Richardson - Gaudin Integrability

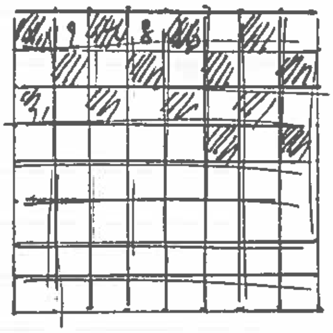
(1) What's our angle! N-body system (but there are other dynamical systems).

$$\boxed{H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{j} V(\vec{r}_i;\vec{r}_j) + \ldots}$$

$H\,\Psi(r_1 \ldots r_N) = E\,\Psi(r_1, \ldots r_N)$.

what's the problem: (*) the size of the Hilbert space
(*) what is the interaction V?

The problem.



The legend of "AMBALAPUZHA PAAL PAYASAM

- LORD KRISHNA challenges the king..
- king loses
- 1 grain of rice on 1st, (2^0)
  2   2nd (2^1)
  4
  ⋮
  2^{63} . . . . 64 th

$$\text{total } \# = \sum_{M=1}^{64} 2^{M-1} = \sum_{M=0}^{63} 2^M = \frac{1-2^{64}}{1-2}$$
$$= 2^{64}-1$$

$$M(\text{grain}) \approx \frac{1}{64} g = 2^{-6} g = 2^{-8}\, kg \rightarrow m(\text{total}): 2^{64} \cdot 2^{-8}\, kg = 2^{56}\, kg.$$
$$\approx 7.205 \cdot 10^{16}\, kg.$$

compare with: $738 \cdot 10^9\, kg$ year production in 2012.

→ Lord Krishna is a considerable god: the king would pay over time, so there are still offerings today

How does this affect a nuclear physicist? plot Wigner's heads

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We need a framework.

$$H = \sum_i \frac{p_i^2}{2m_i} + \sum_j V(\vec{r}_i;\vec{r}_j) + w(\vec{r}_i) - w(\vec{r}_i).$$

$$= \underbrace{\sum_i \frac{p_i^2}{2m_i} + w(\vec{r}_i)}_{\text{mean field}} + \underbrace{\sum_j V(\vec{r}_i;\vec{r}_j) - w(\vec{r}_i)}_{\text{residual interactions. (hopefully small}}$$

How to tackle the problem? → use approximations
→ use symmetries!

(*) obvious symmetries : - particle number
- spin (Coulomb systems non-relativistic)
- parity
- ...

(*) non-obvious symmetries : $\boxed{\text{INTEGRABILITY}}$

Q: what is it? In the classical sense: LIOUVILLE.

(*) A ~~system~~ dynamical system in $2n$-dimensional phase space is integrable if it ~~ac~~ supports n integrals of motion (*)
(2n integrals of
~~eg~~ examples : (*) Non-interacting systems. (time independent)

(*) also: constants of motion
- conserved charges.
- ...

$$H = \sum_{i=1} H_i(\vec{r}_i) \cdot = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(\vec{r}_i).$$

⇒ all H_i are ct. of motions. (**) (ENERGY)

**) Important:
all 1-body Hamiltonians are integrable

(*) Harmonic oscillator systems (coupled).

$$H = \sum_{i=1}^{N} \frac{1}{2m} p_i^2 + \frac{1}{2}\sum K_{ij}(x_i - x_j)^2.$$

$$= \sum_{i=1}^{N} \frac{1}{2m} p_i^2 + \frac{1}{2}\sum_{i,j} K_{ij}(x_i^2 - 2x_i x_j + x_j^2)$$

(*) Calogero–Sutherland models. in 1D.

$$H = \sum_{i=1}^{N} \frac{1}{2m} p_i^2 + \frac{1}{2} m x_i^2 + \sum_{i\neq j} \frac{\varepsilon}{(x_i - x_j)^2}$$

Harmonic oscillator. constants of motion

$$H = \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} \sum_j K_{ij} (x_i - x_j)^2 \qquad K_{ij} = K_{ji}$$

$$= \frac{1}{2m} \sum_i p_i^2 + \frac{1}{2} \sum_{ij} K_{ij} (x_i^2 - 2x_i x_j + x_j^2)$$

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$$= \frac{1}{2m}\sum_i^N p_i^2 + \sum_{ij}^N K_{ij} x_i^2 - \sum_{ij} K_{ij} x_i x_j$$

$$= \frac{1}{2m}\sum_i^N p_i^2 + x^T \tilde{K} x \qquad \text{met } \tilde{K} = \begin{pmatrix} \sum_j K_{1j} & -\frac{1}{2}K_{12} & -\frac{1}{2}K_{13} & \dots \\ \vdots & \sum_j K_{2j} & & \\ & & \ddots & \\ -\frac{1}{2}K_{N1} & & & \sum_j K_{Nj} \end{pmatrix}$$

Let $O^T \tilde{K} O = \Lambda$

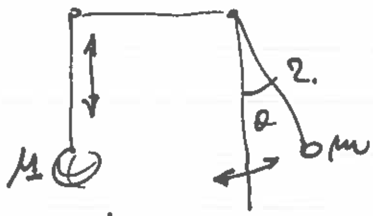
$$= \frac{1}{2m} p^T p + x^T O \Lambda \underbrace{O^T x}_{y} \qquad y = O^T x \text{ canonical transf.}$$

$$\Rightarrow p^T p = p'^T p'$$

$$= \frac{1}{2m}\sum_{i=1}^N p_i'^2 + \sum_i \Lambda_i y_i^2.$$

Decoupled! [note that some people consider an integrable model those that can be mapped on a set of HO].

(*) Swinging Atwood machine.



integrable for $\frac{M}{m} = 3$.

Mathematically (Liouville integrability).

$$\dot{F}(q,p,t) = 0 \Rightarrow \frac{\partial F}{\partial q}\cdot\dot{q} + \frac{\partial F}{\partial p}\cdot\dot{p} + \frac{\partial F}{\partial t} = 0$$

with Hamilton equations: $\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$

$$\Rightarrow \dot{F}(q,p,t) = \frac{\partial F}{\partial q}\cdot\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial F}{\partial t}.$$

$$= \{F, H\} + \frac{\partial F}{\partial t}$$

For $\frac{\partial F}{\partial t} = 0$: $F(q,p)$ is a constant of motion if it Poisson commutes with the Hamiltonian.

quantum analog: $\{\cdot;\cdot\} \to -\frac{i}{\hbar}[\,;\,]$

⇒ (a) A system is quantum Liouville integrable if there exist N commuting operators with the Hamiltonian

example: non-interacting systems.

$$H = \sum_i \left(\hat T_i + V(\vec r_i)\right) = \sum_i H_i$$

$$[H; H_i] = 0 \quad \& \quad [H_i; H_j] = 0 \quad \forall_{i,j} \quad ✓$$

However: TOO NAIVE!

$$H = \sum_{m=1}^{\dim \mathcal{H}} E_m |m\rangle\langle m| \qquad \dim(\mathcal{H}) > N$$

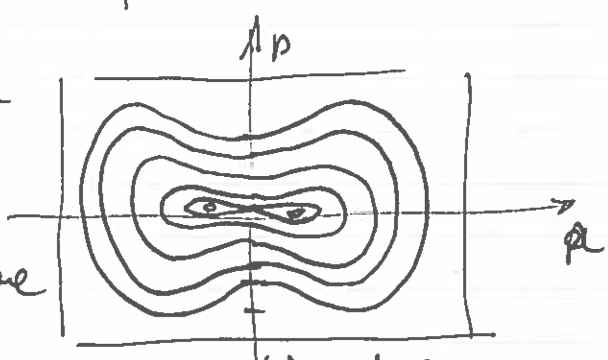
$$[|m\rangle\langle m|; |n\rangle\langle n|] = 0 \quad \& \quad [H; |m\rangle\langle m|] = 0 \quad \forall_{m,n}$$

→ All quantum systems are integrable ?! However, you get the idea

other definitions

(b) ergodicity: Integrable systems form closed orbits in phase space

e.g.: $H = \frac{1}{2m}p^2 + V(x) = E$



$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$

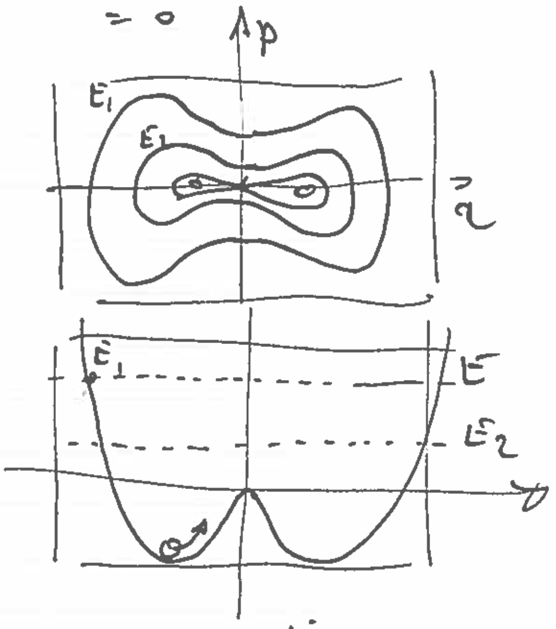
you never reach all of phase space
compare with a kicked system

(non conservative).

Normal

$$H = \frac{1}{2m}p^2 + V(x)$$

$$\dot H = \frac{\partial H}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial x} = 0$$

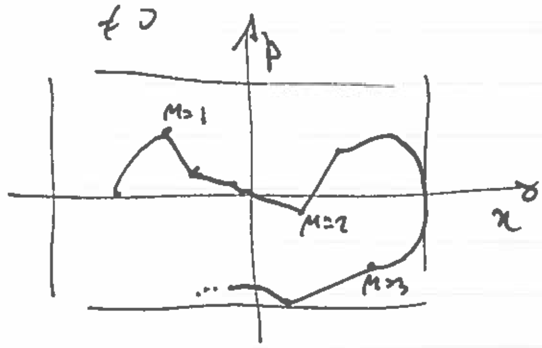


non ergodic

KICKED

$$H = \frac{1}{2m}p^2 + V(x) + \sum_{n=0}^{\infty}\delta(t-n\tau) f(x)$$

$$\dot H = \{H,H\} + \frac{\partial H}{\partial t} \neq 0$$



ergodic

→ to the quantum case:

quantum ergodicity $\quad \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,\langle \Delta A(0)\,\Delta A(t)\rangle \to 0$ $\quad$ "It must have reached each value of $\langle A\rangle$ ..."

$$\Delta A = A - \langle A\rangle$$

unpractical to define integrability

(c) level statistics (from Random Matrix Theory) $\quad$ Poisson vs Gaussian orthogonal ensemble (GOE)

Poisson: more 'degeneracy', so more 'symmetry'

i-).

(*) only well-defined for ∞ dimensional Hilbert space
(*) more of a heuristic tool than a practical rule
(*) 'breakable' by construction for RO.

(d) "solvable": ~~[illegible]~~ i.e.: solvable by means of "algebraic" means.
→ refers to the use of a Bethe Ansatz (and an R-matrix).

also: ill defined because every finite-dimensional Hilbert space is "algebraically solvable"

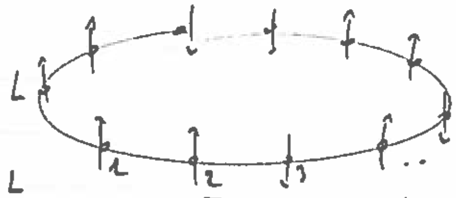
i.e.:
$$\begin{vmatrix} H_{11}-\lambda & H_{12} & \cdots & \\ & H_{22}-\lambda & & \\ & & H \ldots & \\ & & & H_{MM}-\lambda \end{vmatrix} = 0$$

$$P_M(\lambda) = 0$$

Result: It is a mixture of all these features, depending on the model. Integrability is a historical |

Historically:

1931: Bethe solves the $S=1/2$ Heisenberg model (LOCAL). (coordinate BA).



$$H_{XXX} = \sum_{i=1}^{L} J \, \vec{S}_i \cdot \vec{S}_{i+1}$$ with periodic boundaries: $N+1 = 1$

1958 Around ORBACH generalizes to XXZ

$$H_{XXZ} = J \sum_{i=1}^{L} \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \underbrace{\Delta}_{\text{anisotropy}} \left(S_j^z S_{j+1}^z - 1/4 \right) \right].$$

1963: LIEB & LINIGER solve 1d $\delta(x)$ ~~[illegible]~~ bosonic gas.

$$H_{LL} = \sum_{i=1}^{N} \frac{1}{2m} p_i^2 + c \sum_{i \neq j} \delta(x_i - x_j).$$

Field Theory: $$H_{LL} = \int dx \, \partial\psi^\dagger(x) \partial\psi(x) + c \, \psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x)$$

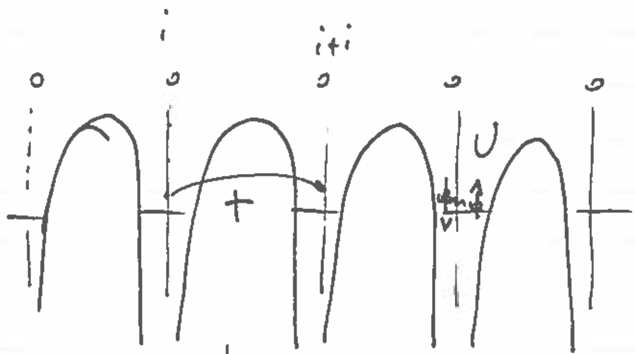
1963: Richardson solves BCS Hamiltonian (reduced)

1968: Lieb and Wu solve the 1d Hubbard model. (LOCAL).

$$H = -t\underbrace{\sum_{i=1}^{L}\left(a_{i\uparrow}^{\dagger} a_{i+1\uparrow} + a_{i\downarrow}^{\dagger} a_{i+1\downarrow}\right)}_{(*)} + h.c. + U\underbrace{\sum_{i=1}^{L} n_{i\uparrow} n_{i\downarrow}}_{(**)}$$

(*) t: Energy of fermions hopping through a lattice

(**) U: Coulomb repulsion on-site.



1972 Gaudin calculates $\langle \Psi_{XXX} | \Psi_{XXX} \rangle$ (Soulay report) :-)

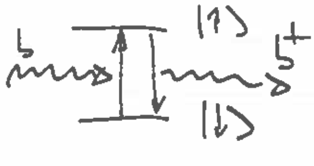
⋮

etc: other quantum integrable models.

(*) Jaynes-Cummings models (RG models) (Quantum optics).

~~(*) Sheetry~~

$$H = \hbar\omega\, b^{\dagger} b + \Delta S_z + g\left(S^{+} b + S\, b^{+}\right)$$



(*) Calogero-Sutherland-Moser models

$$H = \cancel{\frac{1}{2m}\sum_i p_i^2 + \frac{1}{2m}\omega^2 x_i^2}$$

$$H = \sum_i \left(\frac{1}{2m_i} p_i^2 + \frac{1}{2} m\omega_i^2 x_i^2\right) + \sum_{i\neq j} \frac{c}{(x_i - x_j)^2}$$

cfr: SHASTRY HALDANE models (long-range!)

$$H = \sum_{\lambda \neq j} \frac{z_\lambda z_j}{(z_\lambda - z_j)^2}\, \vec{S}_\lambda \cdot \vec{S}_j \qquad z_\lambda = e^{\frac{2\pi i \lambda}{N}}$$

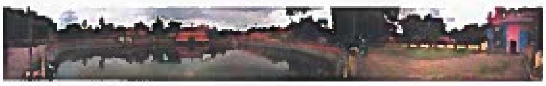
Ambalappuzha Sri Krishna Temple

Ambalappuzha Sri Krishna Temple (Malayalam: അമ്പലപ്പുഴ ശ്രീകൃഷ്ണ ക്ഷേത്രം) is a Hindu temple in Ambalappuzha, Alapuzha district of Kerala, in south India.

The Ambalappuzha Sri Krishna Temple is believed to have been built during 15th – 17th AD by the local ruler *Chembakasserry Pooradam Thirunal-Devanarayanan Thampuran.*

The idol at Ambalapuzha is likened to Parthasarthi with a whip in the right hand and a Shankhu (sacred conch) in the left. This temple is directly associated to the Guruvayoor Sree Krishna Temple. During the raids of Tipu Sultan in 1789, the idol of Sri Krishna from the Guruvayoor Temple was brought to the Ambalappuzha Temple for safe keeping.

The *payasam* served in the Ambalappuzha Temple is famous among Hindu devotees. This sweet pudding made of rice and milk has an interesting mythological legend behind it.It is believed that Guruvayoorappan reaches here daily at the time of Palpayasa Nedyam to have it.



panoramic view of Ambalappuzha Sri Krishna Temple and pool

1 Legend of the Ambalappuzha Paal Payasam

According to the legend, God Krishna once appeared in the form of a sage in the court of the king who ruled the region and challenged him for a game of chess (or *chaturanga*). The king being a chess enthusiast himself gladly accepted the invitation. The prize had to be decided before the game and the king asked the sage to choose his prize in case he won. The sage told the king that he had a very modest claim and being a man of few material needs, all he wished was a few grains of rice. The amount of rice itself shall be determined using the chess-board in the following manner. One grain of rice shall be placed in the first square, two grains in the second square, four in the third square, eight in the fourth square, sixteen in 5th square and so on. Every square will have double of its predecessor.

Upon hearing the demand, the king was unhappy since the sage requested only a few grains of rice instead of other riches from the kingdom which the king would have been happy to donate. He requested the sage to add other items to his prize but the sage declined.

So the game of chess started and needless to say the king lost the game. It was time to pay the sage his agreed-upon prize. As he started adding grains of rice to the chess board, the king soon realised the true nature of the sage's demands. By the 20th square, the number had reached one million grains of rice and by the 40th square, it became one million million. The royal granary soon ran out of grains of rice. The king realised that even if he provides all the rice in his kingdom and his adjacent kingdoms, he will never be able to fulfill the promised reward. The number of grains was increasing as a geometric progression and the total amount of rice required to fill a 64-squared chess board is ((2^64) - 1) which is equal to the number 18,446,744,073,709,551,615 translating to trillions of tons of rice.

Upon seeing the dilemma, the sage appeared to the king in his true-form, that of God Krishna. He told the King that he did not have to pay the debt immediately but could pay him over time. The king would serve *paal-payasam* (made of rice) in the temple freely to the pilgrims every day until the debt was paid off.

2 Festival

The Amabalapuzha Temple Festival was established during the fifteenth century A.D. At this time, a part of the Travancore, was ruled by the Chembakassery Devanarayana Dynasty. The rulers of this dynasty were highly religious and decided that an idol of Lord Krishna was to be brought to the Amabalapuzha Sree Krishna Swamy Temple from the Karinkulam temple. The celebration in commemoration of the bringing of this idol of Lord Krishna is the origin of the Amabalapuzha Temple Festival, also referred to as the Chambakulam Moolam water festival. This festival is conducted every year on the Moolam day of the Mithunam month of the Malayalam era.

The Aaraattu festival commences with the flag hoisting ceremony on the Atham star in Meenam (March–April). The important Aaraattu festival takes place on the Thiruvonam day of the same month.In this temple 'Pallipana' is performed by 'Velans' (sorcerers) once in twelve years. Human sacrifice was conducted in ancient times. However, cocks have now replaced humans on the sacrificial altar. Kalakkaththu Kunchan Nambiar(1705-1770) also