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On the ellipsoidal correction to the spherical Stokes solution of the gravimetric geoid

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Abstract. The solutions of four ellipsoidal approximations for the gravimetric geoid are reviewed: those of Molodenskii et al., Moritz, Martinec and Grafarend, and Fei and Sideris. The numerical results from synthetic tests indicate that Martinec and Grafarend's solution is the most accurate, while the other three solutions contain an approximation error which is characterized by the first-degree surface spherical harmonic. Furthermore, the first 20 degrees of the geopotential harmonic series contribute approximately 90% of the ellipsoidal correction. The determination of a geoid model from the generalized Stokes scheme can accurately account for the ellipsoidal effect to overcome the first-degree surface spherical harmonic error regardless of the solution used.

Keywords: Ellipsoidal correction – Geoid – Geodetic boundary value problem

1 Introduction

The geoid can be determined from the disturbing potential, denoted by T, that is defined as the difference between the Earth's gravity potential W and the normal potential U at a point (r, Ω) in space

$$T(r,\Omega) = W(r,\Omega) - U(r,\Omega) \tag{1}$$

where Ω is the solid angle and r is the geocentric radius of the point. If the disturbing potential T is known on the geoid, the geoid height N can be determined by Bruns' formula [Heiskanen and Moritz 1967, Eq. (2–144)]

$$N = \frac{T(r_G,\Omega)}{\gamma(r_E,\Omega)} \tag{2}$$

where r_G is the geocentric radius of a point on the geoid, γ is normal gravity on the ellipsoid and r_E is the geocentric radius of the corresponding point on the reference ellipsoid. Grafarend et al. (1999) give a discussion on Bruns' formula.

If the disturbing potential T is harmonic above the geoid, it can be determined by solving the third boundary value problem that can be prescribed as follows. Find the disturbing potential T by solving the Laplace differential equation

$$\Delta T(r,\Omega) = 0 \tag{3}$$

subject to the boundary condition (Heiskanen and Moritz 1967)

$$\left(\frac{\partial T(r,\Omega)}{\partial h} - \frac{T(r,\Omega)}{\gamma(r_E,\Omega)}\frac{\partial \gamma(r,\Omega)}{\partial h}\right)_G = -\Delta g(r_G,\Omega) + \epsilon$$
$$T(r,\Omega) \to 0 \quad \text{for} \quad r \to \infty \tag{4}$$

where h is the normal to the ellipsoidal surface E, and $\Delta g(r_G,\Omega)$ is the gravity anomaly on the geoid, defined as the difference between the normal gravity at a point Q on the ellipsoid and the actual gravity at the corresponding point P on the geoid. The point P is on the normal line to the ellipsoid from Q

$$\Delta g = g_P - \gamma_Q \tag{5}$$

In Eq. (4), the term ϵ accounts for the difference of the derivative of the gravity potential W with respect to the normal to the ellipsoid and the plumb line. It can be expressed as

$$\epsilon = \frac{\partial W}{\partial h} - \frac{\partial W}{\partial H} \tag{6}$$

where H is normal to the geoid. Within the precision adopted in this paper, this term can be neglected for a Somigliana–Pizzetti reference field (Cruz 1986; Moritz 1990).

Stokes' integral (Stokes 1849) represents a spherical approximation to the solution of the disturbing potential T on the geoid, introducing a relative geoid

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error of the order of the flattening ($f \doteq 3 \times 10^{-3}$) which causes an absolute geoid error of the order of at most 0.2 m globally. In an effort to improve the accuracy of the geoid result from Stokes' integral, several derivations have been attempted based on ellipsoidal approximation (see e.g. Sagrebin 1956; Molodenskii et al. 1962; Bjerhammar 1966; Koch 1968; Moritz 1974; Rapp 1981; Zhu 1981; Martinec and Grafarend 1997; Fei and Sideris 2000, 2001; Brovar et al. 2001; Heck and Seitz submitted; Sjöberg submitted). These derivations split the ellipsoidal solution into two parts: the spherical approximation solution and the ellipsoidal correction. The ellipsoidal correction may reduce the relative error of Stokes' spherical solution to the order of the square of the flattening, which does not exceed 2 mm in absolute value (Martinec and Grafarend 1997).

One question naturally arises: To what extent are these solutions equivalent and, if not, which one is best? In order to answer this question, only four solutions, namely those derived by Molodenskii et al. (1962), Moritz (1974), Martinec and Grafarend (1997) and Fei and Sideris (2000, 2001), are considered due to time constraints, leaving others for future studies. The choice of the four methods above is somewhat arbitrary, without a sound justification. Interested readers are encouraged to study other methods that provide profound insights into the geodetic boundary value problem (GBVP) from different aspects. First, the four solutions are theoretically reviewed. Second, a synthetic approach is adopted to compare them numerically. Subsequently, numerical results for the ellipsoidal correction over North America are presented. Finally, a procedure for the practical evaluation of the ellipsoidal correction is proposed.

2 Review

2.1 Molodenskii et al. solution

Molodenskii et al. (1962, pp 53–59) start with Green's third identity on an ellipsoidal surface E

$$-2\pi T(r_E, \phi, \lambda) = \int_E \left(\frac{1}{\ell} \frac{\partial T(r, \phi, \lambda)}{\partial h} - T(r, \phi, \lambda) \frac{\partial}{\partial h} \frac{1}{\ell} \right) \mathrm{d}E \tag{7}$$

subject to the boundary condition

$$\left(\frac{\partial T(r, \phi, \lambda)}{\partial h} - \frac{T(r, \phi, \lambda)}{\gamma(r, \phi, \lambda)} \frac{\partial \gamma(r, \phi, \lambda)}{\partial h} \right)_E = -\Delta g_h(r_E, \phi, \lambda) \tag{8}$$

where ϕ and λ are the geocentric latitude and longitude, respectively, and ℓ is the distance between the computation point on the surface E and the integration element $\mathrm{d}E$ on this surface.

The solution for the disturbing potential with a relative error of the order of e'^4 can be expressed as

$$T(r_E, \phi, \lambda) = T_0(\phi, \lambda) + e'^2 \Bigg[\frac{1}{4} T_0(\phi, \lambda) \sin^2 \phi + \chi(\phi, \lambda) + \frac{3}{8\pi} \int_{\Omega'} \left(\chi(\phi', \lambda') - \frac{1}{2} a \Delta g_h(r_E, \phi', \lambda') \times \sin^2 \phi' \right) S(\psi) \mathrm{d}\Omega' \Bigg] \tag{9}$$

where

$$T_0(\phi, \lambda) = \frac{a}{4\pi} \int_{\Omega'} \Delta g_h(r_E, \phi', \lambda') S(\psi) \mathrm{d}\Omega' \tag{10}$$

$$\chi(\phi, \lambda) = \frac{1}{4\pi} \int_{\Omega'} T_0(\phi', \lambda') \left(1 - \frac{5}{2} \sin^2 \phi' + \frac{2m}{e'^2} \right) \csc \frac{\psi}{2} \mathrm{d}\Omega' - \frac{1}{4\pi} \int_{\Omega'} \frac{T_0(\phi', \lambda')(\sin \phi' - \sin \phi)^2}{8 \sin^3 \frac{\psi}{2}} \mathrm{d}\Omega' \tag{11}$$

$S(\psi)$ is the spherical Stokes function, e' $[e'^2 = (a^2 - b^2)/b^2]$ is the second numerical eccentricity of the reference ellipsoid, $\mathrm{d}\Omega' = \cos \phi' \, \mathrm{d}\phi' \, \mathrm{d}\lambda'$, a is the semi-major axis of the meridian ellipse, and ψ is the angular distance between the geocentric directions of the computation point and the integration element $\mathrm{d}\Omega'$.

2.2 Moritz solution

Moritz (1974) defines a transformation

$$T_0(\theta, \lambda) = T(r_E, \theta, \lambda) \tag{12}$$

where $T_0(\theta, \lambda)$ is a surface spherical harmonic representation of $T(r_E, \theta, \lambda)$ and θ is the geodetic co-latitude (i.e. $90 - \Phi$; Φ is the geodetic latitude). $T_0(\theta, \lambda)$ can be determined through mapping of $\Delta g_h(r_E, \theta, \lambda)$ onto $\Delta g_0(\theta, \lambda)$ that generates $T_0(\theta, \lambda)$ on the sphere of radius R, followed by the spherical Stokes integration over $\Delta g_0(\theta, \lambda)$. The solution for the geoid height is expressed as

$$T_0(\theta, \lambda) = \frac{R}{4\pi} \int_{\sigma'} S(\psi) \Delta g_0(\theta, \lambda) \mathrm{d}\sigma' \tag{13}$$

where

$$\Delta g_0(\theta, \lambda) = \Delta g_h(r_E, \theta, \lambda) - e^2 \Delta g^1(\theta, \lambda) \tag{14}$$

$\mathrm{d}\sigma' = \sin \theta' \, \mathrm{d}\theta' \, \mathrm{d}\lambda'$, $e\,[e^2 = (a^2 - b^2)/a^2]$ is the first numerical eccentricity of the reference ellipsoid, and R is the mean radius of the Earth. Note that the term N^1 defined by Moritz (1974) is not present here, because in Eq. (2) the normal gravity at the computation point is used rather than the mean normal gravity on the ellipsoid.

Stokes' integral over Δg^1 in Eq. (13) can be directly evaluated by the spherical harmonic expansion

$$\frac{R}{4\pi\gamma}\int_{\sigma}\Delta g^1(\theta',\lambda')S(\psi)\mathrm{d}\sigma$$
$$= R\sum_{n=2}^{\infty}\frac{1}{n-1}\sum_{m=0}^{n}[G_{nm}\bar{R}_{nm}(\theta,\lambda)+H_{nm}\bar{S}_{nm}(\theta,\lambda)] \tag{15}$$

with

$$\begin{aligned} G_{nm} &= \kappa_{nm}u_{nm}c_{n-2,m}+\lambda_{nm}c_{nm}+\mu_{nm}v_{nm}c_{n+2,m} \\ H_{nm} &= \kappa_{nm}u_{nm}d_{n-2,m}+\lambda_{nm}d_{nm}+\mu_{nm}v_{nm}d_{n+2,m} \end{aligned} \tag{16}$$

and

$$\begin{aligned} \kappa_{nm} &= -\frac{3(n-3)(n-m-1)(n-m)}{2(2n-3)(2n-1)} \\ \lambda_{nm} &= \frac{n^3-3m^2n-9n^2-6m^2-10n+9}{3(2n+3)(2n-1)} \\ \mu_{nm} &= -\frac{(3n+5)(n+m+2)(n+m+1)}{2(2n+5)(2n+3)} \\ u_{nm} &= \left[\frac{(2n-3)(n+|m|-1)(n+|m|)}{(2n+1)(n-|m|-1)(n-|m|)}\right]^{\frac{1}{2}} \\ v_{nm} &= \left[\frac{(2n+5)(n-|m|+1)(n-|m|+2)}{(2n+1)(n+|m|+1)(n+|m|+2)}\right]^{\frac{1}{2}} \end{aligned} \tag{17}$$

c_{nm} and d_{nm} are the fully normalized ellipsoidal geopotential coefficients. In Moritz' derivation, the coefficients c_{nm} and d_{nm} are not fully normalized, so his expressions for coefficients G_{nm} and H_{nm} differ from the expressions given here. $\bar{R}_{nm}$ and $\bar{S}_{nm}$ are the fully normalized spherical harmonics. In the actual computation, the spherical geopotential coefficients can be used to approximate the ellipsoidal geopotential coefficients c_{nm} and d_{nm} to evaluate the ellipsoidal correction term in Eq. (14) accurate to $O(e^4)$, since the ellipsoidal corrections to the spherical geopotential coefficients are of order $O(e^2)$.

2.3 Martinec and Grafarend solution

Martinec and Grafarend (1997) formulate the ellipsoidal correction based on the ellipsoidal harmonic solution for the Laplace equation [Heiskanen and Moritz 1967, Eq. (1-111b)]

$$T(u,\vartheta,\lambda)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{Q_{nm}\left(i\frac{u}{\mathcal{E}}\right)}{Q_{nm}\left(i\frac{b}{\mathcal{E}}\right)}[a_{nm}P_{nm}(\cos\vartheta)\cos m\lambda + b_{nm}P_{nm}(\cos\vartheta)\sin m\lambda] \tag{18}$$

where ϑ is the reduced co-latitude, the triplet (u,ϑ,λ) forms the ellipsoidal coordinate system, P_{nm} is Legendre's function of the first kind, Q_{nm} is Legendre's function of the second kind, b is the semi-minor axis of the meridian ellipse, $\mathcal{E}$ ($\mathcal{E}^2=a^2-b^2$) is the linear eccentricity, and a_{nm} and b_{nm} are the geopotential coefficients of the ellipsoidal harmonics. The boundary values are defined as

$$\left(\frac{\partial T(u,\vartheta,\lambda)}{\partial u}+\frac{2}{u}T(u,\vartheta,\lambda)\right)_E = -\Delta g_u(b,\vartheta,\lambda) \tag{19}$$

Note that Δg_u in Eq. (19) differs from the gravity anomaly Δg_h in Eq. (8). Grafarend (pers. commun. 2002) reminds us that the exact formula is in Grafarend et al. (1999, Table 4, Model 3: SOM-PI). The solution for the ellipsoidal geodetic boundary value problem can be written as [Martinec and Grafarend 1997, Eq. (12)]

$$T(b,\vartheta,\lambda)=\frac{b}{4\pi}\int_{\Gamma'}\Delta g_u(b,\vartheta',\lambda')S^E(\vartheta,\lambda;\vartheta',\lambda')\mathrm{d}\Gamma' \tag{20}$$

where $\mathrm{d}\Gamma'=\sin\vartheta'\,\mathrm{d}\vartheta'\,\mathrm{d}\lambda'$, S^E denotes the Stokes function for the ellipsoidal approximation in the form of a spherical harmonic product series. By neglecting the terms of powers greater than e^2, S^E is approximated by the spherical Stokes function corrected by an ellipsoidal correction term

$$S^E(\vartheta,\lambda;\vartheta',\lambda')\doteq S(\psi)-e^2S^{\mathrm{ell}}(\vartheta,\lambda;\vartheta',\lambda') \tag{21}$$

S^{ell} [Martinec and Grafarend 1997, Eq. (50)] is the kernel for the ellipsoidal correction. Note that the boundary values Δg_u in Martinec and Grafarend's formula are different from the gravity anomaly Δg_h. The following equation was derived to transform Δg_h to Δg_u:

$$\Delta g_u(b,\vartheta,\lambda)=\Delta g_h(r_E,\vartheta,\lambda)+\epsilon_T+\epsilon_u+O(e^4) \tag{22}$$

where

$$\epsilon_T=\frac{1}{2}e^2\sin^2\vartheta\frac{\partial T}{\partial r}\bigg|_E \tag{23}$$

$$\epsilon_u=-e^2\cos 2\vartheta\frac{T}{b}\bigg|_E \tag{24}$$

The derivations of these two terms are shown in Appendix A. Heck (1991) gave similar expressions to these two terms. In order to evaluate these two correction terms, we need to know the disturbing potential T that is to be determined. Theoretically, they can be evaluated iteratively, i.e. the spherical approximation of T is used in the first iteration, then the improved T is accordingly used in subsequent iterations. In practice, the spherical approximation of T is good enough to give accurate estimation of these two terms with an error of order $O(e^4)$.

2.4 Fei and Sideris solution

Fei and Sideris' (2000, 2001) derivation follows a similar approach to Molodenskii's, but uses the Pizzetti kernel function $S(r,\psi,r')$ [Moritz 1980, Eq. (44–11)] in place of the Newtonian kernel $\ell^{-1}(r,\psi,r')$ in Green's second identity. The solution for the geoid height is written as

$$N(\phi,\lambda)=N_0(\phi,\lambda)+e^2N_1(\phi,\lambda) \tag{25}$$

where

$$N_0(\phi,\lambda) = \frac{R}{4\pi\gamma(r_E,\phi,\lambda)} \int_{\Omega'} S(\psi)\Delta g_h(r_E,\phi',\lambda')\mathrm{d}\Omega' \quad (26)$$

$$N_1(\phi,\lambda) = N_{11}(\phi,\lambda) + \frac{1}{4\pi}\int_{\Omega'} N_0(\phi',\lambda')f_0(\phi,\lambda;\phi',\lambda')\mathrm{d}\Omega' \quad (27)$$

where N_{11} is defined in Eq. (2.44) and f_0 is defined in Eq. (2.31) of Fei and Sideris (2001). The formulae of Fei and Sideris (2000) had mistakes (Huang et al. 2000). After we pointed out the errors, the formulae were re-derived and corrected (see Fei and Sideris 2001).

2.5 Theoretical contrast

From the above formulae, two different definitions of the boundary condition have been seen, even though the same quantity T is sought. Martinec and Grafarend (1997) prescribe a purely mathematical boundary condition [Eq. (19)] on the ellipsoidal surface E without a direct physical meaning, while the other methods invariably define the boundary condition by Eq. (8), which approximately represents the gravity anomaly on the geoid.

In the derivation of the ellipsoidal correction terms, the disturbing potential (or the geoid height) is mathematically expressed as a series expansion with respect to a small parameter ϵ. This parameter can be either the square of the first numerical eccentricity e^2 or the square of the second numerical eccentricity e'^2.

$$T = T_0 + \epsilon T^{(1)} + O(\epsilon^2) \quad (28)$$

in which the first term represents the spherical approximation to the disturbing potential T, and the second term accounts for the 'ellipsoidal correction' that leads to a solution of T accurate to $O(e^4)$. When 1-cm geoid accuracy is sought, terms with squares and higher powers of ϵ can be safely neglected so that Eq. (28) reduces to

$$T \doteq T_0 + \epsilon T^{(1)} \quad (29)$$

Even though all derivations conform to Eq. (29), their mathematical approaches and final expressions are different. The ellipsoidal correction is applied only to the boundary values in Moritz' approach (Moritz 1974), while it is applied to both the kernel and the boundary values in Martinec and Grafarend's approach (Martinec and Grafarend 1997). In the cases of the Molodenskii et al. (1962) and Fei and Sideris (2000, 2001) methods, the ellipsoidal correction terms are expressed as integral functions of the spherical approximation of the disturbing potential T and the boundary values. Both methods are based on Green's identity.

Three coordinate systems were used in these derivations. Molodenskii et al. (1962) and Fei and Sideris (2000, 2001) choose the geocentric latitude ϕ and longitude λ, while Moritz refers to the geodetic co-latitude θ and longitude λ. Martinec and Grafarend adopt the elliptical coordinate system (u, ϑ, λ) for convenience in their derivation.

As far as the mathematical complexity is concerned, Moritz' method provides the simplest solution while that of Martinec and Grafarend represents the most complex one.

It should be noted that the ellipsoidal correction is a relative quantity dependent on how the spherical solution is defined. By observing the four methods above, it can be found that they formulate the spherical solution differently; therefore, the ellipsoidal corrections are not directly comparable. We can only compare the complete geoid solutions (i.e. the spherical solution plus the ellipsoidal correction) arising from these methods in order to be able to verify their equivalence.

3 Numerical comparisons

3.1 Synthetic data

One experimental approach regarding the evaluation of accuracy for the ellipsoidal solution is the synthetic test, for which a synthetic gravity model is required to allow us to generate synthetic input and output. In the determination of the geoid on the ellipsoidal boundary, the input is the gravity anomaly on the ellipsoidal surface, while the output is the geoid height (or the height anomaly). If a solution is exact, it should give exactly the same result as the synthetic geoid height when the synthetic gravity anomaly is used for evaluation of the solution. Because of the presence of numerical errors such as round-off errors, sampling errors and approximation errors, the geoid result from the computation may conform to the synthetic geoid only within an acceptable error interval.

The synthetic geoid (or the height anomaly) can be generated on the ellipsoid by

$$N(r_E,\Omega) = \frac{GM}{a\gamma_E}\sum_{n=0}^{\infty}\left(\frac{a}{r_E}\right)^{n+1}\sum_{m=-n}^{n}\bar{C}_{nm}\bar{Y}_{nm}(\Omega) \quad (30)$$

where Ω is the full solid angle indicating the pair (ϕ, λ), GM is the geocentric gravitational constant, $\bar{C}_{nm}$ are the fully normalized spherical geopotential coefficients that have been reduced by the even zonal harmonic coefficients of the (Somiliana–Pizzetti) reference field. In addition

$$\bar{Y}_{nm}(\Omega) = \bar{P}_{n|m|}(\sin\phi)\begin{cases}\cos m\lambda & \text{if } m \geq 0\\ \sin|m|\lambda & \text{if } m < 0\end{cases} \quad (31)$$

where $\bar{P}_{n|m|}$ are the fully normalized associated Legendre functions of the first kind; n and m are the degree and order of harmonic series. It is worth pointing out that Jekeli (1981) developed transformation formulae between the spherical and the ellipsoidal coefficients (Gleason 1988). The ellipsoidal coefficients can therefore be used to produce the synthetic geoid on the ellipsoid in terms of ellipsoidal harmonic series equivalently.

The synthetic gravity anomaly can be evaluated on the ellipsoid by

$$\Delta g_h(r_E,\Omega) = \Delta g_r(r_E,\Omega) - \epsilon_h(r_E,\Omega) - \epsilon_\gamma(r_E,\Omega) + O(e^4) \tag{32}$$

where

$$\Delta g_h(r_E,\Omega) = \left(-\frac{\partial T(r,\Omega)}{\partial h} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T(r,\Omega) \right)_{r=r_E} \tag{33}$$

$$\begin{aligned}\Delta g_r(r_E,\Omega) &= \left(-\frac{\partial T(r,\Omega)}{\partial r} - \frac{2}{r} T(r,\Omega) \right)_{r=r_E} \\ &= \frac{GM}{a^2}\sum_{n=0}^{\infty}(n-1)\left(\frac{a}{r_E}\right)^{n+2}\sum_{m=-n}^{n}\bar{C}_{nm}\bar{Y}_{nm}(\Omega)\end{aligned} \tag{34}$$

The two ellipsoidal correction terms in Eq. (32) were derived by Jekeli (1981), and simplified by Cruz (1986) to give

$$\epsilon_h(r,\Omega) = e^2 \sin\phi\cos\phi \frac{\partial T(r,\Omega)}{r\partial\phi} \tag{35}$$

and

$$\epsilon_\gamma(r,\Omega) = e^2(2 - 3\sin^2\phi)\frac{T(r,\Omega)}{r} \tag{36}$$

Equations (35) and (36) are valid for any point above the ellipsoid. Here, the synthetic gravity anomaly on the ellipsoid is needed, thus the two ellipsoidal terms must be evaluated on the ellipsoid as well. According to Moritz (1980, pp 39–46)

$$\begin{aligned}\sin\phi\cos\phi\frac{\partial Y_{nm}(\Omega)}{\partial\phi} &= f_{nm}Y_{n+2,m}(\Omega) + g_{nm}Y_{nm}(\Omega) \\ &\quad + h_{nm}Y_{n-2,m}(\Omega)\end{aligned} \tag{37}$$

where Y_{nm} are spherical harmonics

$$f_{nm} = -\frac{n(n-|m|+1)(n-|m|+2)}{(2n+1)(2n+3)} \tag{38}$$

$$g_{nm} = \frac{n^2 - 3m^2 + n}{(2n+3)(2n-1)} \tag{39}$$

$$h_{nm} = \frac{(n+1)(n+|m|)(n+|m|-1)}{(2n+1)(2n-1)} \tag{40}$$

Considering Cruz [1986, Eq. (5.10)], Eq. (35) can be written on the ellipsoid as

$$\epsilon_h(r_E,\Omega) = e^2\frac{GM}{a^2}\sum_{n=0}^{\infty}\left(\frac{a}{r_E}\right)^{n+2}\sum_{m=-n}^{n}\bar{C}_{nm}F_{nm}(\Omega) \tag{41}$$

where

$$\begin{aligned}F_{nm}(\Omega) &= f_{nm}u_{n+2,m}\bar{Y}_{n+2,m}(\Omega) + g_{nm}\bar{Y}_{nm}(\Omega) \\ &\quad + h_{nm}v_{n-2,m}\bar{Y}_{n-2,m}(\Omega)\end{aligned} \tag{42}$$

Equation (36) can be directly written on the ellipsoid as

$$\begin{aligned}\epsilon_\gamma(r_E,\Omega) &= e^2(2-3\sin^2\phi)\frac{GM}{a^2}\sum_{n=0}^{\infty}\left(\frac{a}{r_E}\right)^{n+2} \\ &\quad \times \sum_{m=-n}^{n}\bar{C}_{nm}\bar{Y}_{nm}(\Omega)\end{aligned} \tag{43}$$

The term ϵ_h is called the ellipsoidal correction to the gravity disturbance, and ϵ_γ is called the ellipsoidal correction to the spherical approximation by Vaníček et al. (1999).

A global geopotential model can be used to derive the synthetic data sets (see e.g. Novák et al. 2000). Here, EGM96 (Lemoine et al. 1998) was adopted in generating the synthetic data sets. The synthetic data were produced on a grid of $30' \times 30'$ covering the entire globe from the first 20 degrees of EGM96. These data allow us to perform global integration in a reasonable time. A spherical case test showed that when the $30' \times 30'$ grid and the first 20 degrees of EGM96 were used, the numerical errors on the derived geoid ranged from about -1 to 1 cm globally. In the test, the degree-banded Stokes kernel was used to eliminate the aliasing error (-3 to 3 cm) in the numerical Stokes integration (Huang et al. 2001). This numerical error range represents the integration noise level. Figure 1 illustrates the procedure of the synthetic test.

3.2 Numerical results and discussion

Figures 2 and 3 show the ellipsoidal corrections and the differences between the synthetic geoid and the one computed from each method, respectively. It can be found that the ellipsoidal corrections are highly correlated to the global geoid undulations even though they show significantly different detailed features from one another. Due to the differences in the mathematical definitions for the 'spherical Stokes' term and the

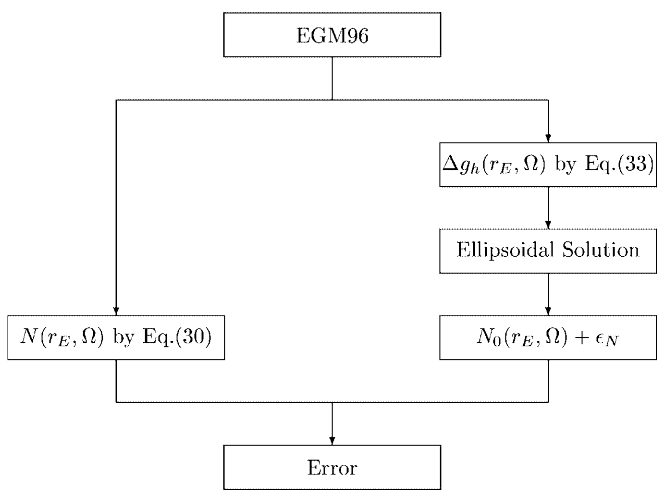

Fig. 1. The synthetic test procedure

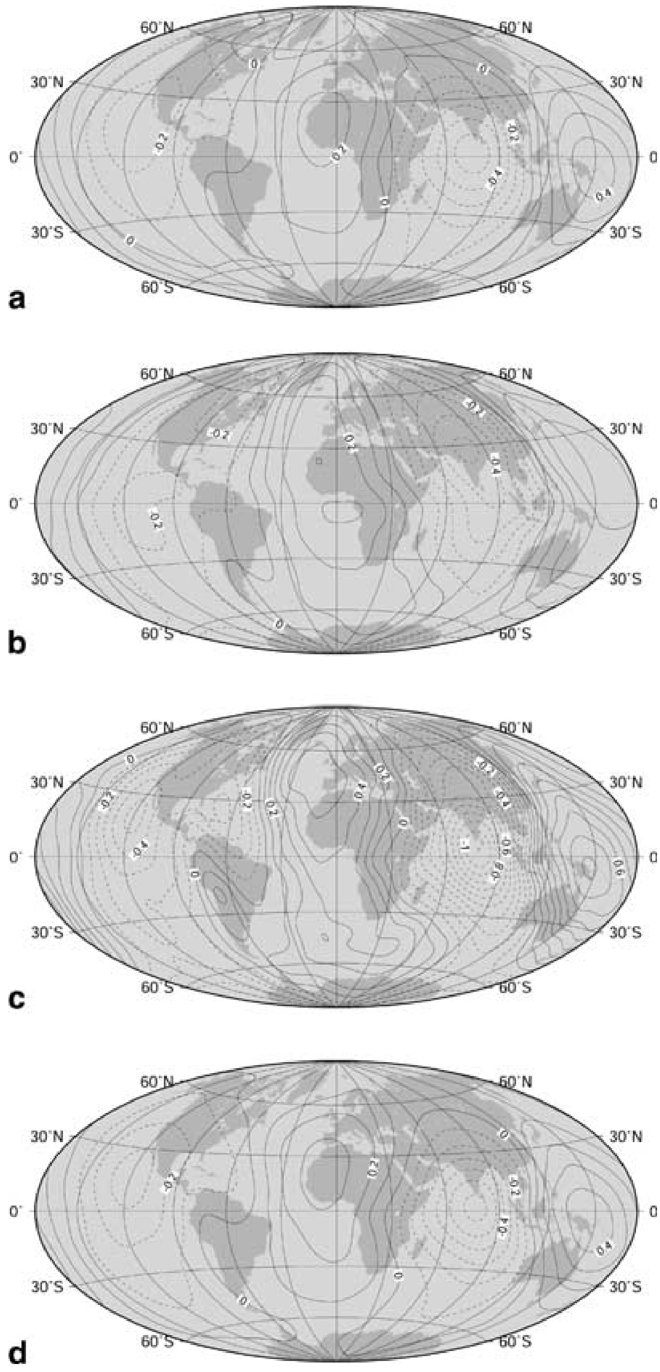

Fig. 2a–d. The ellipsoidal corrections from different solutions. Contour interval: 0.1 m (Hammer equal-area projection). **a** Molodenskii et al.; **b** Moritz; **c** Martinec and Grafarend; **d** Fei and Sideris

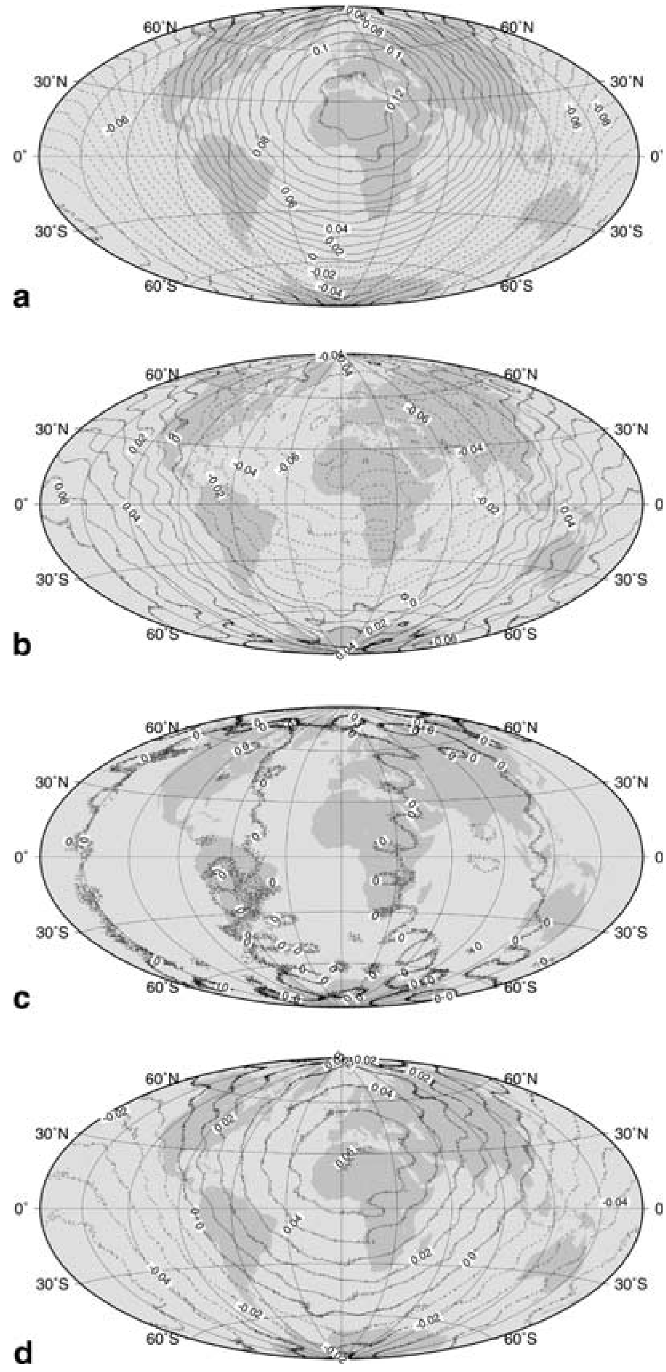

Fig. 3a–d. The differences between the synthetic geoid and the geoid computed from the synthetic gravity anomaly for each method. Contour interval: 0.01 m (Hammer equal-area projection). **a** Molodenskii et al.; **b** Moritz; **c** Martinec and Grafarend; **d** Fei and Sideris

ellipsoidal correction between the four methods, the correction magnitude varies significantly from method to method, which is to be expected. Table 1 summarizes ellipsoidal correction results from the four methods.

Comparing the differences in Table 2, it can be found that Martinec and Grafarend's solution demonstrates the best conformity to the synthetic geoid with an acceptable level of numerical errors, while the other three solutions contain errors beyond the integration noise range (−1, 1 cm).

By observing Fig. 3 and Table 2, it can be found that, excluding Martinec and Grafarend's method, the other three methods give ellipsoidal solutions which differ from the synthetic geoid by the first-degree components of the *surface* spherical harmonics based on their spatial patterns. In other words, if the ellipsoidal solution of $T(r_E, \Omega)$ for each solution is transformed into the surface spherical harmonics as follows:

Table 1. Numerical statistics of the ellipsoidal corrections for the four methods. The input synthetic gravity anomaly was generated from the first 20 degrees of EGM96. Unit: m

Method	Min	Max	Mean	Standard deviation	RMS
Molodenskii	−0.555	0.461	0.008	0.146	0.146
Moritz	−0.525	0.384	−0.011	0.177	0.177
Martinec and Grafarend	−1.101	0.818	−0.005	0.298	0.298
Fei and Sideris	−0.635	0.486	0.007	0.159	0.159

Table 2. Numerical statistics of the differences between the synthetic geoid and the computed geoid from each method. The synthetic geoid and gravity anomaly were generated from the first 20 degrees of EGM96. Unit: m

Method	Min	Max	Mean	Standard deviation	RMS
Molodenskii	−0.132	0.128	0.000	0.071	0.071
Moritz	−0.082	0.079	0.000	0.042	0.042
Martinec and Grafarend	−0.013	0.010	0.000	0.004	0.004
Fei and Sideris	−0.057	0.062	0.000	0.031	0.031

$$T(r_E,\Omega) = T^s(\Omega) = \sum_{n=0}^{\infty} T_n(\Omega) \tag{44}$$

the four solutions give different T_1, while only the T_1 term for Martinec and Grafarend's solution conforms to the synthetic field. The T_1 can be explicitly expressed as (see e.g. Heiskanen and Moritz 1967)

$$T_1(\Omega) = y_1 \sin\theta\cos\lambda + y_2 \sin\theta\sin\lambda + y_3\cos\theta \tag{45}$$

which has a global zero mean and a dipole pattern similar to ones shown in Fig. 3a, b, d. Note that the T_1 term in Eq. (44) does not have a direct physical meaning, and y_1, y_2 and y_3 in Eq. (45) should not be interpreted as the rectangular coordinate differences between the Earth's centre of gravity and the centre of the reference ellipsoid. The solid disturbing potential $T(r,\Omega)$ on any star-shaped surface can be represented by surface spherical harmonics in which the first-degree term is usually not equal to zero, even though the centre of the Earth's gravity coincides with the centre of the reference ellipsoid. A more through discussion from a theoretical point of view is given in Heck and Seitz (submitted).

In order to verify the 'observed' first-degree differences, or more precisely errors, the first-degree surface spherical harmonic term is excluded from the synthetic data. In Eq. (30), by taking the maximum degree as 20, it can be written as

$$\left(\frac{a}{r_E}\right)^{n+1} \doteq 1 + e^2\frac{n+1}{2}\sin^2\phi \tag{46}$$

Then, Eq. (30) reduces to

$$N(r_E,\Omega) = \frac{GM}{a\gamma_E}\sum_{n=0}^{20}\sum_{m=-n}^{n}\bar{C}_{nm}\bar{Y}_{nm}(\Omega) + \epsilon_N \tag{47}$$

where

$$\epsilon_N = e^2\frac{GM}{a\gamma_E}\sum_{n=0}^{20}\frac{n+1}{2}\sum_{m=-n}^{n}\bar{C}_{nm}\bar{Y}_{nm}(\Omega)\sin^2\phi \tag{48}$$

According to the relation [Moritz 1980, Eq. (39–76)]

$$Y_{nm}\sin^2\phi = \alpha_{nm}Y_{n+2,m} + \beta_{nm}Y_{nm} + \gamma_{nm}Y_{n-2,m} \tag{49}$$

and considering Cruz [1986, Eq. (5.10)], Eq. (48) can be written as

$$\epsilon_N = e^2\frac{GM}{a\gamma_E}\sum_{n=0}^{20}\sum_{m=-n}^{n}\Pi_{nm}\bar{Y}_{nm}(\Omega) \tag{50}$$

where

$$\Pi_{nm} = \frac{n-1}{2}\alpha_{n-2,m}u_{nm}\bar{C}_{n-2,m} + \frac{n+1}{2}\beta_{nm}\bar{C}_{nm} + \frac{n+3}{2}\gamma_{n+2,m}v_{nm}\bar{C}_{n+2,m} \tag{51}$$

Equations (47) and (50) represent the surface spherical harmonic expression of $N(r_E,\Omega)$ for the first 20 degrees. If we let $\bar{C}_{nm} = 0$ for $n < 4$ and $m < 4$ in Eq. (30), the first-degree term of the surface spherical harmonic expansion for $N(r_E,\Omega)$ is excluded. The same method can be used to exclude the first-degree surface spherical harmonics for $\Delta g(r_E,\Omega)$.

Table 3 shows the test results from degrees 4 to 20. They suggest that the four methods give the same result if the first-degree surface spherical harmonic components are removed from the synthetic data. Therefore, it can be concluded that only Martinec and Grafarend's method gives a complete ellipsoidal approximation solution, while the other three solutions contain approximation and omission errors that manifest in the first-degree term from their derivations.

Table 4 shows that the contribution of degree 21 to degree 360 of EGM96 to the geoid only accounts for about 10% of the total ellipsoidal correction based on the EGM96 model globally.

4 The method for ellipsoidal correction

4.1 The correctional method

The above numerical analysis has shown that the dominant components of the ellipsoidal correction arise

Table 3. Numerical statistics of the differences between the synthetic geoid and the computed geoid from each method. The synthetic geoid and gravity anomaly were generated by using the coefficients of degrees 4–20 of EGM96. Unit: m

Method	Min	Max	Mean	Standard deviation	RMS
Molodenskii	−0.004	0.007	0.000	0.001	0.001
Moritz	−0.005	0.006	0.000	0.001	0.001
Martinec and Grafarend	−0.003	0.007	0.000	0.001	0.001
Fei and Sideris	−0.004	0.007	0.000	0.001	0.001

from the low degrees of the geopotential harmonics. The terrestrial gravity, which suffers from irregular distribution, datum biases and height errors, will introduce considerable biases into the low-degree components when being used to evaluate the ellipsoidal correction. The best source available for the low-degree components is the satellite-only solution, because of its superior sensitivity to the long-wavelength components of the geopotential. The generalized Stokes scheme (Vaníček and Sjöberg 1991) has been suggested for the determination of a precise geoid in order to fully make use of information from both the terrestrial data and the satellite solution. Proper use of the generalized Stokes scheme can reduce the ellipsoidal effect on the geoid to a few centimetres (Vaníček and Sjöberg 1991; Featherstone et al. 2001). A straightforward procedure is as follows.

Table 4. Numerical statistics of the ellipsoidal correction for Martinec and Grafarend's method using EGM96. Unit: m

Data	Min	Max	Mean	Standard deviation	RMS
Degrees 21 to 360	−0.124	0.109	0.000	0.011	0.011
Degrees 2 to 360	−1.112	0.859	0.005	0.298	0.298

Table 5. Numerical statistics of the ellipsoidal correction over North America by using Martinec and Grafarend's method. Unit: m

Data	Min	Max	Mean	Standard deviation	RMS
Degrees 2 to 20	−0.532	0.578	−0.021	0.285	0.286
Degrees 21 to ∞	−0.077	0.083	0.001	0.015	0.015
Total	−0.528	0.603	−0.020	0.286	0.286

1. Remove the low degrees of the gravity anomalies evaluated on the ellipsoid using the satellite-only geopotential model.
2. Estimate the residual geoid and the residual ellipsoidal correction from the residual gravity anomalies via the generalized Stokes formula, or modification thereof and an ellipsoidal correction formula.
3. Restore the low-degree terms of the geoid evaluated on the ellipsoid from the satellite-only geopotential model.

With the generalized Stokes scheme, all four solutions will produce the same geoid results because the way in which the low-degree components of the geoid are pre-evaluated independent of each method avoids the 'first-degree' error. In the implementation of the ellipsoidal correction computation, the coordinate system must be completely in accordance with the coordinate system defined with the corresponding method. For example, if Molodenskii's method is used, the gravity anomaly must be referenced to the geocentric latitude.

4.2 The ellipsoidal correction over North America

Martinec and Grafarend's solution was chosen to evaluate the ellipsoidal correction over North America. This area is delimited by 20°–84° latitude and 170°W–10°W longitude. The $2' \times 2'$ Helmert gravity anomaly and EGM96 were used as input data.

Table 5 shows the statistical results of the ellipsoidal correction over the area. Figures 4–6 show the spatial distribution of the ellipsoidal correction. Again, it can be seen that the first 20 degrees of the geopotential model account for about 90% of the total ellipsoidal correction. Furthermore, the ellipsoidal correction influences mainly the long-wavelength components of the geoid,

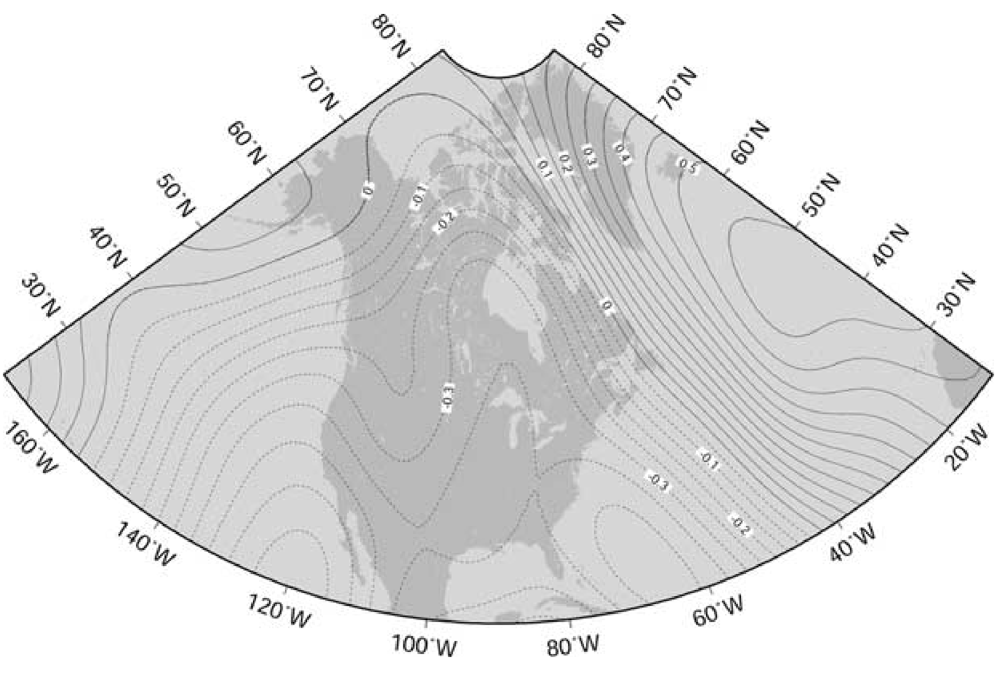

Fig. 4. The ellipsoidal correction to the spherical Stokes solution geoid over North America from degree 2 to degree 20 of EGM96 by using Martinec and Grafarend's method. Contour interval: 0.05 m (Lambert conic conformal projection)

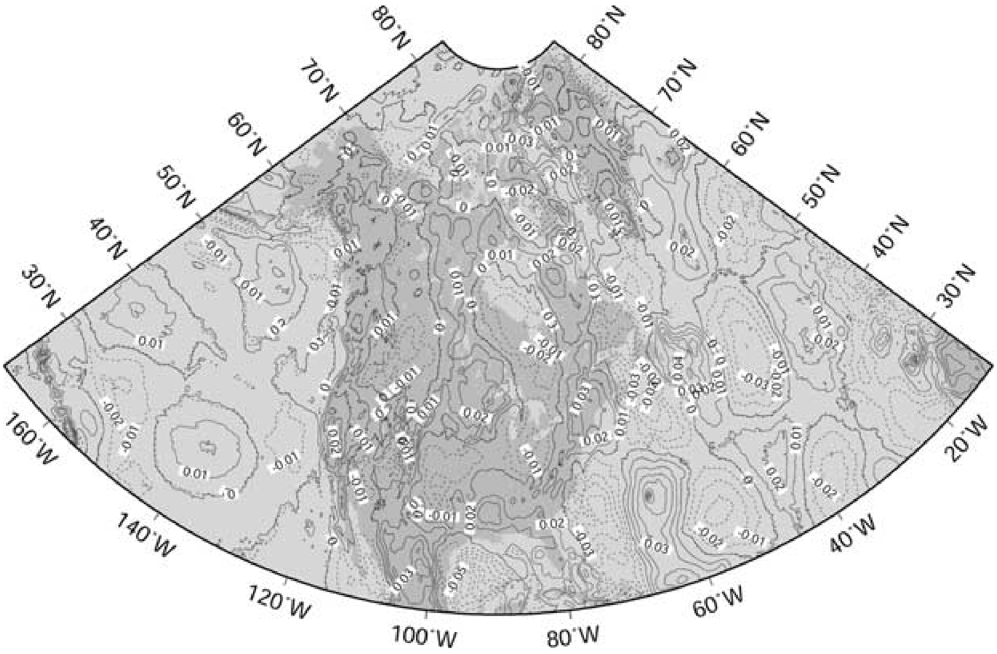

Fig. 5. The ellipsoidal correction to the spherical Stokes geoid over North America from the residual gravity anomaly above degree 20 of EGM96 by using Martinec and Grafarend's method. Contour interval: 0.01 m (Lambert conic conformal projection)

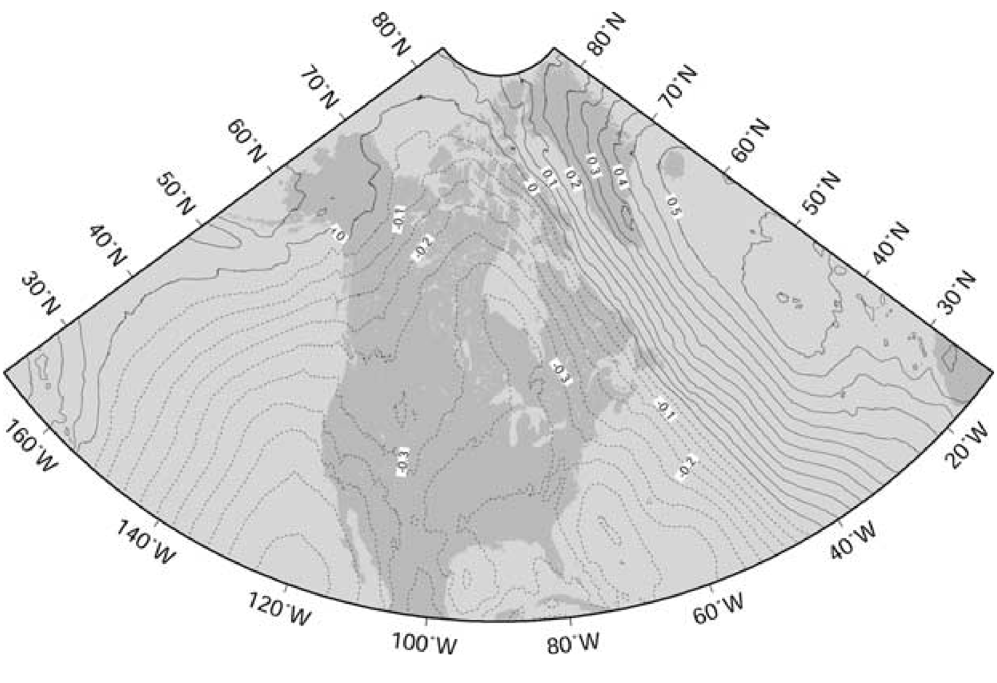

Fig. 6. The total ellipsoidal correction to the spherical Stokes geoid over North America by using Martinec and Grafarend's method. Contour interval: 0.05 m (Lambert conic conformal projection)

and has minor effects on the local features of the geoid. The residual ellipsoidal correction above degree 360 was computed from the residual Helmert gravity anomaly according to Sect. 4.1. It ranges from 1.9 to 2.3 cm over the area of interest.

5 Summary

In this contribution, four different ellipsoidal solutions of the Stokes geodetic boundary value problem are discussed. These solutions are classified into three categories: Green's identity method (Molodenskii et al. 1962; Fei and Sideris 2000, 2001), mapping of boundary values (Moritz 1974) and construction of the ellipsoidal kernel function (Martinec and Grafarend 1997).

All are expressed as the spherical Stokes integration plus an ellipsoidal correction of the order of the flattening. Moritz's method provides the simplest solution while Martinec and Grafarend give the most complex one. Since the boundary value condition for Martinec and Grafarend's solution is not directly related to observables, two ellipsoidal correction terms were derived to make this method practically usable.

Due to differences in definition for the spherical part and the ellipsoidal corrections, the resulting ellipsoidal correction from one method is not in itself comparable with the others. Three coordinate systems (geocentric for Molodenskii et al., Fei and Sideris; geodetic for Moritz; and elliptical for Martinec and Grafarend) have been used in the derivations of the ellipsoidal solution for the convenience of each derivation.

A test procedure was designed to verify their equivalence based on a synthetic field. Numerical tests show that Martinec and Grafarend's solution is the most accurate, while the other three solutions are subject to an approximation error which is related to the first-degree components of the surface spherical harmonic expansion for the geoid solution.

The computational results show that 90% of the ellipsoidal correction is described by the first 20 degrees of the geopotential harmonic series. Thus, the proper use of the generalized Stokes scheme, combining the terrestrial data and the satellite solution, can accurately include the ellipsoidal correction to the geoid in the light of the superior sensitivity of satellites to the lower-degree components of the geopotential, and also provide the correct geoid result regardless of which method is used.

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Appendix A

On the reference ellipsoid, we can write (Heiskanen and Moritz 1967, Sect. 2–8)

$$\left.\frac{\partial T}{\partial h}\right|_E = \frac{1}{w_0}\left.\frac{\partial T}{\partial u}\right|_E \tag{A1}$$

where

$$\begin{aligned}\frac{1}{w_0} &= \frac{a}{(a^2\sin^2\beta + b^2\cos^2\beta)^{\frac{1}{2}}}\\ &= \frac{1}{(1-e^2\cos^2\beta)^{\frac{1}{2}}}\\ &= 1+\frac{1}{2}e^2\cos^2\beta + O(e^4)\end{aligned} \tag{A2}$$

β is the reduced latitude. Substituting $1/w_0$ into Eq. (A1), we obtain

$$\left.\frac{\partial T}{\partial h}\right|_E = \left.\frac{\partial T}{\partial u}\right|_E + \frac{1}{2}e^2\cos^2\beta\left.\frac{\partial T}{\partial u}\right|_E + O(e^4) \tag{A3}$$

By using the following equation (Heiskan and Moritz 1967, Sect. 2–8):

$$r = (u^2 + \mathcal{E}^2\cos^2\beta)^{\frac{1}{2}} \tag{A4}$$

we obtain

$$\left.\frac{\partial T}{\partial u}\right|_E = \left.\left(\frac{\partial T}{\partial r}\frac{\partial r}{\partial u}\right)\right|_E = \left(1-\frac{1}{2}e^2\cos^2\beta\right)\left.\frac{\partial T}{\partial r}\right|_E + O(e^4) \tag{A5}$$

Therefore, Eq. (A1) can be expressed as

$$\left.\frac{\partial T}{\partial h}\right|_E = \left.\frac{\partial T}{\partial u}\right|_E + \epsilon_T + O(e^4) \tag{A6}$$

where

$$\epsilon_T = \frac{1}{2}e^2\cos^2\beta\left.\frac{\partial T}{\partial r}\right|_E \tag{A7}$$

According to Jekeli (1981) and Cruz (1986), we have

$$-\frac{T}{\gamma}\left.\frac{\partial\gamma}{\partial h}\right|_E = \frac{2}{r_E}T + \epsilon_\gamma\big|_E + O(e^4) \tag{A8}$$

Using the following relation on the reference ellipsoid:

$$\frac{1}{r_E} = \frac{(1-e^2\cos^2\phi)^{\frac{1}{2}}}{b} \tag{A9}$$

and considering the following equations:

$$\sin\phi = \sin\beta\left(1-\frac{1}{2}e^2\cos^2\beta\right) + O(e^4) \tag{A10}$$

$$\cos\phi = \cos\beta\left(1+\frac{1}{2}e^2\sin^2\beta\right) + O(e^4) \tag{A11}$$

Eq. (A8) can be written as

$$-\frac{T}{\gamma}\left.\frac{\partial\gamma}{\partial h}\right|_E = \frac{2}{b}T + \epsilon_u + O(e^4) \tag{A12}$$

where

$$\epsilon_u = e^2\cos 2\beta\left.\frac{T}{b}\right|_E \tag{A13}$$

$$\beta = \frac{\pi}{2} - \vartheta \tag{A14}$$

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