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A comparison of Stokes's and Hotine's approaches to geoid computation

Petr Vaníček [1], Zhang Changyou [2], Lars E. Sjöberg [3]

[1] Department of Surveying Engineering, University of New Brunswick, P.O. Box 4400, Fredericton, N.B., Canada E3B 5A3
[2] Department of Surveying Engineering, The University of Calgary, 2500 University Drive N.W., Calgary, Alberta, Canada T2N 1N4
[3] Department of Geodesy, Royal Institute of Technology, S 100-44 Stockholm, Sweden

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Abstract. In the paper, the classical as well as the generalized Stokes techniques — in which use is made of a higher-order reference spheroid defined by satellite determined potential coefficients — are compared with Hotine's technique for geoid determination. While the Stokes techniques use gravity anomalies, Hotine's technique uses gravity disturbances.

Leaving out the complications, we can say that computation of gravity anomalies requires the knowledge of orthometric heights of individual gravity observation points, while gravity disturbances require heights above the reference ellipsoid. The vast majority of the millions of gravity points on land have only the orthometric heights associated with them, thus permitting an evaluation of only gravity anomalies. A systematic use of gravity disturbances on land would require the transformation orthometric heights into heights above the ellipsoid — barring a prohibitively expensive programme of reobservation of the heights at all the gravity points. Such a transformation would call for the geoid-ellipsoid separation, the geoidal height, to be known. Here lies the seeming self-contradiction of Hotine's technique as applied on land: the geoid has to be known to be determinable.

We show that this seeming paradox does not render Hotine's approach meaningless. In our investigation, the geoid (assumed known) is replaced by a higher-order reference spheroid. Two approximate variations of Hotine's technique are then considered, and it is shown that under specific circumstances both of Hotine's variations give more accurate results than the Stokes techniques.

Introduction

With the continuing improvements in accuracy of satellite determined potential coefficients it has become a custom to use the low-frequency satellite information together with terrestrial gravity data for the computation of the geoid. These combined approaches combine the advantages of the two kinds of data to arrive at more equitable solutions: more homogeneously accurate in low frequencies, and more sharply defined in high frequencies.

These combinations can be produced by means of different techniques. In this contribution, we are interested in two such techniques: the Stokes technique and Hotine's technique. More specifically, we are interested in the generalized Stokes technique (Vaníček and Sjöberg 1991) which has been specifically designed to accommodate the two kinds of data, and in approximate Hotine techniques which, unlike the original Hotine formulation, do not require the knowledge of gravity disturbances. Two such approximations are considered in this paper. One may argue that by replacing the requirement for gravity disturbance with something else the formulation can no longer be called a "Hotine" approach. Instead of coining some new names, we decided to call the formulations "approximate Hotine" to help the reader bear in mind the point of departure for their development.

The more narrow aim of this contribution is to show the circumstances under which one of the approximate Hotine techniques will give more precise (and more accurate) results than the generalized Stokes technique. For this goal, the expected biases of the approximate Hotine solutions are investigated and the expected variances of the different kinds of solutions are derived.

The comparison of precisions is carried out by means of expected mean square errors. These turn out to be functions of error degree variances of both the satellite-derived gravity and terrestrial gravity as well as the gravity degree variances which describe the average behaviour of the magnitude of the harmonic coefficients of gravity.

The effects — and corrections for removal of these effects — of both the spherical approximation and the flaws in the modelling of the physics of the geodetic

Offprint requests to: Petr Vaníček

gravity features from satellite orbit analysis has become quite accurate. For example, the error (standard deviation) in the first (20, 20) geopotential coefficients of the GEM T1 model (Marsh et al. 1988) is estimated to be only about 85 cm (Vaníček et al. 1990), certainly better than can be obtained from observed terrestrial gravity.

It thus makes sense to replace the values of the first M terms in equation (5), where the optimum choice of M should be somewhere around 20, by the corresponding values obtained from satellites. Denoting the satellite-determined gravity spherical harmonics by γ_n^M we obtain a different expression (N') for geoidal heights:

$$N' \doteq 2C\left[\sum_{n=2}^{8} \frac{\gamma_n^M - \gamma_n}{n-1} + \sum_{n=9}^{M} \frac{\gamma_n^M}{n-1} + \sum_{n=M+1}^{\infty} \frac{g_n}{n-1}\right] , \tag{9}$$

where we have assumed that $M > 8$. We note that if the expected values $E(\gamma_n^M)$ and $E(g_n)$ are equal for all $n \leq M$, then the expected values of N and N' obtained from equations (5) and (9), respectively, will be the same, as they should be.

On the other hand, assuming again that equation (7) is valid, denoting the satellite-determined gravity degree error variances by $\sigma_n^2(\gamma^M)$ and assuming γ_n^M uncorrelated with g_n we get from equation (9):

$$\sigma^2(N') \doteq 4C^2\left[\sum_{n=2}^{M} \frac{1}{(n-1)^2}\sigma_n^2(\gamma^M) + \sum_{n=M+1}^{\infty} \frac{1}{(n-1)^2}\sigma_n^2(g)\right] . \tag{10}$$

Comparison with equation (8) shows clearly that if

$$\forall n \leq M : \quad \sigma_n^2(\gamma^M) < \sigma_n^2(g) \quad , \tag{11}$$

then

$$\sigma^2(N') < \sigma^2(N) \ , \tag{12}$$

as we have stated above.

Formula (9) can be rewritten in a closed form. Denoting the satellite-determined (model) gravity by $(\gamma)_M$ we have

$$(\gamma)_M = \sum_{n=2}^{M} \gamma_n^M \tag{13}$$

and the spheroid of degree M, defined by the first M degree potential coefficients is given as

$$(N)_M = \sum_{n=2}^{M} N_n^M = 2C\sum_{n=2}^{M} \frac{\gamma_n^M - \gamma_n}{n-1} \ . \tag{14}$$

Now we can redefine the gravity anomaly as the difference of actual gravity on the geoid, g_G, and the model gravity, $(\gamma)_M$, on the reference spheroid, i.e.,

$$\Delta g^M = g_G - (\gamma)_M \ . \tag{15}$$

We can write also

$$\Delta g^M = g_G - \gamma_E - [(\gamma)_M - \gamma_E] = \Delta g - (\Delta g)_M \ , \tag{16}$$

where $(\Delta g)_M$ is what one may call the satellite-determined gravity anomaly. Finally, we get (see, e.g., Vaníček and Krakiwsky (1986)):

$$N' \doteq (N)_M + \kappa \oint\!\!\oint_{\mathcal{E}} S^M \Delta g^M \, d\mathcal{E} \ , \tag{17}$$

where

$$S^M = \sum_{n=M+1}^{\infty} \frac{2n+1}{n-1} P_n \ . \tag{18}$$

Realizing now that

$$N^M = N' - (N)_M \tag{19}$$

is the height of the geoid above the reference spheroid $(N)_M$, we can see that equation (17) can be used to evaluate the geoidal height with respect to a reference spheroid of degree M. Equation (17) looks much the same as the original Stokes formula (1) with even the gravity anomaly Δg^M having a parallel meaning with Δg. This is why we call this approach a "generalized Stokes approach " (Vaníček and Sjöberg 1991). Its accuracy is, of course, described by equation (10) and we see that under the assumption (11), the generalized Stokes approach is more accurate than the classical Stokes approach.

We note that the use of higher-degree gravity field models such as, e.g., Rapp's OSU'89 models (Rapp and Pavlis 1990), arising from the combination of satellite and terrestrial data as reference models, is not considered here. The reason for this is that we do not wish to become entangled in the discussion of correlations which would appear once such a model is further combined with terrestrial data — see, e.g., Vaníček and Sjöberg (1990).

The Hotine Approach

In the middle of this century, another Englishman, cartographer Martin Hotine, formulated and solved the fixed boundary value problem: Given the geoid and the actual gravity on the geoid, determine the disturbing potential on and above the geoid. His solution is subject to the same qualifications as the Stokes solution described above. Hotine's formula (Hotine 1969),

$$N \doteq \kappa \oint\!\!\oint_{\mathcal{E}} H \, \delta g \, d\mathcal{E} \ , \tag{20}$$

where

$$H = \sum_{n=2}^{\infty} \frac{2n+1}{n+1} P_n \tag{21}$$

is known as Hotine's function of a spherical distance and

$$\delta g = g_G - \gamma_G \ , \tag{22}$$

which is easily evaluated as

$$\mathrm{Bias}\,(\tilde{N}) = -\,2C \sum_{M+1}^{\infty} \frac{2}{(n-1)(n+1)}\, E(g_n)\ . \tag{33}$$

This bias contains only higher frequencies — from wave number M up — as one would expect from the approximation of N by $(N)_M$ in equation (26).

The variance of $\tilde{N}$ is easily derived from equation (28). Considering again the γ_n components of normal gravity errorless, we obtain

$$\sigma^2(\tilde{N}) \doteq 4C^2 \left[\sum_{n=2}^{M} \frac{4}{(n-1)^2(n+1)^2}\, \sigma_n^2(\gamma^M) + \sum_{n=2}^{\infty} \frac{1}{(n+1)^2}\, \sigma_n^2(g) \right] . \tag{34}$$

Taking the MSE (an error 'centred' on 0) as a more appropriate measure of accuracy in the presence of a bias, we get

$$\begin{aligned} \mathrm{MSE}(\tilde{N}) &= \mathrm{Bias}^2(\tilde{N}) + \sigma^2(\tilde{N}) \\ &\doteq 4C^2 \left[\sum_{n=2}^{M} \frac{4}{(n-1)^2(n+1)^2}\, \sigma_n^2(\gamma^M) + \sum_{n=2}^{\infty} \frac{1}{(n+1)^2}\, \sigma_n^2(g) + \sum_{n=M+1}^{\infty} \frac{4}{(n-1)^2(n+1)^2}\, c_n \right] , \end{aligned} \tag{35}$$

where by $c_n = E^2(g_n)$ we denote the "degree variances" (squares of values of spherical harmonics in the sense of actual, physical averages) of gravity.

The Second Approximate Hotine Approach

As an alternative, let us take directly the value defined by equation (30) from the previous paragraph and rewrite it as,

$$\approx\!\!N \doteq (N)_M + \kappa \oint\!\!\oint_{\mathcal{E}} H^M\, \Delta g^M\, d\mathcal{E}\ . \tag{36}$$

Its expected value is, of course, the same as that of the first approximation, i.e.,

$$E(\approx\!\!N) = E\,(\tilde{N}) \tag{37}$$

and so is therefore its bias

$$\mathrm{Bias}\,(\approx\!\!N) = \mathrm{Bias}\,(\tilde{N})\ . \tag{38}$$

The spectral expression for $\approx\!\!N$ is different, however, and can be written right away as

$$\approx\!\!N \doteq 2C \left[\sum_{n=2}^{M} \frac{\gamma_n^M - \gamma_n}{n-1} + \sum_{n=M+1}^{\infty} \frac{g_n}{n+1} \right] . \tag{39}$$

From the spectral form, we obtain the following expression for the variance

$$\sigma^2(\approx\!\!N) \doteq 4C^2 \left[\sum_{n=2}^{M} \frac{1}{(n-1)^2}\, \sigma_n^2(\gamma^M) + \sum_{n=M+1}^{\infty} \frac{1}{(n+1)^2}\, \sigma_n^2(g) \right] , \tag{40}$$

which is also different from the variance of the first approximation. The MSE of N is given by

$$\mathrm{MSE}(\approx\!\!N) \doteq 4C^2 \left[\sum_{n=2}^{M} \frac{1}{(n-1)^2}\, \sigma_n^2(\gamma^M) + \sum_{n=M+1}^{\infty} \frac{1}{(n+1)^2}\, \sigma_n^2(g) + \sum_{n=M+1}^{\infty} \frac{4}{(n-1)^2(n+1)^2}\, c_n \right] . \tag{41}$$

We can see that while both approximations should give the same biased result, they have different error characteristics. In the next paragraph, we shall attempt to compare the performance of both these approximate techniques with the performance of the Stokes techniques.

Comparison of Stokes's and Hotine's Approaches

Since both of Hotine's approximations, $\tilde{N}$ and $\approx\!\!N$, are biased with respect to the correct Hotine approach and the two Stokes approaches, we have to use the MSEs to compare their respective accuracies. The method we shall use will be to investigate the magnitude of the differences of the MSEs, realizing that the biases in the two Stokes techniques are both zero, making their MSEs equal to their variances.

The difference between the errors in generalized Stokes's formula and in the first Hotine approximation — it makes no sense to evaluate errors in the accurate Hotine formula since it cannot be used directly — is given by the difference between equations (10) and (35). After a little mathematical development, we obtain:

$$\begin{aligned} &\mathrm{MSE}(N')-\mathrm{MSE}(\tilde{N}) \\ &\doteq 4C^2 \left\{ \sum_{n=2}^{M} \frac{1}{(n+1)^2} \left[\frac{(n+1)^2-4}{(n-1)^2}\, \sigma_n^2(\gamma^M) - \sigma_n^2(g) \right] + \sum_{n=M+1}^{\infty} \frac{4}{(n-1)^2(n+1)^2} \left[n\sigma_n^2(g) - c_n \right] \right\} = \tilde{A}_1' + \tilde{A}_2'\ . \end{aligned} \tag{42}$$

Clearly, if $\tilde{A}_1' + \tilde{A}_2'$ is positive, then the first approximate Hotine approach gives a smaller error than the generalized Stokes approach.

If in equation (42) N' is replaced by N, i.e., if the first Hotine approximation is compared against the classical Stokes approach, then the expression changes to

$$\mathrm{MSE}(N) - \mathrm{MSE}(\tilde{N}) = \tilde{A}_1 + \tilde{A}_2\ , \tag{43}$$

while the first author was staying at the Geodetic Institute of the University of Stuttgart as a recipient of a Distinguished Scientist award from the Humboldt Foundation. Useful discussions with the gracious host, Prof. Dr. Ing. E.W. Grafarend, have led to the elimination of some problems and better elucidation of some explanations in the earlier drafts of the paper. We wish to thank Dr. C. Jekeli for his thoughtful comments on our manuscript. Last but not least, we wish to acknowledge that the editing and flawless word processing of the paper was done by Ms. W. Wells at the University of New Brunswick.

References

Almqvist G (1960) Mathematical methods for determination of the gravity potential. In: Communications from The Royal Institute of Technology to the Twelfth General Assembly of the IUGG, Helsinki, Department of Geodesy, Stockholm

Heiskanen W, Moritz H (1967) Physical geodesy. W.H. Freeman and Co., San Francisco

Hotine M (1969) Mathematical geodesy. U.S. Department of Commerce, Washington, D.C.

Jekeli C (1980) Comparison of undulation difference accuracies using gravity anomalies and gravity disturbances. Bulletin Géodésique 54:137-147

Marsh JG, Lerch FJ, Putney BH, Christodoulidis DC, Smith DE, Felsentreger TL, Sanchez BV, Klosko SM, Pavlis EC, Martin TV, Robbins JW, Williamson RG, Colombo OL, Rowlands DD, Eddy WF, Chandler NL, Rachlin KE, Patel GB, Bhati S, Chinn DS (1988) A new gravitational model for the earth from satellite tracking data: GEM-T1. Journal of Geophysical Research, 93(B6):6169-6215

Mather RS (1973) A solution of the geodetic boundary value problem to order e^3. Goddard Space Flight Center preprint X-592-73-11, Greenbelt, Md.

Rapp RH, Pavlis NK (1990) The development and analysis of geopotential coefficient models to spherical harmonic degree 360. Journal of Geophysical Research 95(B13):21,885-21,911

Sjöberg LE (1989) Integral formulas for geopotential coefficient determination from gravity anomalies versus gravity disturbances. Bulletin Géodésique 64:213-221

Sjöberg LE, Nord T (1991) Geoidal undulation computation by modifying Stokes's kernel versus Hotine's kernel from gravity anomalies. Submitted to Manuscripta Geodaetica

Stokes GG (1849) On the variation of gravity at the surface of the earth. Trans. Cambridge Philosophical Society VIII:672-695

Strang van Hees G (1986) Precision of the geoid computed from terrestrial gravity measurements. Manuscripta Geodaetica 11:1-14

Vaníček P, Kleusberg A (1987) The Canadian Geoid — Stokesian approach. Manuscripta Geodaetica 12(2):86-98

Vaníček P, Krakiwsky EJ (1986) Geodesy: The Concepts. 2nd rev. ed., North Holland, Amsterdam

Vaníček P, Sjöberg LE (1990) Kernel modification in generalized Stokes's technique for geoid determination. In: Sunkel H, Baker T (eds) Sea Surface Topography and the Geoid, Proceedings of a General Meeting of the IAG, Edinburgh, Scotland, 3-12 August 1989, Springer, pp 31-38

Vaníček P, Sjöberg LE (1991) Reformulation of Stokes's theory for higher than second-degree reference field and modification of integration kernels. Accepted for publication by *Journal of Geophysical Research*, February 1991

Vaníček P, Zhang C, Ong P (1990) Computation of a file of geoidal heights using Molodenskij's truncation method. Department of Surveying Engineering Technical Report No. 147, University of New Brunswick, Fredericton, N.B., September