Journal of Geodesy (1996) 70: 724–730

Best Wishes, from Lars

Journal of Geodesy

On the error of analytical continuation in physical geodesy

Lars E. Sjöberg

Royal Institute of Technology, Department of Geodesy and Photogrammetry, S-100 44 Stockholm, Sweden

Received 26 July 1995; Accepted 20 May 1996

ABSTRACT

Analytical continuation of gravity anomalies and height anomalies is compared with Helmert's second condensation method. Assuming that the density of the terrain is constant and known the latter method can be regarded as correct. All solutions are limited to the second power of H/R, where H is the orthometric height of the terrain and R is mean sea-level radius. We conclude that the prediction of free-air anomalies and height anomalies by analytical continuation with Poisson's formula and Stokes's formula goes without error. Applying the same technique for geoid determination yields an error of the order of H^2, stemming from the failure of analytical continuation inside the masses of the Earth.

1. Introduction

Analytical continuation is frequently used to solve Molodensky's boundary value problem in physical geodesy. In this way the gravity potential and gravity itself can be estimated on and outside the surface of the Earth from gravity related data on and outside the irregular surface of the Earth. Downward continuation of the data to the Earth's surface is most essential in the application of satellite and airborne techniques for gravity field determination.

The analytical continuation of a gravity related quantity means that it is continued downward or upward in free-air, i.e. the Earth's topography is disregarded. This approach permits the use of classical formulas, such as Poisson's and Stokes's formulas, to solve the modern Molodensky's problem for the irregular surface of the Earth. For details see Bjerhammar (1963), (1964), (1969) and (1975), Moritz (1966 a, b) and (1980), Heiskanen and Moritz (1967) and Bjerhammar and Svensson (1979).

Frequently analytical continuation is justified by reference to the approximation theorems of Krarup-Runge (Krarup, 1969) and Keldysh-Laurientiev (Bjerhammar, 1975). However, although these theorems postulate that analytical continuation is possible, they do not provide the tool for it. Thus it is still an open question whether Bjerhammar's method and/or least-squares collocation yield such approximations.

A major problem with the analytical continuation is that the method can hardly be applied in a strict sense, as the downward continuation may not converge at or below the Earth's surface. This problem may partly be circumvented by a suitable approximation of the strict analytical continuation. Bjerhammar (1963), (1964) and (1969) emphasized that there are merely a finite set of observations, and the corresponding system of discretized integral equations can always be solved. In one way or another the analytical coninuation necessitates that the external gravity potential is represented by a smoothed approximation. The problem is solely due to the contribution of the Earth's topography. The problem was recently studied by Wang (1995), who gave an example of a uniform approximation under planar approximation. Cf. also Wang (1990).

Our approach to study the error of the analytical continuation is the following. We assume that the density of the terrain is constant and known. Then we can solve Molodensky's problem by Helmert's second condensation method. By comparing this solution with that of the analytical continuation we get the possible error. The study is limited to the second order approximation, i.e. the reciprocal distance is expanded in a Taylor series to second power of terrain elevation. This implies that the terrain is smoothed to have slopes within 45°.

2. The analytical continuation approach

Consider Poisson's integral equation:

$$\Delta g_P = \frac{R^2(r_P^2 - R^2)}{4\pi r_P} \iint_\sigma \frac{\Delta g_Q^*}{r_{PQ}^3} d\sigma_Q, \qquad (1)$$

which relates gravity anomalies Δg^* on a sphere of radius R to gravity anomalies Δg_P at radius $r_P > R$, σ being the unit sphere and r_{PQ} the distance from point Q on the sphere to P. This formula is strictly valid in free-air, i.e. if there are no masses outside the sphere of radius R. Bjerham-mar (1962), (1963), (1964) and (1969) considered Poisson's formula with the sphere of radius R ("the Bjerhammar sphere") embedded in the Earth. In that case Δg^* is the downward continued, fictitious gravity anomaly. Then the strict solution for Δg^*, of the integral equation (1), i.e. the downward continuation of Δg, is most doubtful, being dependent on the smoothness of the terrain and the gravity anomaly. However, Bjerhammar emphasized, that in practice there is only a limited set of observations Δg, and in.this discrete case a solution to Δg^* (not necessarily unique!) always exists. In this case eqn. (1) can be approximately solved iteratively. See for example Bjerhammar (1969), Moritz (1966 a, b) and Heiskanen and Moritz (1967, ch. 8).

Once Δg^* is determined it can be used in Poisson's integral (1) and the extended Stokes's formula for upward continuation of the gravity anomaly and the disturbing potential (T), respectively, to any external point P. In particular, the latter formula, combined with Bruns's formula, yields the height anomaly

$$\zeta_P = \frac{R}{4\pi\gamma} \iint_\sigma S(r_P, \psi) \Delta g_Q^* \, d\sigma_Q \qquad (2)$$

where

$S(r_P, \psi)$ = extended Stokes's function

γ = normal gravity at the normal height of P.

In formulas (1) and (2) we should regard Δg and Δg^* as free-air anomalies.

3. Helmert's condensation approach

In Helmert's second condensation method (Helmert, 1884) the terrain is condensed into a surface layer at sea-level. The free-air gravity anomaly corrected for its direct terrain (reduction) effect ($\delta\Delta g$) we call the Helmert anomaly

$$\Delta g^H = \Delta g + \delta \Delta g \ . \qquad (3)$$

As the terrain is reduced to sea-level, the Helmert anomaly strictly obeys Poisson's formula (1):

$$\Delta g^H = \frac{R^2(r_P^2 - R^2)}{4\pi r_P} \iint_\sigma \frac{(\Delta g^H)^*_Q}{r_{PQ}^3} d\sigma_Q . \qquad (4)$$

where Δg^* is the downward continued free-air anomaly.

Formula (6) shows that $\delta\Delta g$ is of order H^2. Moreover, considering that

$$\delta\Delta g = \delta A + 2\frac{\delta V_I}{R} , \tag{11}$$

where δA is the direct gravity effect and δV_I is the indirect effect on potential, with

$$\delta A = -\frac{\partial \delta V_I}{\partial H} , \tag{12}$$

one obtains

$$\frac{\partial}{\partial H}\delta\Delta g = \frac{\partial}{\partial H}\delta A - \frac{2}{R}\delta A \tag{13}$$

From Sjöberg {1995a, formulas (A.6), (21) and (23)} follows that for each point Q at the Earth's surface it holds that

$$\left(\frac{\partial \delta A}{\partial H}\right)_Q = O(H_Q^2) , \tag{14}$$

implying that

$$\left(\frac{\partial}{\partial H}\delta\Delta g\right)_Q = O(H_Q^2) \tag{15a}$$

and

$$\left(\frac{\partial^2}{\partial H^2}\delta\Delta g\right)_Q = O(H_Q) . \tag{15b}$$

Inserting formulas (15) into (10) we finally arrive at the following relation between the downward continued Helmert and free-air anomaly (to order H^2):

$$(\Delta g^H)^* = \Delta g^* + \delta\Delta g + O(H^3) , \tag{16}$$

where $\delta\Delta g$ is applied at the Earth's surface.

Inserting (16) into formula (4) we get (to order H^2)

$$\Delta g_P = \\ = \frac{R^2(r_P^2 - R^2)}{4\pi\, r_P} \iint_\sigma \frac{\Delta g^* + \delta\Delta g}{r_{PQ}^3} d\sigma_Q - \delta\Delta g_P \tag{17a}$$

or

$$\Delta g_P = \\ = \frac{R^2(r_P^2 - R^2)}{4\pi\, r_P} \iint_\sigma \frac{\Delta g^*}{r_{PQ}^3} d\sigma + \delta\Delta g_{total}(P), \tag{17b}$$

where

$$\delta\Delta g_{total}(P) = \\ = -\delta\,\Delta g_P + \frac{R^2(r_P^2 - R^2)}{4\pi\, r_P} \iint_\sigma \frac{\delta\Delta g}{r_{PQ}^3} d\sigma . \tag{17c}$$

$\delta\Delta g_{total}$ is the total correction to Poisson's formula (1).

Rewriting Poisson's kernel in the form

$$\frac{R^2(r_P^2 - R^2)}{r_P\, r_{PQ}^3} = \\ = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\frac{R}{r_P}\right)^{n+4} Y_{nm}(P) Y_{nm}(Q) \tag{18}$$

and inserting formula (6), (17c) becomes

$$\delta\Delta\gamma_{total}(P) = \tag{19a}$$
$$= -\frac{2\pi\mu}{R} \sum_{n=2}^{\infty} \left\{\frac{(n+2)(n-1)}{2n+1}\right\} \left\{\left(\frac{R}{r_P}\right)^{n+4} - 1\right\} (H^2)_n$$

Adding (25) and (27) we thus obtain

$$\delta N_{total} = -\frac{2\pi\mu}{\gamma}\sum_{n=2}^{\infty}(H^2)_n = -\frac{2\pi\mu}{\gamma}\tilde{H}^2, \qquad (28)$$

where $\tilde{H}^2$ was defined in (9). This means that formula (23) is in error by $2\pi\mu\tilde{H}^2/\gamma$.
This result agrees with Wang {(1990), eq. (41)}.

6. Concluding remarks

We have shown that the error of analytical continuation is of order less than H^2/R for gravity anomaly and height anomaly estimation, while the error is of order H^2 for the geoid. The reason that the analytical continuation fails for the continental geoid is, of course, that the continuation procedure provides merely fictitious values inside the terrain. The above results are in agreement with the downward continuation of a satellite derived geopotential harmonic series as presented by Sjöberg (1994) and (1995 b).

Our study suggests that geoidal undulations are estimated by the formula

$$N = \frac{R}{4\pi\gamma}\iint_{\sigma} S(\psi)\Delta g^{*} d\sigma + \delta N_{total}, \qquad (29)$$

where the integral is the outcome of the analytical continuation and δN_{total} is a correction given by formula (28). Subsequently the frequently applied remove-restore technique of the terrain appears obsolete. However, the latter technique might be justified from a numerical point of view, as the removal of the terrain stabilizes the solution for Δg^{*}.

As a final remark we like to point out that the direct gravity anomaly effect {formula (5) or (11)} is the sum of the direct gravity effect and the so-called secondary indirect effect. However, as these effects are applied at the Earth's surface (and not at the geoid), it implies that the secondary indirect effect includes $\delta\zeta_I$ and not δN_I, as usually suggested in the literature. The latter choice of the effect would cause a significant error both in geoid and height anomaly determination.

REFERENCES

Bjerhammar, A. (1962): Gravity reduction to a spherical surface. Royal Institute of Technology, Division of Geodesy, Stockholm.

Bjerhammar, A. (1963): A new theory of gravimetric geodesy. Royal Institute of Technology, Division of Geodesy, Stockholm.

Bjerhammar, A. (1964): A new theory of geodetic gravity. Royal Institute of Technology, Division of Geodesy, Stockholm.

Bjerhammar, A. (1969): On the boundary value problem of physical geodesy. Tellus XXI, 4.

Bjerhammar, A. (1975): Discrete approaches to the solution of the boundary value problem in physical geodesy. Boll. Geod. Sci. Affini, Vol. 34, pp. 185-240.

Bjerhammar, A. and L. Svensson (1979): The convergence of harmonic reduction to an internal sphere. Bull. Géod., Vol, 53, pp. 247-257.

Helmert, F.R. (1884): Die mathematischen und physikalischen Theorien der höheren Geodäsie, Vol. 2, B.G. Teubner, Leipzig.

Heiskanen, W. and H. Moritz (1967): Physical Geodesy. W.H Freeman and company, San Fransisco and London.

Krarup, T. (1969): A contribution to the Mathematical Foundation of Physical Geodesy, Meddelelse No. 44, Geodaetisk Institut, Copenhagen.

Moritz, H. (1966 a): Methods for downward continuation of gravity. DGK, Series A, Vol. no. 50, Munieh.

Moritz, H. (1966 b): Linear solutions of the geodetic boundary value problem. The Department of Geodetic Science Report, No. 79, The Ohio State University, Columbus, Ohio.

Moritz, H. (1980): Advanced Physical Geodesy. Herbert Wichmann Verlag, Karlsruhe

Sjöberg, L. E. (1994): On the terrain effects in geoid and quasigeoid determinations using Helmert's second condensation method. Division of Geodesy Report No. 36, (TRITA-GEOD 1036), KTH, Stockholm.