PROOF OF EQUIVALENCE BETWEEN POISSON'S AND ANALYTICAL CONTINUATION AND THE EQUIVALENCE OF INTEGRAL MEAN OF GRAVITY AND THE INTEGRAL MEAN OF ANALYTICALLY CONTINUED GRAVITY

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Abstract: Since the geoid-generated gravity disturbance is harmonic above the geoid surface if multiplied by a geocentric radius, its mean value can be defined so that Poisson's and analytical upward continuation is applied in the integral mean. The proof of equivalence between Poisson's and analytical upward continuation and the application of this equivalence in the evaluation of the integral mean of the geoid-generated gravity disturbance is discussed in this paper. To evaluate the mean topography-generated gravitational attraction within the topography, the analytical continuation can be used only if the gravitational attraction itself is analytical, i.e., if the vertical derivatives of all orders exist. In this paper the laterally varying topographical density distribution is used to make the gravitational attraction analytical. The gravitational attraction in the spectral form is then introduced, and the proof of equivalence between the integral mean of gravitational attraction and the integral mean of analytically continued gravitational attraction is derived.

Key words: Dirichlet's problem – Gravity – Newton integral – Taylor's series

1. Introduction

To determine the orthometric heights, the mean value of gravity along the plumbline has to be computed. The mean gravity is defined as the integral mean of gravity, i.e., as the averaged value of gravity along the plumbline within the topography. The gravity can be decomposed into the geoid-generated gravity disturbance, normal gravity and gravitational attraction of topographical and atmospheric masses (Vaníček et al., 2004). To evaluate the mean gravity this decomposition can then be applied to the integral mean of gravity so that the mean values of these gravity components are computed separately.

The mean value of normal gravity can be computed according to well-known Somigliana-Pizzeti's theory (Pizzetti, 1911; Somigliana, 1929), while the effect of atmosphere on the orthometric height is negligible (Tenzer et al., 2004).

Since the geoid-generated gravity disturbance (multiplied by a geocentric radius) is harmonic above the geoid surface, i.e., satisfies the Laplace equation, Poisson's solution to the Dirichlet boundary value problem can be applied in the integral mean to evaluate the mean value of the geoid-generated gravity disturbance. Considering that any harmonic function is also analytical (see e.g., Rektorys, 1968), the analytical upward continuation of the geoid-generated gravity disturbance can also be applied in the integral mean. Poisson's solution to the Dirichlet problem is described by the Poisson integral (e.g., Kellogg, 1929), while for the analytical upward continuation the expansion of the geoid-generated gravity disturbance into the Taylor series is used.

The different conditions arise when the mean value of the topography-generated gravitational attraction is evaluated within the topography. In this case, the gravitational attraction is not harmonic. Moreover, the analytical continuation can be applied only if the gravitational attraction is analytical, so that all its derivatives with respect to the plumbline exist between the geoid and the earth surface. Such analytical continuation is achieved, for example, if the vertical distribution of topographical density is continuous.

The analytical continuation for the evaluation of the mean geoid-generated gravity disturbance and mean topography-generated gravitational attraction within the topography has been used for the computation of the mean gravity in the determination of the orthometric heights in Tenzer et al. (2003). The analytical continuation for the evaluation of the mean gravity disturbance was used before also by Hwang and Hsiao (2003), and the comparison of the analytical and Poisson's downward continuation of the gravity anomalies can be found in Huang (2002). In Tenzer et al. (2004), the integral mean of gravitational attraction of topographical masses has is used for the computation of the mean value of the topography-generated gravitational attraction and Poisson's continuation has been applied in the integral mean of the geoid-generated gravity disturbance. In this paper it is shown that these methods are equivalent and thus if some difference of results occur, they are caused by the different accuracy of the numerical methods used for computations.

2. Mean gravity disturbance

Let us begin with the definition of the mean value of gravity disturbance within the topography, i.e., between the geoid surface of which the geocentric radius is denoted by $\forall\,\Omega\in\Omega_{\mathrm{O}} : r_g(\Omega)$ and the physical surface of the earth $\forall\,\Omega\in\Omega_{\mathrm{O}} : r_t(\Omega)\cong r_g(\Omega)+H^{\mathrm{O}}(\Omega)$, where $H^{\mathrm{O}}(\Omega)$ stands for the orthometric height. The (integral) mean value of the gravity disturbance $\overline{\delta g}(\Omega)$ reads

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad \overline{\delta g}(\Omega)=\frac{1}{H^{\mathrm{O}}(\Omega)}\int_{r=r_g(\Omega)}^{r_g(\Omega)+H^{\mathrm{O}}(\Omega)}\delta g(r,\Omega)\,\mathrm{d}r\,, \tag{1}$$

where $\delta g(r,\Omega)$ is the gravity disturbance at point (r,Ω).

The geocentric position is described by the geocentric spherical coordinates ϕ and λ; $\Omega=(\phi,\lambda)$, $\Omega\in\Omega_{\mathrm{O}}$ ($\Omega_{\mathrm{O}}\in\langle-\pi/2\le\phi\le\pi/2;0\le\lambda<2\pi\rangle$), and the geocentric radius r; $r\in\Re^+$ $(\Re^+\in\langle\,0,+\infty\,))$.

According to Heiskanen and Moritz (1967, eqn. 2-142), the gravity disturbance $\delta g(r,\Omega)$ is defined as the difference between the actual gravity $g(r,\Omega)$ and the normal gravity $\gamma(r,\phi)$, so that $\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\in\Re^+$: $\delta g(r,\Omega)=g(r,\Omega)-\gamma(r,\phi)$. The gravity $g(r,\Omega)$ can further be decomposed into the geoid-generated gravity $g^{\mathrm{NT}}(r,\Omega)$ and the gravitational attraction of topographical and atmospheric masses $g^{t}(r,\Omega)$ and $g^{a}(r,\Omega)$, respectively, i.e., $\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\in\Re^+$: $g^{\mathrm{NT}}(r,\Omega)=g^{\mathrm{NT}}(r,\Omega)+g^{t}(r,\Omega)+g^{a}(r,\Omega)$. The gravity disturbance on the right-hand side of eqn. (1) then becomes

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\in\Re^+: \qquad \delta g(r,\Omega)=\delta g^{\mathrm{NT}}(r,\Omega)+g^{t}(r,\Omega)+g^{a}(r,\Omega), \tag{2}$$

where $\delta g^{\mathrm{NT}}(r,\Omega)$ is the geoid-generated gravity disturbance for which the following equation is valid (Vaníček et al., 2004)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\in\Re^+: \qquad \delta g^{\mathrm{NT}}(r,\Omega)=g^{\mathrm{NT}}(r,\Omega)-\gamma(r,\phi). \tag{3}$$

Regarding eqns. (1) and (2), the mean gravity disturbance $\overline{\delta g}(\Omega)$ can be evaluated so that the mean geoid-generated gravity disturbance $\overline{\delta g}^{\mathrm{NT}}(\Omega)$ and the mean values of the gravitational attraction of topographical and atmospheric masses $\overline{g}^{t}(\Omega)$ and $\overline{g}^{a}(\Omega)$ are computed separately.

The mean geoid-generated gravity disturbance $\overline{\delta g}^{\mathrm{NT}}(\Omega)$ is computed from (Tenzer et al., 2004)

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad \overline{\delta g}^{\mathrm{NT}}(\Omega)\cong\frac{1}{H^{\mathrm{O}}(\Omega)}\int_{r=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega)}\delta g^{\mathrm{NT}}(r,\Omega)\,\mathrm{d}r\,. \tag{4}$$

The mean value of the topography-generated gravitational attraction $\overline{g}^{t}(\Omega)$ is given by (Niethammer, 1932)

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad \overline{g}^{t}(\Omega)\cong\frac{1}{H^{\mathrm{O}}(\Omega)}\int_{r=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega)}g^{t}(r,\Omega)\,\mathrm{d}r\,. \tag{5}$$

The mean value of the gravitational attraction of atmospheric masses $\overline{g}^{a}(\Omega)$ is defined analogically with eqn. (5).

In the above equations and in all the equations in the sequel, the spherical approximation of the geoid surface by the mean radius of the earth R (Bomford, 1971) is used, and the deflection of the vertical from the geocentric radial direction (see Vaníček and Krakiwsky, 1986) is not considered. As explained in Vaníček et al. (2004), the spherical approximation of the geoid surface causes an error of the topography-generated gravitational attraction, and correspondingly an error of its mean value, at the most of a few microgals. Approximately the same order of error may be expected in the computation of the mean geoid-generated gravity disturbance. The ellipsoidal correction to the mean gravity disturbance, i.e., the mean value of the deflection of the vertical from the geocentric radial direction within the topography, has been investigated in Tenzer (2004). According to the numerical estimation of this correction, it can reach up to a few hundred microgals.

As mentioned in the Introduction, the geoid-generated gravity disturbance $\delta g^{\mathrm{NT}}(r,\Omega)$ multiplied by the geocentric radius r satisfies the Laplace equation for the exterior of the geoid (see Vaníček et al., 2004), i.e., $\forall\,\Omega\in\Omega_{\mathrm{O}},\ r>\mathrm{R}: \Delta\left[r\,\delta g^{\mathrm{NT}}(r,\Omega)\right]=0$. The geoid-generated gravity disturbance $\delta g^{\mathrm{NT}}(r,\Omega)$ on the right-hand side of eqn. (4) can then be evaluated applying Poisson's solution to the Dirichlet boundary value problem.

The Poisson integral reads (Kellogg, 1929)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\ r\geq\mathrm{R}: \qquad \delta g^{\mathrm{NT}}(r,\Omega)=\frac{1}{4\pi}\frac{\mathrm{R}}{r}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\mathrm{K}(r,\Omega;\mathrm{R},\Omega')\,\delta g^{\mathrm{NT}}(\mathrm{R},\Omega')\,\mathrm{d}\Omega'. \tag{6}$$

The spectral form of the Poisson integral kernel $\mathrm{K}(r,\Omega;\mathrm{R},\Omega')$ in eqn. (6) is given by (ibid)

$$\forall\,\Omega,\Omega'\in\Omega_{\mathrm{O}},\ r\geq\mathrm{R}: \qquad \mathrm{K}(r,\Omega;\mathrm{R},\Omega')=\sum_{n=0}^{\infty}(2n+1)\left(\frac{\mathrm{R}}{r}\right)^{n+1}\mathrm{P}_{\mathrm{n}}(\cos\psi), \tag{7}$$

where $\mathrm{P}_{\mathrm{n}}(\cos\psi)$ are the Legendre polynomials (Hobson, 1931; Mac Millan, 1930) for the argument of cosine of the spherical distance ψ.

Inserting the Poisson integral from eqn. (6) into eqn. (4), the mean value of the geoid-generated gravity disturbance $\overline{\delta g}^{\mathrm{NT}}(\Omega)$ becomes (Tenzer et al., 2004)

$\forall\,\Omega\in\Omega_{\mathrm{O}}$:

$$\overline{\delta g}^{\mathrm{NT}}(\Omega)=\frac{1}{4\pi}\frac{\mathrm{R}}{H^{\mathrm{O}}(\Omega)}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\int_{r=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega)}r^{-1}\,\mathrm{K}(r,\Omega;\mathrm{R},\Omega')\,\mathrm{d}r\;\delta g^{\mathrm{NT}}(\mathrm{R},\Omega')\,\mathrm{d}\Omega', \tag{8}$$

where $\frac{1}{H^{\mathrm{O}}(\Omega)}\int_{r=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega)}r^{-1}\,\mathrm{K}(r,\Omega;\mathrm{R},\Omega')\,\mathrm{d}r$ is the "mean Poisson's kernel" that has to be convolved with the $\delta g^{\mathrm{NT}}(\mathrm{R},\Omega)$ over the whole earth. The mean value $\overline{\delta g}^{\mathrm{NT}}(\Omega)$ can also be evaluated applying the analytical upward continuation in the integral mean. The analytical upward continuation is obtained by expanding the gravity disturbance $\delta g^{\mathrm{NT}}(r,\Omega)$ into the Taylor series starting at the geoid, so that

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\ r\geq\mathrm{R}: \qquad \delta g^{\mathrm{NT}}(r,\Omega)=\sum_{k=0}^{\infty}\frac{(r-\mathrm{R})^{k}}{k!}\left.\frac{\partial^{\mathrm{k}}\,\delta g^{\mathrm{NT}}(r,\Omega)}{\partial\,r^{\mathrm{k}}}\right|_{r=\mathrm{R}}. \tag{9}$$

Substituting eqn. (9) back to eqn. (4), the following solution is obtained

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad \overline{\delta g}^{\mathrm{NT}}(\Omega)=\delta g^{\mathrm{NT}}(\mathrm{R},\Omega)+\sum_{k=1}^{\infty}\frac{\left[H^{\mathrm{O}}(\Omega)\right]^{k}}{(k+1)k!}\left.\frac{\partial^{\mathrm{k}}\,\delta g^{\mathrm{NT}}(r,\Omega)}{\partial\,r^{\mathrm{k}}}\right|_{r=\mathrm{R}}. \tag{10}$$

Since the mean value $\overline{\delta g}^{\mathrm{NT}}(\Omega)$ in eqn. (10) is defined within the topography, the term $r-\mathrm{R}$ from eqn. (9) is equal to the orthometric height $H^{\mathrm{O}}(\Omega)$. The first term on the right-hand side of eqn. (10), i.e., the geoid-generated gravity disturbance $\delta g^{\mathrm{NT}}(\mathrm{R},\Omega)$ at the geoid surface, can be expressed in terms of the surface spherical harmonics $\delta\mathrm{g}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)$ (Heiskanen and Moritz, 1967, eqn. 1-86)

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad \delta g^{\mathrm{NT}}(\mathrm{R},\Omega)=\sum_{n=0}^{\infty}\delta\mathrm{g}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)=\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\delta g^{\mathrm{NT}}(\mathrm{R},\Omega')\,\mathrm{P}_{\mathrm{n}}(\cos\psi)\,\mathrm{d}\Omega'. \tag{11}$$

Computing the mean geoid-generated gravity disturbance $\overline{\delta g}^{\,\mathrm{NT}}(\Omega)$ according to eqns. (8) and (10), the geoid-generated gravity disturbances must be known at the geoid surface. To obtain the geoid-generated gravity disturbances $\delta g^{\mathrm{NT}}(\mathrm{R},\Omega)$ on the geoid from its values $\delta g^{\mathrm{NT}}[r_t(\Omega)]$ at the earth surface, the inverse Dirichlet's problem (described by the Poisson integral equation, see e.g., Vaníček et al., 1996) has to be solved. The different method for a computation of the mean geoid-generated gravity disturbance has been introduced in Tenzer et al. (2004). According to this method, the mean value is obtained directly from the values $\delta g^{\mathrm{NT}}[r_t(\Omega)]$ at the earth surface so that the Poisson integral equation is applied in the integral mean. As follows from eqns. (9) and (10), the radial derivatives of the geoid-generated gravity disturbances referred to the geoid surface are required in order to evaluate the analytical upward continuation. As shown in the next paragraph (see eqn. 27), these derivatives can be expressed in terms of spherical harmonics of the geoid-generated gravity disturbance. Thereby, all orders of the radial derivatives of $\delta g^{\mathrm{NT}}(r,\Omega)$ are defined as a function of the geoid-generated gravity disturbances $\delta g^{\mathrm{NT}}(\mathrm{R},\Omega)$ at the geoid surface, and thus can be computed from them for instance by numerical integration.

In Martinec (1998), the gravitational attraction $g^t(r,\Omega)$ is defined as a negative radial derivative of the gravitational potential $V^t(r,\Omega)$ of topographical masses. Thereby, the mean value $\bar{g}^t(\Omega)$ in eqn. (5) takes the following form

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad \bar{g}^t(\Omega) = -\frac{1}{H^{\mathrm{O}}(\Omega)}\int_{r=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega)} \frac{\partial\, V^t(r,\Omega)}{\partial\, r}\,\mathrm{d}r\,. \tag{12}$$

Furthermore, evaluating the integral on the right-hand side of eqn. (12), the equation can be rewritten as (Santos et al., 2003)

$$\forall\,\Omega\in\Omega_{\mathrm{O}}: \qquad \bar{g}^t(\Omega) = \frac{V^t(\mathrm{R},\Omega) - V^t[r_t(\Omega)]}{H^{\mathrm{O}}(\Omega)}\,. \tag{13}$$

The above equation states that the mean value of the gravitational attraction $\bar{g}^t(\Omega)$ is defined as the difference of the gravitational potentials $V^t(\mathrm{R},\Omega)$ and $V^t[r_t(\Omega)]$, which are referred to the geoid and the earth surface, and divided by the orthometric height $H^{\mathrm{O}}(\Omega)$.

Considering the laterally varying distribution of the topographical density $\rho(\Omega)$, for the definition of the laterally varying topographical density see Martinec (1993), the gravitational potential of topographical masses $V^t(r,\Omega)$ is defined by the Newton volume integral (e.g., Martinec, 1998, eqn. 3.3), which reads

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\in\Re^+: \qquad V^t(r,\Omega) \cong \mathrm{G}\iint_{\Omega'\in\Omega_{\mathrm{O}}} \rho(\Omega')\int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \ell^{-1}(r,\Omega;r',\Omega')\, r'^2\,\mathrm{d}r'\,\mathrm{d}\Omega'\,, \tag{14}$$

where G denotes Newton's gravitational constant, and $\ell(r,\Omega;r',\Omega')$ is the spatial distance of a two points (r,Ω) and (r',Ω').

The gravitational attraction of topographical masses $g^t(r,\Omega)$ is given similarly by (ibid)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\in\Re^+: \qquad g^t(r,\Omega) \cong -\mathrm{G}\iint_{\Omega'\in\Omega_{\mathrm{O}}} \rho(\Omega')\int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \frac{\partial\,\ell^{-1}(r,\Omega;r',\Omega')}{\partial\, r}\, r'^2\,\mathrm{d}r'\,\mathrm{d}\Omega'. \tag{15}$$

For the laterally varying distribution of topographical density, the radial derivative of topography-generated gravitational attraction $\partial\, g^t(r,\Omega)/\partial\, r$ together with the higher-order radial derivatives of $g^t(r,\Omega)$ exist at the interval $r\in\left\langle \mathrm{R},\, \mathrm{R}+H^{\mathrm{O}}(\Omega)\right\rangle$, if the radial derivatives $\forall\, k\in\left\langle 1, +\infty\right)$: $\partial^{\mathrm{k}+1}\ell^{-1}(r,\Omega;r',\Omega')/\partial\, r^{\mathrm{k}+1}$ of reciprocal value of the spatial distance exist at this interval. It has been shown in Tenzer et al. (2003) that all orders of the radial derivatives of the reciprocal spatial distance exist. Thus, the gravitational attraction $g^t(r,\Omega)$ as defined in eqn. (15) is analytical and for the evaluation of the mean value $\bar{g}^{t}(\Omega)$ the analytical continuation can be applied. The analytical continuation of gravitational attraction $g^{t}(r,\Omega)$ can also be formulated for any density distribution that is continuous in the radial direction.

By analogy with eqn. (9), the analytical upward continuation of gravitational attraction $g^{t}(r,\Omega)$ is given by

$$\forall\, \Omega\in\Omega_{\mathrm{O}},\, r\geq \mathrm{R}: \qquad g^{t}(r,\Omega)=\sum_{k=0}^{\infty}\frac{(r-\mathrm{R})^{k}}{k!}\left.\frac{\partial^{\mathrm{k}} g^{t}(r,\Omega)}{\partial r^{\mathrm{k}}}\right|_{r=\mathrm{R}} = -\sum_{k=0}^{\infty}\frac{(r-\mathrm{R})^{k}}{k!}\left.\frac{\partial^{\mathrm{k+1}} V^{t}(r,\Omega)}{\partial r^{\mathrm{k+1}}}\right|_{r=\mathrm{R}}. \tag{16}$$

Substituting eqn. (16) to eqn. (5), and integrating with respect to the geocentric radial distance r, the following result is obtained

$$\forall\, \Omega\in\Omega_{\mathrm{O}}: \qquad \bar{g}^{t}(\Omega)=g^{t}(\mathrm{R},\Omega)-\sum_{k=1}^{\infty}\frac{\left[H^{\mathrm{O}}(\Omega)\right]^{k}}{(k+1)\,k!}\left.\frac{\partial^{\mathrm{k+1}}\, V^{t}(r,\Omega)}{\partial r^{\mathrm{k+1}}}\right|_{r=\mathrm{R}}, \tag{17}$$

where (with in accordance with eqn. 10) the difference $r-\mathrm{R}$ is equal to the orthometric height $H^{\mathrm{O}}(\Omega)$.

3. Equivalence between Poisson's and analytical continuation

To proof the equivalence between Poisson's and analytical upward continuation as described by eqns. (6) and (9) respectively, the geoid-generated disturbing gravity potential $T^{\mathrm{NT}}(r,\Omega)$ is first expanded into a series of solid spherical harmonics $\mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)$. According to Heiskanen and Moritz (1967, eqn. 1-87b) the series reads

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\geq\mathrm{R}: \qquad T^{\mathrm{NT}}(r,\Omega)=\sum_{n=0}^{\infty}\left(\frac{\mathrm{R}}{r}\right)^{n+1}\mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega), \tag{18}$$

where

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, n\in\Im^{+}\ \left(\Im^{+}=0,1,\ldots,+\infty\right): \quad \mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)=\frac{2n+1}{4\pi}\iint_{\Omega'\in\Omega_{\mathrm{O}}} T^{\mathrm{NT}}(\mathrm{R},\Omega')\,\mathrm{P}_{\mathrm{n}}(\cos\psi)\,\mathrm{d}\Omega'. \tag{19}$$

Performing the negative radial derivative of eqn. (18), the relation between the geoid-generated gravity disturbance $\delta g^{\mathrm{NT}}(r,\Omega)$ and the solid spherical harmonics $\mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)$ is obtained

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\geq\mathrm{R}: \qquad \delta g^{\mathrm{NT}}(r,\Omega)=-\frac{\partial T^{\mathrm{NT}}(r,\Omega)}{\partial r}=\sum_{n=0}^{\infty}\frac{n+1}{\mathrm{R}}\left(\frac{\mathrm{R}}{r}\right)^{n+2}\mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega). \tag{20}$$

By analogy with eqn. (18), the geoid-generated gravity disturbance $\delta g^{\mathrm{NT}}(r,\Omega)$ can be expressed in terms of solid spherical harmonics $\delta \mathrm{g}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)$ as follows

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r\geq\mathrm{R}: \qquad r\,\delta \mathrm{g}^{\mathrm{NT}}(r,\Omega)=\mathrm{R}\sum_{n=0}^{\infty}\left(\frac{\mathrm{R}}{r}\right)^{n+1}\delta \mathrm{g}_{\mathrm{n}}^{\mathrm{NT}}(\Omega). \tag{21}$$

Comparing eqns. (20) and (21), the well-known relation between the spherical harmonics $\mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)$ and $\delta \mathrm{g}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)$ is obtained (Heiskanen and Moritz, 1967)

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, n\in\Im^{+}: \qquad \delta \mathrm{g}_{\mathrm{n}}^{\mathrm{NT}}(\Omega)=\frac{n+1}{\mathrm{R}}\mathrm{T}_{\mathrm{n}}^{\mathrm{NT}}(\Omega). \tag{22}$$

Substituting Poisson's integral kernel from eqn. (7) back to eqn. (6), the Poisson integral becomes

$$\forall\,\Omega\in\Omega_{\mathrm{O}},\, r>\mathrm{R}: \qquad \delta g^{\mathrm{NT}}(r,\Omega)=\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\left(1+\frac{r-\mathrm{R}}{\mathrm{R}}\right)^{-n-2}\iint_{\Omega'\in\Omega_{\mathrm{O}}}\delta g^{\mathrm{NT}}(\mathrm{R},\Omega')\,\mathrm{P}_{\mathrm{n}}(\cos\psi)\,\mathrm{d}\Omega'. \tag{23}$$

The interchange of summation and integration is permissible in eqn. (23), because the expansion into a series of the Legendre polynomials converges uniformly and absolutely for $r>\mathrm{R}$ (see, Moritz, 1980).

Considering the expression for the surface spherical harmonics $\delta g_n^{NT}(\Omega)$ in eqn. (11) and taking into account also the relation between the spherical harmonics $T_n^{NT}(\Omega)$ and $\delta g_n^{NT}(\Omega)$ in eqn. (22), the Poisson integral from eqn. (23) takes the following form

$$\forall\, \Omega \in \Omega_O,\, r > R: \qquad \delta g^{NT}(r,\Omega) = \sum_{n=0}^{\infty} \delta g_n^{NT}(\Omega)\left(1+\frac{r-R}{R}\right)^{-n-2} = \sum_{n=0}^{\infty} \frac{n+1}{R} T_n^{NT}(\Omega)\left(1+\frac{r-R}{R}\right)^{-n-2}. \qquad (24)$$

Assuming that $r-R$ is much smaller than R, the binomial theorem (e.g., Rektorys, 1968) can be applied in eqn. (24). It yields

$$\forall\, \Omega \in \Omega_O,\, r > R: \qquad \delta g^{NT}(r,\Omega) = \sum_{n=0}^{\infty} \frac{n+1}{R} T_n^{NT}(\Omega) \sum_{k=0}^{n} \binom{-n-2}{k}\left(\frac{r-R}{R}\right)^{k}. \qquad (25)$$

The first-three terms of the development with respect to k in eqn. (25) read

$$\forall\, \Omega \in \Omega_O,\, r > R: \qquad \delta g^{NT}(r,\Omega) = \sum_{n=0}^{\infty} T_n^{NT}(\Omega)\frac{(n+1)}{R} - \frac{r-R}{R}\sum_{n=0}^{\infty} T_n^{NT}(\Omega)\frac{(n+1)(n+2)}{R} + \left(\frac{r-R}{R}\right)^{2}\sum_{n=0}^{\infty} T_n^{NT}(\Omega)\frac{(n+1)(n+2)(n+3)}{2R} - \ldots, \qquad (26)$$

where the first term is (according to eqn. 11) equal to the geoid-generated gravity disturbance $\delta g^{NT}(R,\Omega)$ referred to the geoid surface.

The analytical upward continuation of the geoid-generated gravity disturbance is defined by eqn. (9), where the radial derivatives of $\delta g^{NT}(r,\Omega)$ can be written as follows

$$\forall\, \Omega \in \Omega_O,\, k \in \Im^{+}: \quad \left.\frac{\partial^k\, \delta g^{NT}(r,\Omega)}{\partial r^k}\right|_{r=R} = \sum_{n=0}^{\infty} \delta g_n^{NT}(\Omega)\binom{-n-2}{k}\frac{k!}{R^k} = \sum_{n=0}^{\infty}\frac{n+1}{R} T_n^{NT}(\Omega)\binom{-n-2}{k}\frac{k!}{R^k}. \qquad (27)$$

Substitution of eqn. (27) into eqn. (9) yields the expression identical to the expression in eqn. (25), which is obtained from the Poisson integral. Thereby, the proof of equivalence between Poisson's and analytical upward continuation is concluded.

Inserting eqn. (27) into the expression in eqn. (10), which is obtained so that the analytical upward continuation of the geoid-generated gravity disturbance is applied in the integral mean, the mean value of the geoid-generated gravity disturbance $\overline{\delta g}^{NT}(\Omega)$ can be described in terms of the solid spherical harmonics $T_n^{NT}(\Omega)$ as follows

$$\forall\, \Omega \in \Omega_O: \qquad \overline{\delta g}^{NT}(\Omega) = \sum_{n=0}^{\infty}\frac{n+1}{R} T_n^{NT}(\Omega)\sum_{k=0}^{\infty}\binom{-n-2}{k}\left(\frac{H^O(\Omega)}{R}\right)^{k}\frac{1}{k+1}. \qquad (28)$$

Since Poisson's and analytical upward continuation of the geoid-generated gravity disturbance is equivalent, the same formula as in eqn. (28) is obtained if the Poisson integral in eqn. (25) is substituted to the integral mean in eqn. (4).

4. Equivalence between the integral mean of gravitational attraction and the integral mean of analytically continued gravitational attraction

Studying Newton's volume integral in the spectral domain, the requirement of convergence of the series expansion into the Legendre polynomials has to be taken into account. Therefore, the radial component of the volume domain of Newton's integral is separated into the sub-intervals for $r < r'$ and $r > r'$, corresponding to the two convergence domains: the external and the internal. The reciprocal value of the spatial distance $\ell^{-1}(r,\Omega;r',\Omega')$ is then expanded into a series of the Legendre polynomials $P_n(\cos\psi)$ as follows (Hobson, 1931; see also Pick et al., 1973, eqn. D-14; 4 and 5, and D-18; 1)

$$\forall\, \psi \in \langle 0, \pi \rangle;\, r, r' > 0: \qquad \ell^{-1}(r,\Omega;r',\Omega') = \begin{cases} \dfrac{1}{r}\displaystyle\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n} \mathrm{P_n}(\cos\psi), & r \geq r', \\ \dfrac{1}{r'}\displaystyle\sum_{n=0}^{\infty}\left(\frac{r}{r'}\right)^{n} \mathrm{P_n}(\cos\psi), & r < r'. \end{cases} \tag{29}$$

From eqn. (29), the radial derivatives of the reciprocal spatial distance with respect to r are found to be $\forall\, \psi \in \langle 0, \pi \rangle;\, r, r' > 0,\, k \in \Im^{+}$:

$$\frac{\partial^{\mathrm{k}}\, \ell^{-1}(r,\Omega;r',\Omega')}{\partial\, r^{\mathrm{k}}} = \begin{cases} (-1)^{k}\, \dfrac{k\,!}{r'^{\,k+1}} \displaystyle\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n+k+1} \binom{n+k}{k} \mathrm{P_n}(\cos\psi)\,, & r > r', \\ \dfrac{k\,!}{r'^{\,k+1}} \displaystyle\sum_{n=k}^{\infty}\left(\frac{r}{r'}\right)^{n-k} \binom{n}{k} \mathrm{P_n}(\cos\psi)\,, & r < r'. \end{cases} \tag{30}$$

Inserting the first radial derivative of $\ell^{-1}(r,\Omega;r',\Omega')$ from eqn. (30) into eqn. (15), the spectral form of the topography-generated gravitational attraction $g^{t}(r,\Omega)$ is obtained

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r \in \Re^{+}:\;\; g^{t}(r,\Omega) = \mathrm{G} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \begin{cases} \displaystyle\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^{n+2} (n+1)\, \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r > r', \\ -\displaystyle\sum_{n=1}^{\infty}\left(\frac{r}{r'}\right)^{n-1} n\, \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r < r'. \end{cases} \tag{31}$$

The above expression for the gravitational attraction $g^{t}(r,\Omega)$ is herein defined only for the laterally varying topographical density distribution. If more precise information about topographical density exist, the 3D model can for instance be defined so that the vertical density distribution is a function of the geocentric radius r within each topographical column $r \in \langle \mathrm{R}, \mathrm{R} + H^{\mathrm{O}}(\Omega) \rangle$. A similar approach has been formulated for the atmospheric effect on gravitational potential and attraction by Sjöberg (1998 and 1999), and later investigated by Novák (2000). According to this approach, the vertical distribution of atmospheric density is defined as a function of atmospheric density at the sea surface and the height above the sea.

At the exterior of the geoid $(\Omega \in \Omega_{\mathrm{O}} \cap r > \mathrm{R})$, the gravitational attraction in eqn. (31) can be rewritten as $\forall\, \Omega \in \Omega_{\mathrm{O}},\, r > \mathrm{R}$:

$$g^{t}(r,\Omega) = \mathrm{G} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \begin{cases} \displaystyle\sum_{n=0}^{\infty}\left(\frac{r'}{\mathrm{R}}\right)^{n+2} \left(1 + \frac{r-\mathrm{R}}{\mathrm{R}}\right)^{-n-2} (n+1)\, \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r > r', \\ -\displaystyle\sum_{n=1}^{\infty}\left(\frac{\mathrm{R}}{r'}\right)^{n-1} \left(1 + \frac{r-\mathrm{R}}{\mathrm{R}}\right)^{n-1} n\, \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r < r'. \end{cases} \tag{32}$$

Applying the binomial theorem, eqn. (32) takes the following form $\forall\, \Omega \in \Omega_{\mathrm{O}},\, r > \mathrm{R}$:

$$g^{t}(r,\Omega) = \mathrm{G} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \begin{cases} \displaystyle\sum_{n=0}^{\infty}\left(\frac{r'}{\mathrm{R}}\right)^{n+2} \sum_{k=0}^{n} \binom{-n-2}{k} \left(\frac{r-\mathrm{R}}{\mathrm{R}}\right)^{k} (n+1)\, \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r > r', \\ -\displaystyle\sum_{n=1}^{\infty}\left(\frac{\mathrm{R}}{r'}\right)^{n-1} \sum_{k=0}^{n} \binom{n-1}{k} \left(\frac{r-\mathrm{R}}{\mathrm{R}}\right)^{k} n\, \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r < r'. \end{cases} \tag{33}$$

As mentioned in the Introduction, the analytical upward continuation of gravitational attraction is permissible only under a certain assumption for the topographical density distribution, while for the mean topography-generated gravitational attraction defined as the integral mean in eqn. (13) no assumption about topographical density distribution is needed. In this case, the proof of equivalence between the integral mean of gravitational attraction and the integral mean of the analytically upward continued gravitational attraction will be derived under the assumption of laterally varying topographical density.

The gravitational attraction $g^t(r,\Omega)$ for the lateral model of the topographical density expressed in the spatial domain is introduced in the previous section (see eqn. 33). In the next section, the identical expression is derived by using the analytical upward continuation of gravitational attraction. Consequently, the proof of equivalence between the integral mean of gravitational attraction and the integral mean of the analytically upward continued gravitational attraction is finally derived.

According to eqn. (16), the gravitational attraction $g^t(r,\Omega)$ can be evaluated using the analytical upward continuation of the same referred to the geoid surface. Substitution of eqn. (15) into eqn. (16) yields
$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r > \mathrm{R}$:

$$g^t(r,\Omega) = -\mathrm{G} \sum_{k=0}^{\infty} \frac{(r-\mathrm{R})^k}{k!} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \left. \frac{\partial^{\mathrm{k}+1}\, \ell^{-1}(r,\Omega;r',\Omega')}{\partial\, r^{\mathrm{k}+1}} \right|_{r=\mathrm{R}} r'^2\, \mathrm{d}r'\, \mathrm{d}\Omega' . \tag{34}$$

Inserting for the radial derivatives of the reciprocal spatial distance in eqn. (34) from eqn. (30) and considering that they are referred to the geoid surface, the gravitational attraction $g^t(r,\Omega)$ becomes
$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r > \mathrm{R}$:

$$g^t(r,\Omega) = \mathrm{G} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \begin{cases} \displaystyle\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \left(\frac{r'}{\mathrm{R}}\right)^{n+k+2} \left(\frac{r-\mathrm{R}}{r'}\right)^{k} (-1)^k (k+1) \binom{n+k+1}{k+1} \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r > r', \\ \displaystyle -\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty} \left(\frac{\mathrm{R}}{r'}\right)^{n-k-1} \left(\frac{r-\mathrm{R}}{r'}\right)^{k} (k+1) \binom{n}{k+1} \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r < r'. \end{cases} \tag{35}$$

Regarding the following identities

$$\left(\frac{r'}{\mathrm{R}}\right)^{n+k+2} \left(\frac{r-\mathrm{R}}{r'}\right)^{k} \equiv \left(\frac{r'}{\mathrm{R}}\right)^{n+2} \left(\frac{r-\mathrm{R}}{\mathrm{R}}\right)^{k}, \qquad \left(\frac{\mathrm{R}}{r'}\right)^{n-k-1} \left(\frac{r-\mathrm{R}}{r'}\right)^{k} \equiv \left(\frac{\mathrm{R}}{r'}\right)^{n-1} \left(\frac{r-\mathrm{R}}{\mathrm{R}}\right)^{k}, \tag{36}$$

and

$$(-1)^k (k+1) \binom{n+k+1}{k+1} \equiv (n+1) \binom{-n-2}{k}, \qquad (k+1)\binom{n}{k+1} \equiv n \binom{n-1}{k}. \tag{37}$$

it can be seen by comparing eqns. (35) and (33) that they are identical except for the different summation with respect to n starting from $\forall\, k \in \Im^+ : n = k+1$ instead of $n = 0$ in the series expansion for the external convergence domain $r < r'$. Substituting the identities from eqns. (36) and (37) back to eqn. (35), i.e.,
$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r > \mathrm{R}$:

$$g^t(r,\Omega) = \mathrm{G} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \begin{cases} \displaystyle\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \left(\frac{r'}{\mathrm{R}}\right)^{n+2} \left(\frac{r-\mathrm{R}}{\mathrm{R}}\right)^{k} (n+1) \binom{-n-2}{k} \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r > r', \\ \displaystyle -\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty} \left(\frac{\mathrm{R}}{r'}\right)^{n-1} \left(\frac{r-\mathrm{R}}{\mathrm{R}}\right)^{k} n \binom{n-1}{k} \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r < r', \end{cases} \tag{38}$$

and further, expanding the expression in eqn. (38) with respect to k, the following series is found
$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r > \mathrm{R}$:

$$g^t(r,\Omega) = \mathrm{G} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \times$$

$$\times \int_{r'=\mathrm{R}}^{\mathrm{R}+H^{\mathrm{O}}(\Omega')} \begin{cases} \displaystyle\sum_{n=0}^{\infty} \left(\frac{r'}{\mathrm{R}}\right)^{n+2} \left[1 - (n+2)\frac{r-\mathrm{R}}{\mathrm{R}} + \frac{(n+2)(n+3)}{2}\left(\frac{r-\mathrm{R}}{\mathrm{R}}\right)^2 - \ldots\right] (n+1)\, \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r > r', \\ \displaystyle -\sum_{n=1}^{\infty} \left(\frac{\mathrm{R}}{r'}\right)^{n-1} \left[1 + (n-1)\frac{r-\mathrm{R}}{\mathrm{R}} + \frac{(n-1)(n-2)}{2}\left(\frac{r-\mathrm{R}}{\mathrm{R}}\right)^2 + \ldots\right] n\, \mathrm{P_n}(\cos\psi)\, \mathrm{d}r'\, \mathrm{d}\Omega', & r < r', \end{cases} \tag{39}$$

The same result is obtained from eqn. (33). Thereby, the analytically upward continued gravitational attraction $g^t(r,\Omega)$ in eqn. (38) is equivalent to its direct definition by Newton's integral in the spatial domain (eqn. 33) described herein for the laterally varying topographical density.

By analogy with the above proof, the equivalence between the integral mean of gravitational attraction and the mean value of the analytically continued gravitational attraction can be shown.
Regarding eqn. (16), the gravitational potential of topographical masses $V^t[r_t(\Omega)]$ referred to the earth surface in eqn. (13) can be evaluated by applying the analytical upward continuation of gravitational potential, so that

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \quad V^t[r_t(\Omega)] = \sum_{k=0}^{\infty} \frac{[H^{\mathrm{O}}(\Omega)]^k}{k!} \left. \frac{\partial^{\mathrm{k}} V^t(r,\Omega)}{\partial\, r^{\mathrm{k}}} \right|_{r=\mathrm{R}} = V^t(\mathrm{R},\Omega) + \sum_{k=1}^{\infty} \frac{[H^{\mathrm{O}}(\Omega)]^k}{k!} \left. \frac{\partial^{\mathrm{k}} V^t(r,\Omega)}{\partial\, r^{\mathrm{k}}} \right|_{r=\mathrm{R}} . \tag{40}$$

Substituting eqn. (40) into eqn. (13), the integral mean of gravitational attraction of topographical masses takes the following form

$$\forall\, \Omega \in \Omega_{\mathrm{O}}: \quad \bar{g}^t(\Omega) = -\sum_{k=1}^{\infty} \frac{[H^{\mathrm{O}}(\Omega)]^{k-1}}{k!} \left. \frac{\partial^{\mathrm{k}} V^t(r,\Omega)}{\partial\, r^{\mathrm{k}}} \right|_{r=\mathrm{R}} \equiv -\sum_{k=0}^{\infty} \frac{[H^{\mathrm{O}}(\Omega)]^{k}}{(k+1)!} \left. \frac{\partial^{\mathrm{k+1}} V^t(r,\Omega)}{\partial\, r^{\mathrm{k+1}}} \right|_{r=\mathrm{R}} , \tag{41}$$

which is equal to the integral mean of the analytically upward continued gravitational attraction in eqn. (17).

5. Summary and Conclusions

Realizing that the geoid-generated disturbing gravity potential $T^{\mathrm{NT}}(r,\Omega)$ satisfies the Laplace equation within the topography while the topography-generated gravitational potential $V^t(r,\Omega)$ does not, this basic physical property dictates which mathematical method is to be used for the evaluation of the corresponding mean values. To evaluate the mean geoid-generated gravity disturbance within the topography, Poisson's solution to the Dirichlet boundary value problem or the analytical upward continuation are applied. As follows from the proof in section 3, these methods are equivalent.
To evaluate the mean value of the topography-generated gravitational attraction, two methods are available. Integrating the gravitational attraction of topographical masses within topography, its mean value is defined as the difference of the gravitational potential referred to the geoid and the earth surface divided by the orthometric height. The computation is then realized solving the Newton volume integral over the actual, or modeled topographical density distribution. For the analytical continuation approach, the laterally varying model of topographical density, or some such model has to be adopted. In paragraph 4, the proof of equivalence between the integral mean of gravitational attraction and the mean analytically upward continued gravitational attraction has been derived, assuming the lateral density distribution in both cases. This shows the restricted circumstances under which the analytical continuation may be used, while the potential approach does not incur any such restrictions.

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