Truncation of spherical convolution integrals with an isotropic kernel

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Abstract

A truncated convolution integral often has to be used as an approximation of a complete convolution over the sphere in many Earth science or related studies, such as geodesy, geophysics and meteorology. The truncated integration is necessary because detailed input data are not usually available over the entire Earth. In this contribution, a symmetrical mathematical apparatus is presented with which to treat the truncation problem elegantly. Some important aspects are mentioned and one practical example is shown for regional gravimetric geoid determination of Canada.

Introduction

The main topic treated in this paper is the convolution integration over a sphere and how to treat it correctly when data are available only in a certain area around the point of computation. An isotropic, i.e. azimuth independent, integration kernel will be assumed throughout. This problem arose from the common practice in geo-scientific studies, where in many cases it has not been treated correctly. Quite often, the integration over the sphere is reduced to integration over a small region surrounding the computation point, neglecting the integration over the remainder of the sphere. Of course, a physically reasonable integration kernel usually decreases its magnitude with the growing spherical distance from the point of interest, which makes the far-zone contribution relatively small. On the other hand, however, the far-zone integration domain is usually too large to completely neglect its contribution. Some theoretical aspects of this truncation problem have already been treated in *Vaníček and Sjöberg (1991)* and *Vaníček and Featherstone (1998)*, as well as by numerous other authors, e.g. *Evans and Featherstone (2000), Heck and Grüninger (1987), Meissl (1971), Omang* *and Forsberg (2002), Sjöberg (1986; 1991), Sjöberg and Hunegnaw (2000), Wong and Gore (1969), Featherstone et al. (1998).*

This paper consists of two main parts: 1. the mathematical background, and 2. a numerical example. In the first part, it is shown how to perform a convolution integration over the sphere correctly from the mathematical point of view. Symmetrical formulae using a series expansion are derived for the integration over the spherical cap around the point of computation and for the integration over the rest of the sphere. The second part of the paper shows one practical application in geodesy - determination of the precise regional geoid in Canada from a limited gravity data set on the sphere.

Mathematical background

Consider a general convolution-type integral over the unit sphere

$$\int_{\Omega'} K(\psi) f(\Omega') d\Omega', \tag{1}$$

where K is an isotropic kernel, a function of the solid spherical angle ψ between the evaluation and integration points, f is a function on a sphere, continuous together with its first and second derivatives, $\Omega = (\varphi, \lambda)$ is a solid spherical angle denoting the pair of spherical co-ordinates φ, λ, the spherical latitude and longitude. The kernel function K can be expressed as an infinite orthogonal series expansion in terms of Legendre polynomials as the basis functions, which gives

$$K(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} a_n P_n(\cos\psi), \tag{2}$$

where

$$\forall n = 0,1,2,\ldots : a_n = \int_0^{\pi} K(\psi) P_n(\cos\psi) \sin\psi d\psi. \tag{3}$$

If the function K is continuous together with its first and second derivatives, the series in Eq. (2) will be absolutely and uniformly convergent. The same holds for the function f, see *(e.g., Rektorys, 1973, p. 605)*. The integral in Eq. (1) can also be written as an infinite series of the Laplace surface spherical harmonics Y_n of the function f as follows

$$\int_{\Omega'} K(\psi) f(\Omega') d\Omega' = \sum_{n=0}^{\infty} a_n Y_n(\Omega). \tag{4}$$

Equation (4) can be derived by substitution of Eq. (2) into Eq. (1), assuming that for the Laplace surface spherical harmonics of the function f, the following formula holds

$$\forall n = 0,1,2,\ldots : Y_n(\Omega) = \frac{2n+1}{4\pi} \int_{\Omega'} f(\Omega') P_n(\cos\psi) d\Omega', \tag{5}$$

see also *(Heiskanen and Moritz, 1967, Eq. 1-71)*. The Laplace surface spherical harmonics of f can also be written as follows

$$\forall n = 0,1,2,\ldots : Y_n(\Omega) = \sum_{m=0}^{n} \left[\bar{a}_{nm} \bar{R}_{nm}(\Omega) + \bar{b}_{nm} \bar{S}_{nm}(\Omega) \right], \tag{6}$$

see also *(ibid., Eq. 1-66)*. In Eq. (6), $\bar{R}_{nm}$ and $\bar{S}_{nm}$ are the fully normalised spherical harmonic basis functions and $\bar{a}_{nm}$ and $\bar{b}_{nm}$ are the fully normalised spherical harmonic coefficients of f defined by the following expressions *(ibid., Eq. 1-76 and 1-73)*

$$\begin{pmatrix} \bar{a}_{nm} \\ \bar{b}_{nm} \end{pmatrix} = \frac{1}{4\pi} \int_{\Omega'} f(\Omega') \begin{pmatrix} \bar{R}_{nm}(\Omega') \\ \bar{S}_{nm}(\Omega') \end{pmatrix} d\Omega', \tag{7}$$

$$\begin{pmatrix} \bar{R}_{nm}(\Omega) \\ \bar{S}_{nm}(\Omega) \end{pmatrix} = \sqrt{(2-\delta_{0m})(2n+1)\frac{(n-m)!}{(n+m)!}} P_{nm}(\sin\varphi) \begin{pmatrix} \cos m\lambda \\ \sin m\lambda \end{pmatrix}, \tag{8}$$

where δ_{0m} is the Kronecker delta and P_{nm} are the associated Legendre functions. Introducing now a spatial truncation at the solid spherical angle ψ_0 from the computation point, Eq. (1) can be decomposed as

$$\int_{\Omega'} K(\psi) f(\Omega') d\Omega' = \int_{C} K(\psi) f(\Omega') d\Omega' + \int_{\Omega'-C} K(\psi) f(\Omega') d\Omega', \tag{9}$$

where C denotes a spherical cap of an arbitrary radius $0 < \psi_0 < \pi$. In practice, the area C corresponds to the domain where detailed input data are known. The region outside the cap (Ω'-C) corresponds to the domain where detailed data are not available. Associated with this decomposition, a new truncated kernel is defined as

$$K^*(\psi) = \begin{cases} K(\psi), & \text{for} \quad 0 \le \psi \le \psi_0, \\ 0, & \text{for} \quad \psi_0 < \psi \le \pi. \end{cases} \tag{10}$$

This truncated kernel K^* can also be expressed as an orthogonal series expansion in terms of Legendre polynomials, see *(e.g., Molodensky et al., 1962, p. 147)*, as follows

$$K^*(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} s_n(\psi_0) P_n(\cos\psi), \tag{11}$$

where for $\forall n = 0,1,2,...$ and $0 < \psi_0 < \pi$, the expansion coefficients $s_n(\psi_0)$ of this series expansion are given by

$$s_n(\psi_0) = \int_0^{\pi} K^*(\psi) P_n(\cos\psi) \sin\psi d\psi = \int_0^{\psi_0} K(\psi) P_n(\cos\psi) \sin\psi d\psi. \tag{12}$$

The kernel K^* is discontinuous at $\psi = \psi_0$ and therefore the convergence of the series in Eq. (11) at this point is not absolute but mean. Other authors (e.g., *Meissl, 1971*; *Heck and Gruninger, 1987*; *Feathertstone et al., 1998*; *Evans and Featherstone, 2002*) have addressed the problem of the continuity of the kernel function and its effect on the rate of convergence of the series in Eq. (11) so we shall not delve into this problem here.

Inserting Eq. (10), the first integral on the right-hand-side of Eq. (9) can be re-written as

$$\int_C K(\psi) f(\Omega') d\Omega' = \int_{\Omega'} K^*(\psi) f(\Omega') d\Omega', \tag{13}$$

Substituting for K^* from Eq. (11), gives

$$\int_{\Omega'} K^{*}(\psi) f(\Omega') d\Omega' = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} s_n(\psi_0) \int_{\Omega'} f(\Omega') P_n(\cos\psi) d\Omega'. \tag{14}$$

The integral in Eq. (13) can now be derived in terms of the Laplace surface spherical harmonics, similarly to Eq. (4) when taking into account Eq. (5)

$$\int_{C} K(\psi) f(\Omega') d\Omega' = \int_{\Omega'} K^{*}(\psi) f(\Omega') d\Omega' = \sum_{n=0}^{\infty} s_n(\psi_0) Y_n(\Omega). \tag{15}$$

Returning to Eq. (10), a complementary kernel K^{**} can be defined as

$$K^{**}(\psi) = \begin{cases} 0, & \text{for} \quad 0 \le \psi \le \psi_0, \\ K(\psi), & \text{for} \quad \psi_0 < \psi \le \pi, \end{cases} \tag{16}$$

which has the following orthogonal series expansion

$$K^{**}(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} q_n(\psi_0) P_n(\cos\psi), \tag{17}$$

where for $\forall n = 0,1,2,...;\ 0 < \psi_0 < \pi$, the expansion coefficients are

$$q_n(\psi_0) = \int_{0}^{\pi} K^{**}(\psi) P_n(\cos\psi) \sin\psi d\psi = \int_{\psi_0}^{\pi} K(\psi) P_n(\cos\psi) \sin\psi d\psi. \tag{18}$$

The second integral on the right-hand-side of Eq. (9) may then be expressed analogously to Eq. (15) as

$$\int_{\Omega'-C} K(\psi) f(\Omega') d\Omega' = \int_{\Omega'} K^{**}(\psi) f(\Omega') d\Omega' = \sum_{n=0}^{\infty} q_n(\psi_0) Y_n(\Omega). \tag{19}$$

Equations (15) and (19) provide an effective symmetrical apparatus for dealing with any truncation problem encountered in the studies of the Earth, or any other problem on the sphere. It is important to note that the division of the integration area Ω' into a spherical cap and the rest of the sphere does not correspond to an exact separation in the frequency domain: each partial integral must be expressed as a series containing all frequencies.

Since the sum $K^{*}(\psi) + K^{**}(\psi)$ must be equal to $K(\psi)$, the following must also hold, as can be seen comparing Eqs. (12) and (18) with Eq. (3)

$$\forall n = 0,1,2,\ldots; 0 \leq \psi_0 \leq \pi : s_n(\psi_0) + q_n(\psi_0) = a_n. \tag{20}$$

Indeed, putting together Eqs. (9), (15) and (19) gives

$$\int_{\Omega'} K(\psi) f(\Omega') d\Omega' = \sum_{n=0}^{\infty} a_n Y_n, \tag{21}$$

which is the same result as Eq. (4).

A numerical example over Canada

A good example where the above theory is applied is in the determination of the geoid using local gravity data and a global geopotential model, the latter of which is expressed in terms of spherical harmonics. Here, the low frequency part of the function given by Eq. (4) is computed separately from an existing global geopotential model. Then the apparatus described above can be applied to the rest of the frequencies. In such a case, the first term on the right-hand-side of Eq. (9) is computed using the detailed input data and the second term, representing the medium and high-degree part of the far-zone contribution, is estimated from a high-degree global geopotential model.

The geoid is defined as the surface that intersects the direction of gravity at right angles and from which the surface of the world ocean is a part *(Gauss, 1828, p. 49)*, or as the equipotential surface that best approximates the mean sea level over the whole Earth *(Vaníček and Krakiwsky, 1986, p. 87)*, if the sea surface topography is neglected. The geoid is a surface of great theoretical and practical importance, with many practical applications such as: study of the deep Earth mass anomalies, vertical positioning, geodynamics, etc. (e.g., *Vaníček and Christou, 1994*).

A solution for the geoid obtained from terrestrial gravity data, in a spherical approximation, is known as Stokes's integral *(Stokes, 1849)*

$$N(\Omega) = \frac{R}{4\pi\gamma_0} \int_{\Omega'} \Delta g(\Omega') S(\psi) d\Omega'. \tag{22}$$

Equation (22) is a global convolution integral which transforms the gravity anomalies Δg to geoid heights N. In Eq. (22), R is the radius of the reference sphere, γ_0 is normal gravity on the reference ellipsoid, and $S(\psi)$ is Stokes's function. Stokes's function may be represented in a spatial form *(e.g., Heiskanen and Moritz, 1967, Eq. 2-164* or *Vaníček and Krakiwsky, 1986, Eq. 22.15)* as

$$S(\psi) = 1 + \frac{1}{\sin\frac{\psi}{2}} - 6\sin\frac{\psi}{2} - 5\cos\psi - 3\cos\psi \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right), \tag{23}$$

or in a spectral form as

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos\psi), \tag{24}$$

where $P_n(\cos\psi)$ is the Legendre polynomial of degree n. From the point of view of the boundary-value problem solution, Stokes's function is simply a Green's function. It may also be regarded as a homogeneous and isotropic integration kernel. Stokes's integral is thus the Green's type of solution to the geodetic boundary-value problem *(Vaníček and Krakiwsky, 1986, pp. 519-520).*

It is clear that Stokes's integral (Eq. 22) requires knowledge of the gravity anomalies over the whole Earth. However, this poses a problem because there is a lack of data in many regions, due to restricted field access or data confidentiality clauses. *Vaníček and Kleusberg (1987*), for example, suggest separating the summation over n in Stokes's function (Eq. 24) into low- and high-degree frequency parts

$$S(\psi) = \sum_{n=2}^{\ell} \frac{2n+1}{n-1} P_n(\cos\psi) + \sum_{n=\ell+1}^{\infty} \frac{2n+1}{n-1} P_n(\cos\psi) = S_\ell(\psi) + S^\ell(\psi). \tag{25}$$

In this example, the value of $\ell=20$ was chosen since this is the degree beyond which satellite-only global geopotential models become less reliable (excepting those likely to be derived from the current and planned dedicated satellite gravimetry missions, e.g., Rummel et al., 2002). Many authors prefer to use a high-degree global geopotential model based on combined satellite-terrestrial observations up to, typically, ℓ=360. Such an approach would reduce the truncation error, but on the other hand all errors coming from a high-degree geopotential model would affect the solution. While the new generation of global geopotential models based on the current and planned dedicated satellite gravimetry missions will bring a significant improvement in accuracy of the medium degree geopotential coefficients (ibid.), the problems of terrestrial gravity data in combined global geopotential models will remain.

Let us denote the low-degree part of Stokes's kernel by $S_\ell(\psi)$ and the high degree part by $S^\ell(\psi)$. This high degree part of Stokes's kernel is called the *spheroidal Stokes kernel (Vaníček and Kleusberg (1987).* Substituting Eq. (25) into Eq. (22), the geoid height can also be split into a low degree part $N_\ell(\Omega)$, called the *reference spheroid (ibid.)*, and a high-degree contribution $N^\ell(\Omega)$, so that

$$N(\Omega) = N_\ell(\Omega) + N^\ell(\Omega), \tag{26}$$

where, the low-frequency geoid is

$$N_\ell(\Omega) = \frac{R}{4\pi\gamma_0} \int_{\Omega'} \Delta g(\Omega') S_\ell(\psi) d\Omega', \tag{27}$$

and the high-frequency geoid is

$$N^\ell(\Omega) = \frac{R}{4\pi\gamma_0} \int_{\Omega'} \Delta g(\Omega') S^\ell(\psi) d\Omega'. \tag{28}$$

The integration in Eqs. (27) and (28) is still global at this stage of the derivation. It is most sensible to determine the low-frequency part of the geoid $N_\ell(\Omega)$ (Eq. (27) from satellite-only measurements with a sufficient accuracy from Eq. (21) with $K(\psi)=S_\ell(\psi)$, $f(\Omega)=\Delta g(\Omega)$ and $n\in$<0,20>. More specifically, it is evaluated in practice using fully normalised coefficients of a truncated spherical harmonic series of a satellite-only global geopotential model. This long-wavelength part of the geoid over Canada, computed from the GRIM4-S4 satellite-only global geopotential model *(Schwintzer et al., 1997)* is shown in Fig. 1. The modified Stokes integration is then employed for the high-degree part of the geoid height according to Eq. (28). Even though frequency division has been used, the problem with lack of the gravity data in some regions still remains. In order to address this problem, the integration domain is split into a spherical cap of ψ_0=6° and the rest of the reference sphere, as it is defined in the more general sense in Eq. (9). This gives

$$N^{\ell} = \frac{R}{4\pi\gamma_0}\int_C \Delta g(\Omega')S^{\ell}(\psi)d\Omega' + \frac{R}{4\pi\gamma_0}\int_{\Omega'-C} \Delta g(\Omega')S^{\ell}(\psi)d\Omega' = N_C^{\ell} + N_{\Omega'-C}^{\ell}. \quad (29)$$

Integration over the spherical cap (first term on the right-hand-side of Eq. 29) estimates part of the high-frequency components of the geoid height. This part is often called the *residual geoid model* N_C^{ℓ} and it can be obtained in a spatial domain using a numerical integration, or can be computed using Fourier techniques after some approximations. The residual geoid model over Canada, computed from gravity and terrain data available for Canada and nearby seas using the modified spheroidal Stokes's function, which is described below, is shown in Fig. 2. Integration over the rest of the sphere usually yields a much smaller-valued term (typically in decimetres, whereas the contribution from the cap is typically a few metres). Thus, it is possible to determine it from a high-degree global geopotential model with a sufficient accuracy according to Eq. (19) with $K(\psi)=S^{\ell}(\psi)$, $f(\Omega)=\Delta g(\Omega)$ and $n\in$<21,360>. This part is then the *medium-degree far-zone contribution* $N^{\ell}_{\Omega'-C}$. The remaining high-frequency terms (n>360) contribute to geoid height less then 1cm, as it was shown by (*Martinec, 1993*), and can therefore be neglected.

If the global information is in any way of a questionable quality, it is advisable to minimise the far-zone contribution term by modifying the spheroidal Stokes's kernel. One such modification was introduced by *Vaníček and Kleusberg (1987)*, which minimises the truncation term in a least-square sense according to idea of *Molodenskij et al., (1962)*. Other procedures of modification can be found in the articles cited in the Introduction. The result of the *Vaníček and Kleusberg (1987)* modification is called the *Molodenskij-like modified spheroidal Stokes's kernel*. The far-zone contribution computed using the Molodenskij-like modified spheroidal Stokes's kernel and the EGM96 combined satellite-terrestrial global geopotential model *(Lemoine et al., 1998)* to a degree and order 360 is displayed in Fig. 3. From this figure, it can be seen that even the minimised far-zone contribution term reaches the magnitude of a few decimetres over Canada and therefore cannot be neglected. The final geoid height is obtained as a sum of the reference spheroid, residual geoid and the far-zone contribution as:

$$N(\Omega) = N_{\ell}(\Omega) + N_{C}^{\ell}(\Omega) + N_{\Omega'-C}^{\ell}(\Omega). \qquad (30)$$

Finally, several additional correction terms are also required to satisfy the harmonicity of the Stokes solution to the geodetic boundary value problem *(e.g., Vaníček et al.,* *1999)*. These have been included in the computations, but are quite different in scope to the mathematical apparatus presented in this paper and thus not included in the discussion here. The final geoid model over Canada is shown in Fig. 4.

Concluding remark

The mathematical theory described in the first part of this paper, and one application to geodesy presented in the second part of the paper, are meant to be of an assistance to those who deal with convolution integrals over a sphere and who often have to deal with a lack of complete global data coverage. We show herein a mathematical tool for treating this problem correctly not only from the theoretical point of view, but also to certain extent, from the numerical point of view.

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