On the discrete problem of downward continuation of Helmert's gravity

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Abstract. This paper discusses theroretical problems of the existence of downward continuation of Helmert's gravity, singularity of the Poisson kernel and regularity of the Poisson integral. We prove that the downward continuation of Helmert's gravity converges and exists above the geoid. Although the Poisson kernel is singular, the Poisson integral is regular and can be evaluated everywhere without any difficulty. We also numerically investigate the edge effect and the area size limitation. The results show that the edge effect is restricted to less than 1°. A large area can thus be decomposed into something like 3° × 3° overlapping areas, taking only the results from the internal 1° × 1°.

1 Introduction

To determine the geoid by applying Stokes's theory, disturbing potential or observed gravity values have to be reduced from the topographical surface to the geoid. This reduction is the so-called downward continuation. The disturbing potential or gravity values, however, can not be easily reduced since they are not harmonic in the space between the topographical surface and the geoid because of the topographical masses. One way to overcome the difficulty is simply declaring the topographical masses to have a zero density, i.e., the free-air model (Moritz, 1980; Bjerhammar, 1987). Another way to deal with the problem is the Helmert's condensation technique, which condenses the topographical masses onto the geoid by means of one of the condensation techniques, that may preserve either the total mass of the earth, or the location of the centre of mass, or to be just an integral mean of topographical column density (Wichiencharoen, 1982; Vaníček and Martinec, 1994; Martinec and Vaníček, 1994). We are interested in the Helmert model since it seems more physically reasonable than the free-air model. However, does a downward continuation of Helmert's gravity anomaly exist in the Helmert space between topography and the geoid? This is one of topics to be discussed in this paper (section 2). Our discussions show that the downward continuation converges and exists above the geoid.

It is known that the Poisson integral encountered here is a convolution of the Poisson kernel and the gravity anomaly on the geoid. The Poisson kernel increases when the angular distance goes to zero ($\psi \rightarrow 0$), and it also becomes infinit when the topographical height goes to zero ($\bar{H} \rightarrow 0$) . This singular problem was discussed and a variety of numerical procedures was presented (e.g., Shaofeng and Xurong, 1991; Martinec, 1995). We investigate the problem in section 3 and show that although the Poisson kernel is singular, the Poisson integral is regular and can be evaluated everywhere without any difficulty.

Vaníček et al., (1996) have studied the downward continuation of $5' \times 5'$ mean Helmert gravity anomaly. They proposed an iterative scheme to perform the downward continuation to obtain the Helmert gravity anomalies. They claimed that the determination of the downward continuation of mean $5' \times 5'$ Helmert's gravity anomalies is a well posed problem with a unique solution and can be done routinely to any accuracy desired in the geoid computation. However, since we have to perform the downward continuation over a limited area, the Poisson integration along the area boundary is incomplete. In practical calculations, this boundary effect has to be considered. In the section 4, we investigate how far the effect propagates into the area, and discuss the implications for the Poisson integration over a large area.

2 Existence of downward continuation of Helmert's gravity anomaly

We know that Helmert's potential T^h is harmonic everywhere outside the geoid [Heiskanen and Moritz, 1967], i.e.,

$$\forall r \geq r_g, \Omega: \quad T^h(r,\Omega) = \sum_{n=2}^{\infty}\sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} Y_{nm}(\Omega), \quad (1)$$

where a is the radius of Bjerhammar's sphere, r is the radial distance from the centre of the earth, r_g is the radial distance of the geoid, Ω stands for a geocentric direction given by latitude ϕ and longitude λ, and Y_{nm} are the scalar spherical harmonic functions which are expressed by spherical harmonic coefficients C_{nm}, S_{nm} and associated Legendre functions P_{nm} as

$$Y_{nm}(\phi,\lambda) = (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)P_{nm}(\sin\phi). \quad (2)$$

The series (1) is convergent.

To a spherical approximation, we have for the Helmert gravity anomaly

$$\forall r \geq r_g, \Omega: \quad \Delta g^h(r,\Omega) = \sum_{n=2}^{\infty}\sum_{m=0}^{n} \frac{n-1}{r}\left(\frac{a}{r}\right)^{n+1} \cdot Y_{nm}(\Omega). \quad (3)$$

Specifically,

$$\Delta g^h_g(\Omega) = \Delta g^h(r = r_g, \Omega), \quad (4)$$
$$\Delta g^h_t(\Omega) = \Delta g^h(r = r_t, \Omega), \quad (5)$$

are given by convergent series of the type (3). Thus the difference $D\Delta g^h = \Delta g^h_g - \Delta g^h_t$, which is nothing but the downward continuation of Helmert's gravity anomaly from the topographic surface to the geoid, gives

$$\forall r_t \geq r_g, \Omega: \quad D\Delta g^h = \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left[\frac{n-1}{r_g}\left(\frac{a}{r_g}\right)^{n+1} - \frac{n-1}{r_t}\left(\frac{a}{r_t}\right)^{n+1}\right] Y_{nm}(\Omega). \quad (6)$$

In the following, we investigate the convergence of eqn. (6), since convergence implies the existence of the downward continuation. Eqn. (6) can be rewritten as

$$\forall \Omega: \quad D\Delta g^h(\Omega) = \sum_{n=2}^{\infty}\sum_{m=0}^{n} Q_n Y_{nm}(\Omega), \quad (7)$$

where

$$Q_n = \frac{n-1}{r_g}\left(\frac{a}{r_g}\right)^{n+1}\left(1 - \frac{1}{(1+\frac{H}{r_g})^{n+2}}\right) = \frac{n-1}{r_g}\left(\frac{a}{r_g}\right)^{n+1} q_n, \quad (8)$$

where $r_t = r_g + H$ (H is the topographic height), and

$$q_n = 1 - \frac{1}{(1+\frac{H}{r_g})^{n+2}}. \quad (9)$$

Since, for $0 < H < 9$ km:

$$\frac{1}{(1+\frac{H}{r_g})^{n+2}} \in (0.9985, 1], \quad (10)$$

then we have

$$\forall n: \quad q_n \in [0, 1). \quad (11)$$

Let's write eqn. (3) for $r = r_g$ as

$$\forall \Omega: \quad \Delta g^h_g = \sum_{n=2}^{\infty}\sum_{m=0}^{n} \frac{n-1}{r_g}\left(\frac{a}{r_g}\right)^{n+1} Y_{nm}(\Omega) = \sum_{n=2}^{\infty}\sum_{m=0}^{n} R_n(r_g) Y_{nm}(\Omega). \quad (12)$$

Then eqn. (7) can be rewritten as:

$$\forall \Omega: \quad D\Delta g^h(\Omega) = \sum_{n=2}^{\infty}\sum_{m=0}^{n} q_n R_n(r_g) Y_{nm}(\Omega). \quad (13)$$

where

$$\forall n: \quad |q_n R_n(r_g)| < |R_n(r_g)| \quad (14)$$

Since series (12) converges absolutely, so must series (13): its coefficients are systematically smaller because of the ever decreasing quotient q_n.

Observe that for any finite n, $Q_n = q_n R_n(r_g)$ is finite and series (7) truncated at arbitrary high n is finite. Thus a discrete linear system of equations that discretize the Poisson integral always gives a convergent solution, because any cell size l corresponds to a finite number N ($N = 180°/l$), such that $n < N$. Therefore, the convergent series (13) implies that the downward continuation of Helmert's gravity exists above the geoid. It may, however, be unstable, but that is a different question which we will address in our next paper.

3 Singularity of Poisson kernel and regularity of Poisson integral

To get the downward continuation of Helmert gravity anomaly, we have to deal with the following Poisson integral

$$\Delta g^h_t(\Omega) = \frac{R}{4\pi r}\int_{\Omega'} \Delta g^h_g(\Omega') K(r,\psi,R) d\Omega', \quad (15)$$

where

$$K(r,\psi,R) = \sum_{j=2}^{\infty}(2j+1)\left(\frac{R}{r}\right)^{j+1} P_j(\cos\psi) = R\left[\frac{r^2 - R^2}{(R^2 + r^2 - 2Rr\cos\psi)^{\frac{3}{2}}} - \frac{1}{r} - \frac{3R}{r^2}\cos\psi\right] \quad (16)$$

is the Poisson kernel, ψ is the angular distance between geocentric directions Ω and Ω' and $P_j(\cos\psi)$ are the Legendre functions [Heiskanen and Moritz, 1967]. However, the Poisson kernel $K(r, \psi, R)$ is singular when $r = R$ and $\psi = 0$ and we have to check theoretically whether or not eqn. (15) can be integrated at the "singular" point; we have to check whether the Poisson integral is singular or regular. In the following H is the topographical height, and $H = r - R$. Let us first consider the following four cases:

1. $\underline{H \neq 0, \psi \neq 0}$:

   Since $H \neq 0$ and $\psi \neq 0$, eqn. (16) describes certainly a regular kernel. Then the Poisson integral (15) is regular everywhere.

2. $\underline{H \neq 0, \psi = 0}$:

   In this case, from eqn. (16) we have

$$\begin{aligned} K &= R^2 \frac{2 + \frac{H}{R}}{H^{1/2}} - \frac{4 + \frac{H}{R}}{(1 + \frac{H}{R})^2} \\ &\doteq \frac{2R^2}{H^{1/2}} - 4, \end{aligned} \tag{17}$$

   which is finite (for $H \neq 0$). It means that integral (15) is regular when $H \neq 0$.

3. $\underline{H = 0, \psi \neq 0}$:

   In this case, the kernel K becomes

$$K = -1 - 3\cos\psi, \tag{18}$$

   a well behaved function. Then eqn. (15) can be integrated without any problem.

4. $\underline{H = 0, \psi = 0}$:

   As mentioned above, K is singular

$$K = 2\delta(\psi) - 4, \tag{19}$$

   where $\delta(\psi)$ is the Dirac delta function. To discuss the problem conveniently, we divide K into two parts

$$K = K_\delta + K_0 \tag{20}$$
$$K_\delta = 2\delta(\psi) \tag{21}$$
$$K_0 = -4. \tag{22}$$

   We know that K_0 is only a constant and does not cause any problem in integrating eqn. (15). As to K_δ, we have

$$\frac{1}{4\pi}\int_{\Omega'} \Delta g_g^h(\Omega') K_\delta d\Omega' = \Delta g_g^h(\Omega), \tag{23}$$

   in which the reproducing property of δ-function is used

$$\int f(\psi)\delta(\psi)\sin\psi d\psi = f(0). \tag{24}$$

The above discussion indicates that eqn. (15) can be integrated without much of a problem.

Furthermore, when $H = 0$, $D\delta g^h$ vanishes, i.e.,

$$\begin{aligned} \lim_{H\to 0} D\delta g^h(\Omega) &= \lim_{H\to 0}\left[\left.\frac{\partial T^h}{\partial H}\right|_g - \left.\frac{\partial T^h}{\partial H}\right|_t\right] \\ &= \lim_{H\to 0}\left[\left.\frac{\partial^2 T^h}{\partial H^2}\right|_g H + \left.\frac{\partial^3 T^h}{\partial H^3}\right|_g \frac{H^2}{2} + \cdots\right] \\ &= 0, \end{aligned} \tag{25}$$

or

$$\lim_{H\to 0} g_t^h(\Omega) = g_g^h(\Omega). \tag{26}$$

This implies that there is no need to do any downward continuation where we are already on the geoid. So we do not have to actually consider the above cases 3 and 4 at all. We therefore conclude that the Poisson integral (15) is a regular problem and can be integrated everywhere without any difficulty.

4 The boundary effect

Since we have to perform the downward continuation over a limited area, the Poisson integration along the area boundary is incomplete. This boundary effect has to be considered in the computations. In the following, we investigate how far the effect propagates into the computational area, i.e., how much larger the data coverage should be.

To do that, we can perform the downward continuation for a large area and a small area, with the small area being completely immerged in the large area. The difference of the two results in the small area is due to nothing else but the boundary effect. we have taken first a small area of $17° \times 22°$ in the Canadian Rocky Mountains. Then we have extended the $17° \times 22°$ area by 2° in the north and south direction and by 7° in the east and west direction, ending up in a larger area of $21° \times 36°$. The mean $5' \times 5'$ heights in this large area are between 0 m and 3612 m. We have performed the downward continuations for the two areas. Then subtracting the results of downward continuation for the $17° \times 22°$ area from the results obtained for the $21° \times 36°$ area, we get the difference showed in **Figure 1** which represents the boundary effect in the $17° \times 22°$ area. The effect is also showed in profiles at 10 latitudes (**Figure 2**). We see that the effect is height dependent, but it is restricted to less than 1°. **Figure 3** shows the relation between the maximum (for the worst case) absolute effect and the distance from the edge. Therefore the downward continuation is not much of a problem as it can be computed for areas of reasonably small geographical extent.

The point is that a large area, e.g., the whole of Canada, can be decomposed into something like $3° \times 3°$ overlapping areas, taking only the results from the internal $1° \times 1°$ area. This approach significantly reduces the size of the systems of equations that have to be solved

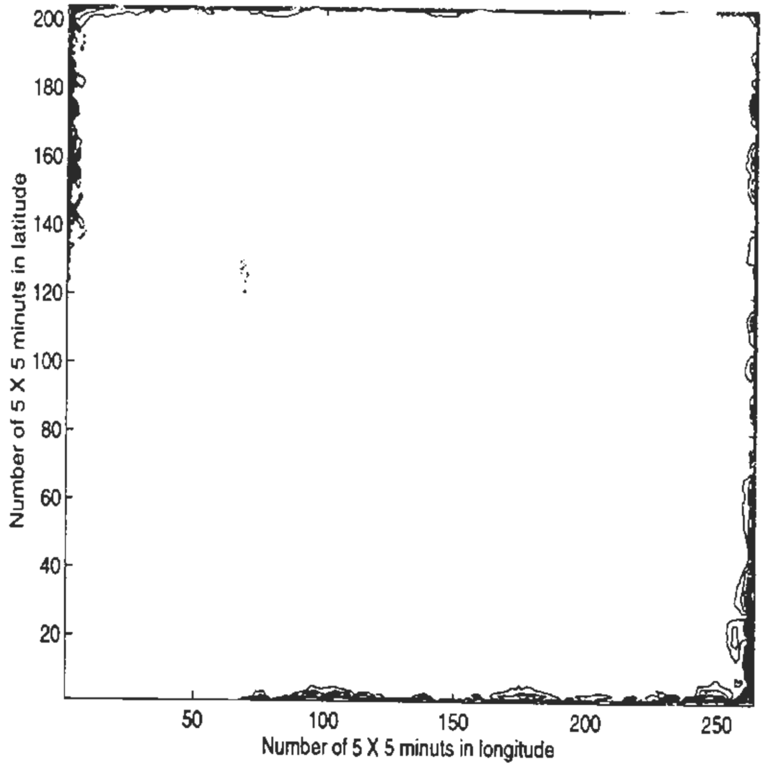


Figure 1: Contour lines of the boundary effect in the 17° × 22° area, contour interval is 0.1 mGal

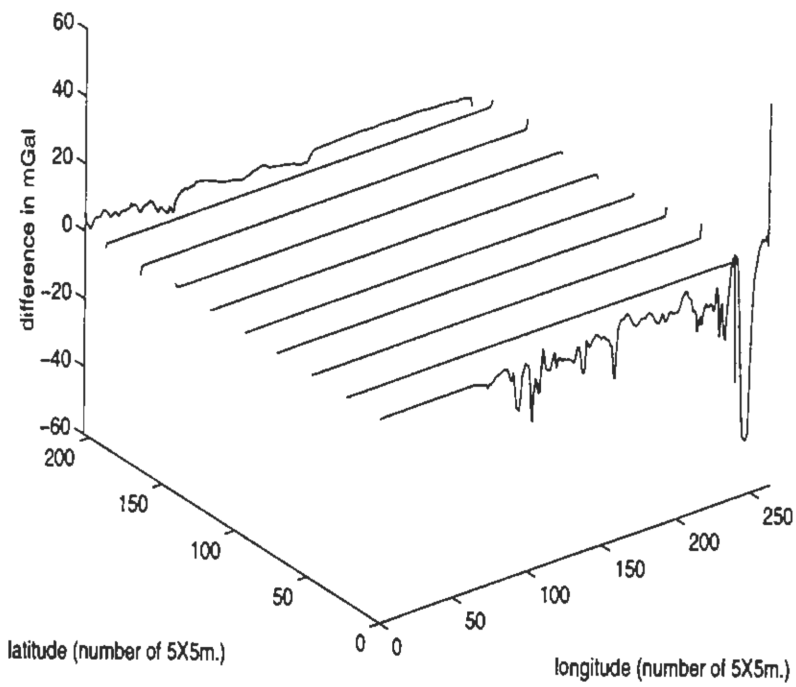
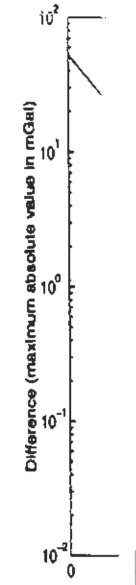


Figure 2: Profiles of the differences at 10 latitudes



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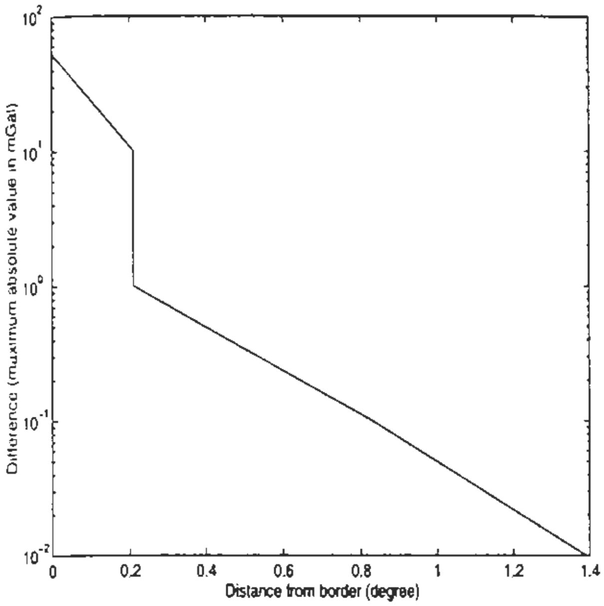

Figure 3: Relation between the maximum absolute difference and the distance from the edge

[Vaníček et al., 1996], which makes the numerical evaluation much more economical.

5 Summary and acknowledgments

Finally, we summarize the above discussions as follows. We have theoretically discussed the existence of downward continuation of Helmert's gravity, the singularity of the Poisson kernel and the regularity of the Poisson integral. We have proved that the downward continuation of Helmert's gravity exists. Discussions showed that although the Poisson kernel is singular, the Poisson integral is regular and can be evaluated everywhere without any difficulty.

We have also numerically investigated the edge effect and the computational area size limitation. The results indicate that the edge effect is restricted to less than 1°. A large computational area can be thus decomposed into something like 3° × 3° overlapping blocks, taking only the results from the internal 1° × 1°.

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