J_{\frac{1}{2}}^{\pm} (c_L + c_1) &= (E + \Delta j) c_1 \\ J_{\frac{1}{2}}^{\pm} (c_{L-1} + c_1) &= (E + \Delta j) c_L \end{aligned}$$

Standard solution: suggest plane-wave solution

PBC (Periodic Boundary Conditions)

$$c_j = e^{i k j}$$

$$\Rightarrow J_{\frac{1}{2}}^{\pm} (e^{i k (j-1)} + e^{i k (j+1)}) = (E + \Delta j) e^{i k j} \quad (j \neq 1, L).$$

$$\Rightarrow J_{\frac{1}{2}}^{\pm} (e^{i k L} + e^{i k 1}) = E + \Delta j$$

$$\Rightarrow J_{\frac{1}{2}}^{\pm} e^{i k L} = E + \Delta j$$

$$\Rightarrow \boxed{E = J(kL - \Delta)}$$

Plugging this solution in the PBC:

$$J_{\frac{1}{2}}^{\pm} (e^{i k L} + e^{i k 1}) = (J(kL - \Delta) + j\Delta) e^{i k L}$$

$$J_{\frac{1}{2}}^{\pm} (e^{i k (L-1)} + e^{i k 1}) = (J(kL - \Delta) + j\Delta) e^{i k L}$$

$$\frac{1}{2} (e^{i k L} + e^{i k 1}) = \frac{1}{2} (e^{-i k} + e^{i k}) e^{i k L}$$

$$\frac{1}{2} (e^{i k (L-1)} + e^{i k 1}) = \frac{1}{2} (e^{-i k} + e^{i k}) e^{i k L}$$

$e^{i k L} = 1$
quantization condition



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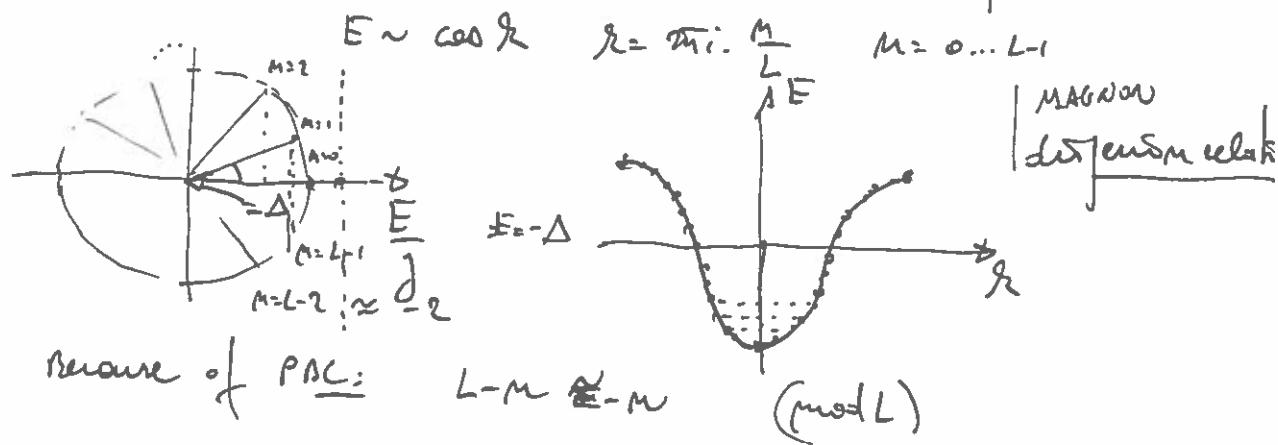
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solution is typical for system with POC:: plane wave

$$|\psi_{AP}\rangle = \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{ijk} |j\rangle = \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{ijk} s_j |\phi\rangle.$$

$\frac{e^{ikr}}{k}$
k: momentum



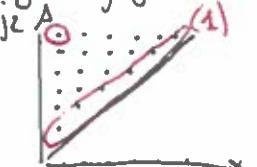
Because of POC: $L-m \equiv -m \pmod{L}$

$$\underline{M_{tot}} = \frac{L}{2} - 2 \quad : \quad 2 \text{ spinless bosons. phys.} \quad |j_1, j_2\rangle = |1, \dots, l_1, \dots, l_2, \dots, r_2\rangle.$$

$$|\psi\rangle = \sum_{j_1, j_2} c_{j_1, j_2} |j_1, j_2\rangle = \sum_{j_1=1}^L \sum_{j_2=1}^{j_1-1} c_{j_1, j_2} |j_1, j_2\rangle.$$

$$H(|\psi\rangle) = \sum_j \left[\frac{1}{2} (s_j^+ s_{j+1}^- + s_j^- s_{j+1}^+) + \Delta (s_j^\rho s_{j+1}^\rho - 1/4) \right] |j_1, j_2\rangle.$$

2 situations. $j_2 = j_1 + 1$. \times $j_2 \neq j_1 + 1$.



$$(1) \therefore H(j_1, j_2) = \sum_j \left[\frac{1}{2} (s_j^+ s_{j+1}^- + s_j^- s_{j+1}^+) + \Delta (s_j^\rho s_{j+1}^\rho - 1/4) \right] |j_1, j_2\rangle$$

$|1, \dots, l_1, \dots, \boxed{l_1, l_2}, \dots, l_{j_1-1}, l_{j_1}, \dots, r_2\rangle$ $|111\boxed{111}111\rangle$ $|111\boxed{1}111\rangle$ (it counts only the borders)

$$= \sum_j \left[\frac{1}{2} (|j_1, j_1+2\rangle + |j_1-1, j_1+1\rangle) + 2\Delta \left(\frac{1}{4} - \frac{1}{4} \right) |j_1, j_1+1\rangle \right]$$

$$= \sum_j \frac{1}{2} (|j_1, j_1+2\rangle + |j_1-1, j_1+1\rangle) - \Delta \sum_j |j_1, j_1+1\rangle$$



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schematically $H(j_1, j_2)$

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$$(2) \approx |1, 1, \dots, 1_{j_1}, 1, \dots, 1_{j_2}, \dots \rangle.$$

$$H|j_1, j_2\rangle = \sum \left[\frac{1}{2} |j_1+1, j_2\rangle + \frac{1}{2} |j_1, j_2+1\rangle + \frac{1}{2} |j_1-1, j_2\rangle + \frac{1}{2} |j_1, j_2-1\rangle \right. \\ \left. + 4\Delta (\frac{1}{4} - \frac{1}{4}) |j_1, j_2\rangle \right].$$

↳ now there are 4 clusters

$$H|q\rangle = H \left\{ \sum_{j_2=1}^L \sum_{j_1=1}^{j_2-2} c_{j_1, j_2} |j_1, j_2\rangle + \sum_{j_1=1}^L c_{j_1, j_1+1} |j_1, j_1+1\rangle \right. \\ \left. \text{because state one is equivalent to } |N, 1\rangle \right. \\ = \sum_{j_2=1}^L \sum_{j_1=1}^{j_2-2} c_{j_1, j_2} \left[\frac{1}{2} |j_1+1, j_2\rangle + \frac{1}{2} |j_1, j_2+1\rangle + \frac{1}{2} |j_1-1, j_2\rangle + \frac{1}{2} |j_1, j_2-1\rangle \right. \\ \left. - 2\Delta |j_1, j_2\rangle \right] \\ + \sum_{j_1=1}^L c_{j_1, j_1+1} \left[\frac{1}{2} |j_1, j_1+2\rangle + \frac{1}{2} |j_1-1, j_1+1\rangle - \Delta |j_1, j_1+1\rangle \right] = E|q\rangle.$$

2-product with $\langle j_1, j_1+1 | :$ $\langle j_1, j_1+1 | \left(\frac{1}{2} c_{j_1+1, j_1+2} + \frac{1}{2} c_{j_1, j_1+2} - \Delta c_{j_1, j_1+1} \right) = E c_{j_1, j_1+1}$

$$\langle j_1, j_2 | : \langle j_1, j_2 | \left(\frac{1}{2} c_{j_1+1, j_2} + \frac{1}{2} c_{j_1, j_2+1} + \frac{1}{2} c_{j_1+1, j_2} + \frac{1}{2} c_{j_1, j_2+1} \right. \\ \left. - 2\Delta c_{j_1, j_2} \right) = E c_{j_1, j_2} \quad (*)$$

These are the "bulk" equations (no PBC yet).

assume a plane wave solution. (inspired by $M = \frac{L}{2}-1$ case)

$$c_{j_1, j_2} = A e^{i(\lambda_1 j_1 + \lambda_2 j_2)} + B e^{i(\lambda_1 j_2 + \lambda_2 j_1)}$$



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play in (*):

$$\begin{aligned} & \frac{1}{2} \sum \left\{ A e^{i(\lambda_1 j_{i-1}) + \lambda_2 j_i} + B e^{i(\lambda_1 j_2 + \lambda_2 j_{i-1})} + A e^{i(\lambda_1 j_i + \lambda_2 j_{i+1})} + B e^{i(\lambda_1 j_{i+1} + \lambda_2 j_i)} \right. \\ & \quad \left. + A e^{i(\lambda_1 j_{i+1}) + \lambda_2 j_i} + B e^{i(\lambda_1 j_2 + \lambda_2 j_{i+1})} + A e^{i(\lambda_1 j_i + \lambda_2 j_{i+1})} + B e^{i(\lambda_1 j_{i+1}) + \lambda_2 j_i} \right\} \\ & = (E + 2j\Delta) (A e^{i(\lambda_1 j_i + \lambda_2 j_i)} + B e^{i(\lambda_1 j_i + \lambda_2 j_i)}) \end{aligned}$$

$$\begin{aligned} & \rightarrow \frac{1}{2} \sum (e^{-i\lambda_1} + e^{-i\lambda_2} + e^{i\lambda_1} + e^{i\lambda_2}) A e^{i(\lambda_1 j_i + \lambda_2 j_i)} \\ & \quad + \frac{1}{2} \sum (e^{-i\lambda_2} + e^{-i\lambda_1} + e^{i\lambda_2} + e^{i\lambda_1}) B e^{i(\lambda_2 j_i + \lambda_1 j_i)} \\ & = (E + 2j\Delta) (A e^{i(\lambda_1 j_i + \lambda_2 j_i)} + B e^{i(\lambda_1 j_i + \lambda_2 j_i)}). \\ & \Rightarrow \boxed{j(\cos \lambda_1 + \cos \lambda_2) = (E + 2j\Delta)} \quad \text{energy expression.} \end{aligned}$$

play in (**).

$$\begin{aligned} & \circ \quad \frac{1}{2} \sum \left\{ A e^{i(\lambda_1(j_{i-1}) + \lambda_2(j_{i+1}))} + B e^{i(\lambda_2(j_{i-1}) + \lambda_1(j_{i+1}))} \right. \\ & \quad \left. + A e^{i(\lambda_1 j_i + \lambda_2(j_{i+2}))} + B e^{i(\lambda_2 j_i + \lambda_1(j_{i+2}))} \right\} = (E + j\Delta) (e^{i(\lambda_1 j_i + \lambda_2(j_{i+1}))} B e^{i(\lambda_2 j_i + \lambda_1(j_{i+1}))} \\ & \rightarrow \cancel{e^{i(\lambda_1 - \lambda_2)}} + \cancel{e^{i(\lambda_2 - \lambda_1)}} \quad \times e^{-i(\lambda_1 + \lambda_2)j} \\ & \quad \frac{1}{2} \sum \left\{ A e^{i(\lambda_1 - \lambda_2)} + B e^{i(\lambda_1 - \lambda_2)} + A e^{i2\lambda_2} + B e^{i2\lambda_1} \right\} = (E + j\Delta) (A e^{i2\lambda_2} + B e^{i2\lambda_1}) \\ & \quad \left[\frac{1}{2} \sum e^{i(\lambda_1 - \lambda_2)} + \frac{1}{2} \sum e^{i2\lambda_2} - (E + j\Delta) e^{i2\lambda_2} \right] A \\ & \quad = - \left[\frac{1}{2} \sum e^{i(\lambda_1 - \lambda_2)} + \frac{1}{2} \sum e^{i2\lambda_1} - (E + j\Delta) e^{i2\lambda_1} \right] B \end{aligned}$$



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$$E + \Delta = J(\omega_1 + \omega_2) - J\Delta$$

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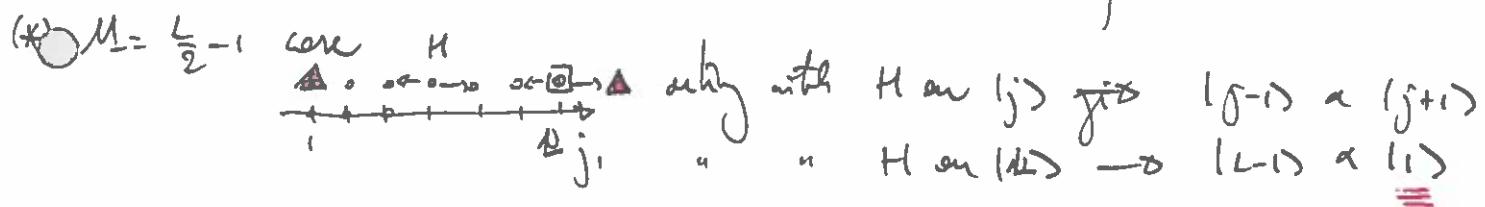
$$\begin{aligned}
 & \left[\frac{1}{2} J e^{i(\omega_1 + \omega_2)} + \frac{1}{2} J e^{i(\omega_1 - \omega_2)} - J \frac{1}{2} (e^{i\omega_1} + e^{-i\omega_1} + e^{i\omega_2} + e^{-i\omega_2} - 2\Delta) e^{i\omega_2} \right] A \\
 &= - \left[\frac{1}{2} J e^{i(\omega_1 - \omega_2)} + \frac{1}{2} J e^{i\omega_1} - J \frac{1}{2} (e^{i\omega_1} + e^{-i\omega_1} + e^{i\omega_2} + e^{-i\omega_2} - 2\Delta) e^{i\omega_1} \right] B \\
 \Rightarrow & E - \frac{1}{2} J (e^{i(\omega_1 + \omega_2)} + 1 - 2\Delta e^{i\omega_2}) A \\
 &= + \frac{J}{2} \left(1 + e^{i(\omega_1 + \omega_2)} - 2\Delta e^{i\omega_1} \right) B \\
 \frac{A}{B} = - & \frac{1 + e^{i(\omega_1 + \omega_2)} - 2\Delta e^{i\omega_1}}{1 + e^{i(\omega_1 + \omega_2)} - 2\Delta e^{i\omega_2}} =: - e^{-i\phi(\omega_1, \omega_2)}
 \end{aligned}$$

SCATTERING PHASE SHIFT

$$\Rightarrow g_{ij} = e^{i(\omega_1 j_1 + \omega_2 j_2)} - \frac{i}{2} \phi(\omega_1, \omega_2) - e^{i(\omega_1 j_1 + \omega_2 j_2)} + \frac{i}{2} \phi(\omega_1, \omega_2).$$

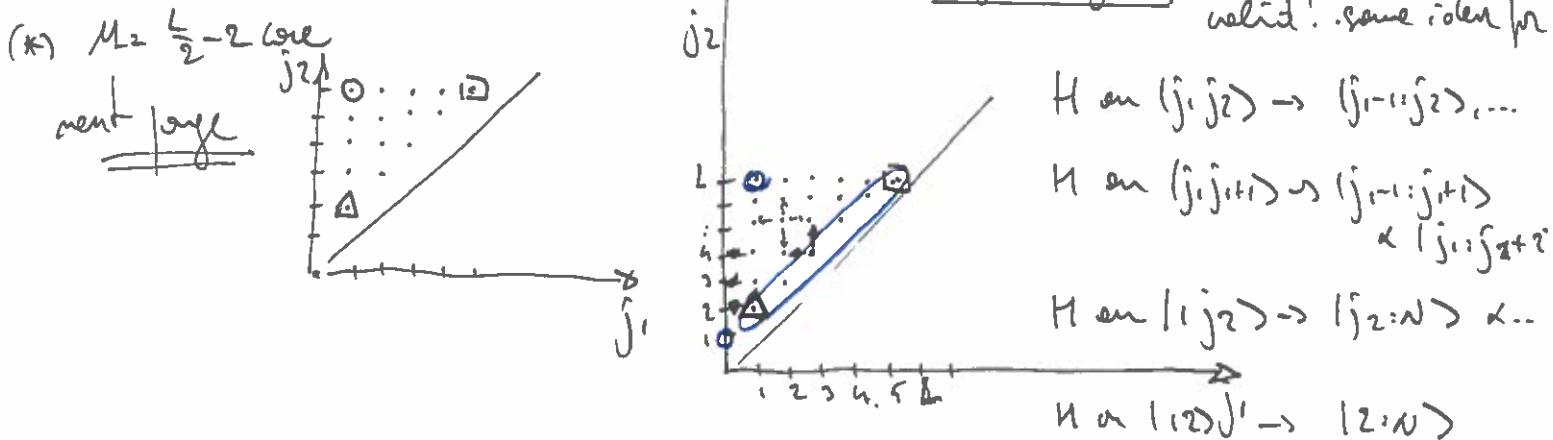
note that $\text{tr}[\text{if}(\omega_1, \omega_2)] = -\phi(\omega_2, \omega_1)$

→ This is just a parametrization of the wave function in terms of 2 variables (ω_1, ω_2) . We need to impose the PBC!



$$\Rightarrow c_j = c_{j+L}$$

PBC as if this is valid! same idea for



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$$H(j_1, j_2) \rightarrow (j_1 \pm i, j_2), (j_1, j_2 \pm i), j_3, \dots$$

$$H(j_1, j_1+i) \rightarrow (j_1-i, j_1), (j_1, j_1+i), \dots$$

$$H(j_1, j_2) \rightarrow (j_1, j_2) = (j_1, j_2)$$

$$H(j_1, N) \rightarrow (j_1, N) = (j_1, j_1)$$

$$H(1, L-i) \rightarrow (L-i, L) = (L-i, L) =$$

$$H(2, N) \rightarrow (1, L), \dots, (2, L+1) = (1, 2) =$$

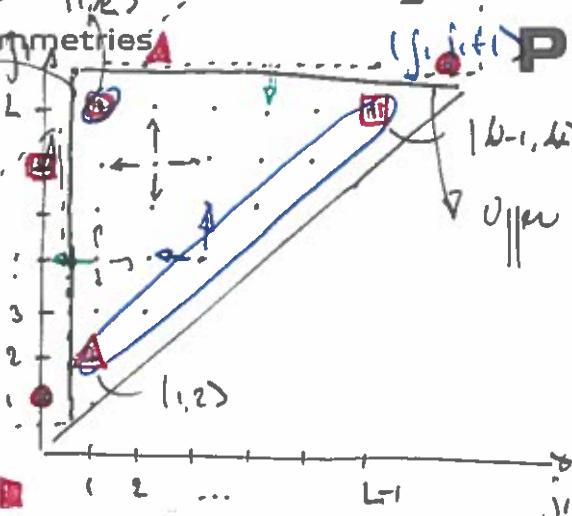
$$H(L-i, N) \rightarrow (L-i, L+1) = (L-i, L)$$

$$H(L-i, L+1) \rightarrow (L-i, L+1)$$

$$(1, L-i) \rightarrow (1, L)$$

$$\Rightarrow c_{0j_2} = c_{j_2L} \quad \text{Left boundary}$$

$$c_{j_1L} = c_{1j_1} \quad \text{Upper boundary}$$



(j₁, j₂)

(j₁-i, j₂)

(j₁, j₂+i)

(j₁-i, j₂+i)

(j₁, j₂-i)

(j₁+i, j₂)

(j₁+i, j₂-i)

(j₁-i, j₂-i)

(j₁, j₂-2i)

(j₁-i, j₂-2i)

(j₁, j₂-3i)

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(j₁, j₂-92i



Integrable systems

and quantum symmetries

Prague

equivalent sch.

$$\begin{cases} e^{i\lambda_1 L} + e^{-i\phi(\lambda_1, \lambda_2)} = 0 \\ e^{i\lambda_2 L} + e^{i\phi(\lambda_1, \lambda_2)} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} e^{i\lambda_1 L} = e^{i\pi - i\phi(\lambda_1, \lambda_2)} \\ e^{i\lambda_2 L} = e^{i\pi + i\phi(\lambda_1, \lambda_2)} \end{cases}$$

$$\rightarrow \begin{cases} \lambda_1 L = \pi - \phi(\lambda_1, \lambda_2) + 2m\pi \\ \lambda_2 L = \pi + \phi(\lambda_1, \lambda_2) + 2n\pi \end{cases}$$

$$\rightarrow \begin{cases} \lambda_1 L + \phi(\lambda_1, \lambda_2) = (2m+1)\pi \\ \lambda_2 L + \phi(\lambda_1, \lambda_2) = (2n+1)\pi \end{cases}$$

~~Physical interpretation: eigenvalues λ_i~~
~~1~~ ~~even entries~~
~~1~~ ~~odd entries~~
~~for general v~~
~~quellen für Interaktion~~

for general v

$$M = \frac{L}{2} - v$$

$$c_{j_1 j_2 \dots j_v} = \sum_{\sigma} (-)^{\sigma} c^{\sigma} \sum_{a=1}^v \lambda_a j_a + \frac{i}{2} \sum_{a \neq b} \phi(\lambda_a : \lambda_b)$$

with periodicity conditions. PBC

observe that we only have even interactions!

$$c_{j_1 j_2 \dots j_v} = c_{j_2 j_3 \dots j_v j_1 + L}$$

leading to:

$$e^{i\lambda_a L} = (-)^{v-1} e^{-i \sum_{b \neq a} \phi(\lambda_a : \lambda_b)} \quad \forall a = 1 \dots v.$$

$$\text{or: } L\lambda_a + \sum_{b \neq a} \phi(\lambda_a : \lambda_b) = 2m\pi \quad (\text{m integer for } v \text{ even})$$

half " for v odd

with $m_a = 0 \dots L-1$ stars

notice that only $\binom{N}{v}$ solutions are inequivalent
 $= \dim \mathcal{H}_v$



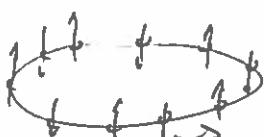
Integrable systems

and quantum symmetries

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Example: The energy is given by:

$$E_V = \sum_{a=1}^L (c_a \bar{r}_a - \Delta)$$



met idem r_a als eer rond de chain
Eng is een entree voor $\sum_{b \neq a} \phi(r_a, r_b)$ gegeven

Spiral cores and solutions

remember: $e^{-i\phi(r_i, r_i)} = \frac{1 + e^{i\theta_i + \theta_i}}{1 + e^{-i\theta_i + \theta_i}} \frac{-i\Delta e^{i\theta_i}}{-i\Delta e^{-i\theta_i}}$

$\Delta=0$: $H_{\text{free}} = \sum_{j=1}^L (S_i^x S_{i+1}^x + S_j^y S_{j+1}^y) \rightarrow \sum (c_i^+ c_{i+1}^- + c_{i+1}^+ c_i^-)$

mapping on free ferier model
by means of Jordan-Wigner transformation

indeed: $\Delta=0$:

$$e^{-i\phi(r_i, r_i)} = 1 \rightarrow |\phi(r_i, r_i)| = 0.$$

$L \theta_i = 2\pi m_i$ free model !!

$\Delta=0$: $H_{\text{Ising}} = \sum_{j=1}^L (S_i^z S_{i+1}^z - 1/4)$. Ising model

* scattering phase shift. $e^{-i\phi(r_i, r_i)} = \frac{e^{i\theta_i}}{e^{-i\theta_i}} = e^{i(\theta_i - \theta_i)}$
 $\rightarrow |\phi(r_i, r_i)| = (\theta_i - \theta_i)$

$\Delta=1$ Kondo model $H_{XXX} = \sum_{i=1}^L (S_i^+ S_{i+1}^-)$

define rapidityes. $n = \frac{1}{2} \cot \frac{\lambda}{2} \Rightarrow \lambda = \pi - \arctan n$

$$(*) \text{ proof: } y = 2 \operatorname{atan} x \quad (7.13) \\ \rightarrow x = \tan \frac{y}{2} \rightarrow x^2 + 1 = \tan^2 \left(\frac{y}{2} \right) +$$

Integrable systems

and quantum symmetries

why? because: $e^{2i\operatorname{atan} x} = \frac{1+ix}{1-ix} \quad (*)$

$$\Rightarrow e^{-i\phi(\lambda_1, \lambda_2)} = \frac{1+e^{i\lambda_1}(e^{i\lambda_2}-2)}{1+e^{i\lambda_2}(e^{i\lambda_1}-2)}$$

$$= \frac{1+(-1)e^{-i\operatorname{atan} \lambda_1}(-e^{-i\operatorname{atan} \lambda_2}-2)}{1+(-1)e^{-i\operatorname{atan} \lambda_2}(-e^{-i\operatorname{atan} \lambda_1}-2)}$$

$$= \frac{1+\frac{1-2i\lambda_1}{1+2i\lambda_2}\left(\frac{1-2i\lambda_2}{1+2i\lambda_1}+2\frac{1+2i\lambda_2}{1-2i\lambda_1}\right)}{1+\frac{1-2i\lambda_2}{1+2i\lambda_1}\left(\frac{1-2i\lambda_1}{1+2i\lambda_2}+2\frac{1+2i\lambda_1}{1-2i\lambda_2}\right)}$$

$$= \frac{(1+2i\lambda_1)(1+2i\lambda_2)+(1-2i\lambda_1)(3+2i\lambda_2)}{(1+2i\lambda_1)(1+2i\lambda_2)+(1-2i\lambda_2)(3+2i\lambda_1)}$$

$$= \frac{1+2i\lambda_1+2i\lambda_2-4\lambda_1\lambda_2+3-6i\lambda_1+2i\lambda_2+3\lambda_1\lambda_2}{1+2i\lambda_1+2i\lambda_2-4\lambda_1\lambda_2+3-6i\lambda_2+2i\lambda_1+4\lambda_1\lambda_2}$$

$$= \frac{4-4i\lambda_1+4i\lambda_2}{4+4i\lambda_1-4i\lambda_2}$$

$$= \frac{1-i(\lambda_1-\lambda_2)}{1+i(\lambda_1-\lambda_2)} = e^{-2i\operatorname{atan}(\lambda_1-\lambda_2)}.$$

\Rightarrow scattering phase shift: $\boxed{\phi(\lambda_1, \lambda_2) = 2\operatorname{atan}(\lambda_1-\lambda_2)}$ is only a $\frac{1}{2}$ of $\Delta \lambda$

$$(e^{i\lambda_a})^L = \epsilon_1^{V-1} e^{-i \sum_{b \neq a} \phi(\lambda_a, \lambda_b)} \quad (\forall a = 1, \dots, V)$$

$$\epsilon_1^L \left(\frac{1-2i\lambda_a}{1+2i\lambda_a} \right)^L = \epsilon_1^{V-1} \frac{\sqrt{V}}{b\pi a} \left[\epsilon_2 \left(\frac{1-2i(\lambda_a-\lambda_b)}{1+i(\lambda_a-\lambda_b)} \right) \right] \quad \forall a = 1, \dots, V$$

$$\epsilon_1^L \left(\frac{i/2 + \lambda_a}{i/2 - \lambda_a} \right)^L = \frac{\sqrt{V}}{b\pi a} \frac{i + (\lambda_a - \lambda_b)}{i - (\lambda_a - \lambda_b)} \epsilon_1^{V-1}$$

Prague

$$\frac{1}{\pi} = \cot \frac{\pi}{2} \rightarrow \frac{1}{x^2+1} = \cot^2 \frac{\pi}{2} + 1 = \cot^2 \left(\frac{\pi}{2} \right)$$

$$\Rightarrow \begin{cases} \omega^2 \left(\frac{\pi}{2} \right) = \frac{1}{1+x^2} \\ \omega^2 \left(\frac{\pi}{2} \right) = \frac{x^2}{1+x^2} \end{cases} \rightarrow \omega g = \omega^2 \frac{\pi}{2} - \omega^2 \frac{\pi}{2} = \frac{1-x^2}{1+x^2}$$

$$\omega g = 2 \operatorname{coth}^2 \frac{\pi}{2} = \frac{2x}{1+x^2}$$

$$e^{ig} = \cos g + i \sin g = \frac{1+x^2 + 2ix}{1+x^2} = \frac{(1+ix)}{(1-ix)}$$

$$x \frac{(1+2i\lambda_1)(1+2i\lambda_2)}{(1+2i\lambda_1)(1+2i\lambda_2)}$$



Integrable systems

and quantum symmetries

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$$\left(\frac{da + i}{da - i} \right)^L = \frac{v}{u} \left(\frac{da - ib + i}{da - ib - i} \right) \quad \forall a = 1 \dots v.$$

- +: simpler structure than the original Bethe equations
- : lost the connection and physical interpretation of winding phase shift and momentum

applications? energy of H_{XXX} :

$$E = \sum_{a=1}^L (\omega_a \omega_{a-1})$$

$$= g \sum_a \frac{(2\lambda)_a^2 - 1}{(2\lambda)_a^2 + 1}$$

$$E = \sum_{a=1}^L -\frac{g}{4\lambda_a^2 + 1}$$

$$\begin{aligned} 2\lambda &= \cot\left(\frac{\theta}{2}\right) \\ \Rightarrow (2\lambda)_a^2 &= \tan^2\left(\frac{\theta}{2}\right) \times (2\lambda)^2 = \\ \Rightarrow \left(\frac{2\lambda}{2\lambda}\right)_a^2 + 1 &= \tan^2\frac{\theta}{2} + 1 = \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} \\ (2\lambda)^2 + 1 &= \cot^2\frac{\theta}{2} + 1 = \frac{L}{\sin^2\left(\frac{\theta}{2}\right)} \\ \text{in } \omega_a \omega_{a-1} &= \omega_a^2 \left(\frac{\theta}{2}\right) - \omega_{a-1}^2 \left(\frac{\theta}{2}\right) \\ &= \frac{(2\lambda)_a^2}{1+(2\lambda)_a^2} - \frac{1}{1+(2\lambda)_a^2} \\ \Rightarrow \omega_a \omega_{a-1} &= \frac{(2\lambda)_a^2 - 1}{(2\lambda)_a^2 + 1} - \frac{(2\lambda)_a^2 + 1}{(2\lambda)_a^2 + 1} \\ &= -\frac{2}{1+(2\lambda)_a^2}. \end{aligned}$$

applications.