THE DOWNWARD CONTINUATION ERROR IN GEOID COMPUTATION FROM SATELLITE DERIVED GEOPOTENTIAL MODELS

by

Lars E. Sjöberg
Royal Institute of Technology
Department of Geodesy and Photogrammetry
S-100 44 STOCKHOLM, Sweden

0. ABSTRACT

This paper deals with the problem of applying satellite derived Earth gravity field models at the geoid. Rigorously, the external harmonic series cannot be used inside the sphere bounding all topographic masses. Its error for geoid computation is derived to the second power of terrain elevation (H) in two ways: a direct approach and a Helmert terrain condensation approach. Assuming a constant density ρ of the terrain both approaches lead to the geoid error

$$\frac{2\pi\mu}{\gamma}\sum_{n=0}^{M}\sum_{m=-n}^{n}(H^2)_{nm}Y_{nm}$$

for an Earth gravity field model complete to degree and order M. Here $\mu = G\rho$, G is the gravitational constant, γ is mean sea-level gravity and $(H^2)_{nm}$ are the coeffients of the terrain elevation squarred in terms of fully normalized spherical harmonics Y_{nm}. As a result, the limiting error of the above formula for $M \to \infty$,

$$2\pi\mu H^2/\gamma,$$

reaches 9 metres for M. Everest. The minus of the formula can be used as a correction.

1. INTRODUCTION

Today the long wavelength features of the Earth's exterior gravity field are successfully determined by dynamic satellite techniques. In the near future we can expect also intermediate wavelengths to be estimated through satellite gradiometry and satellite-to-satellite tracking. The result of these efforts is the representation of the Earth's gravity field by an exterior type of series of solid spherical harmonics truncated at some maximum degree and order. Theoretically, such a representation is correct outside the bounding sphere enclosing all mass of the Earth. As emphasized by Sjöberg (1977, 1980 and 1985) the convergence of such a series inside the bounding sphere is doubtful. Also Jekeli (1981 and 1982) studied the downward continuation error of height anomalies and gravity anomalies. He concluded that "the estimation of point or mean gravity anomalies and geoid undulations (height anomalies) using the outer series expansion to degree 300 anywhere on the earth's surface is practically unaffected by the divergence of the total series".

Grafarend and Engels (1994) presented a convergent series expansion of the gravitational potential. In the topographic domain the series is consistent with the theory developed in Sjöberg (1977).

Below we will firstly approach the downward continuation error following the line of Sjöberg (1977). Secondly we will use Helmert's second condensation method to reduce and restore the terrain masses. Cf. Sjöberg (1993).

2. THE DIRECT APPROACH

The contribution to the geopotential at an arbitrary point P of the masses of the Earth's topography above sea-level of radius R is given by

$$V^t(P) = G \iint_\sigma \int_R^{r_s} \frac{\rho}{\ell} r^2 dr\, d\sigma \tag{1}$$

where

ρ = density of topographic mass

r_s = Earth's surface radius; $r_s \le R_1$ = radius of bounding sphere.

r = radius of volume element under integral

σ = unit sphere

$\ell = (r_P^2 + r^2 - 2 r_P r t)^{1/2}$

$t = \cos\psi$; ψ = geocentric angle.

From now on we will restrict the presentation to the approximation $\mu = G\rho$ = constant. Outside the bounding sphere enclosing all mass of the Earth (i.e. for $r_P \geq R_1$) formula (1) can be expanded in a convergent exterior series of Legendre's polynomials $P_n(t)$:

$$V'(P) = \frac{\mu}{r_P} \sum_{n=0}^{\infty} \iint_{\sigma} \int_{R}^{r_s} \left(\frac{r}{r_P}\right)^n r^2 \, dr \, P_n(t) d\sigma . \tag{2}$$

In practise such an exterior series $\acute{V}(P)$, truncated at some maximum degree M, is used to represent the long wavelength geopotential also below the bounding sphere:

$$\hat{V}'(P) = \frac{\mu}{r_P} \sum_{n=0}^{M} \iint_{\sigma} \int_{R}^{r_s} \left(\frac{r}{r_P}\right)^n r^2 dr \, P_n(t) \, d\sigma . \tag{3}$$

The correct representation between sea-level and the bounding sphere (i.e. $R \leq r_P \leq R_1$) is (cf. Sjöberg 1993 and Grafarend and Engels 1994):

$$V'(P) = \frac{\mu}{r_P} \sum_{n=0}^{M} \left\{ \iint_{\sigma_2} \int_{R}^{r_s} \left(\frac{r}{r_P}\right)^n r^2 dr \, P_n(t) d\sigma + \iint_{\sigma_1} \int_{R}^{r_P} \left(\frac{r}{r_P}\right)^n r^2 dr \, P_n(t) d\sigma + \iint_{\sigma_1} \int_{r_P}^{r_s} \left(\frac{r_P}{r}\right)^{n+1} r^2 dr \, P_n(t) d\sigma \right\}, \tag{4}$$

where σ_1 is that part of σ where $r_P < r_s$ and $\sigma_2 = \sigma - \sigma_1$. Hence the downward continution error of $\hat{V}$.

becomes

$$\delta V(P) = \hat{V}(P) - V(P) = \frac{\mu}{r_P} \sum_{n=0}^{M} \iint_{\sigma_1} \int_{r_P}^{r_s} \left\{ \left(\frac{r}{r_P}\right)^n - \left(\frac{r_P}{r}\right)^{n+1} \right\} P_n(t) \, d\sigma . \tag{5}$$

Inserting

$$P_n(t) = \frac{1}{2n+1} \sum_{m=-n}^{n} Y_{nm}(P) \, Y_{nm}(Q) \tag{6a}$$

where Y_{nm} is a fully normalized spherical harmonic, obeying

$$\frac{1}{4\pi}\iint_{\sigma} Y_{nm} Y_{n'm'} d\sigma = \begin{cases} 1 & \text{if } n = n' \text{ and } m = m' \\ 0 & \text{otherwise} \end{cases} \tag{6b}$$

and Q is the running point under the integral, one obtains (cf. Sjöberg, 1977, p. 16)

$$\delta V(P) = \sum_{n=0}^{M} \delta V_n(P) \ , \tag{7a}$$

where

$$\delta V_n(P) = \sum_{m=-n}^{n} a_{nm}(r_P)\, Y_{nm}(P) \tag{7b}$$

$$a_{nm}(r_P) = \frac{\mu}{2n+1}\iint_{\sigma} I(r_P, r_S) Y_{nm}\, d\sigma \tag{7c}$$

$$I(r_P, r_S) = r_P^2 \begin{cases} 0 & \text{if } r_P \geq r_S \\ \dfrac{(r_S / r_P)^{n+3} - 1}{n+3} + \dfrac{(r_S / r_P)^{-(n-2)} - 1}{n-2} & \text{if } r_P < r_S \text{ and } n \neq 2 \\ \dfrac{(r_S / r_P)^5 - 1}{5} - \ell n(r_S / r_P) & \text{if } r_P < r_S \text{ and } n = 2 \end{cases} \tag{7d}$$

In Sjöberg (ibid) it was shown that formula (7d) can be expanded into the series

$$I(r_P, r_S) = (2M+1)(\Delta H)^2 \left\{ \frac{1}{2} + \frac{1}{3}\frac{\Delta H}{r_P} + \frac{n(n+1)}{2 \cdot 3 \cdot 4}\left(\frac{\Delta H}{r_P}\right)^2 + \ldots \right\}, \tag{8a}$$

where

$$\Delta H = \begin{cases} 0 & \text{if } r_P \geq r_S \\ r_S - r_P & \text{otherwise} \end{cases} \tag{8b}$$

Using the well-known Bruns' formula, formula (6a), (7) and (8a) we thus obtain the approximate downward continuation error for geoid estimation (with $r_P = R$)

$$\delta N_D^M = \frac{1}{\gamma}\sum_{n=0}^{M}\sum_{m=-n}^{n} a_{nm}(R)Y_{nm}(P) \tag{9a}$$

where

$$a_{nm}(R) = 2\pi\mu\frac{1}{4\pi}\iint_\sigma H^2Y_{nm}\,d\sigma = 2\pi\mu(H^2)_{nm} \tag{9b}$$

Letting M go to infinity we get the limiting downward continuation error for the geoid estimate from a complete set of exterior type of coefficients:

$$\delta N_D = \frac{2\pi\mu}{\gamma}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(H^2)_{nm}Y_{nm}(P) = \frac{2\pi\mu}{\gamma}H_P^2 \tag{10}$$

This error is demonstrated in Table 1, revealing that the error may reach 9 metres for M. Everest.

Table 1. The limiting downward continuation error for the geoid. $\rho = 2.67\ g/cm^3, \gamma = 981\ Gal$.

$H_P[km]$	0.1	0.5	1.0	2.0	3.0	4.0	5.0	6.0	8.848
$\delta N_D[m]$	0.001	0.029	0.114	0.456	1.027	1.825	2.852	4.106	8.93

Note that the finite error demonstrated in formula (10) and Table 1 is based on the omission of all but the first term of eq. (8a). A more rigorous error estimate is obtained from formula (7), if we allow μ to vary laterally:

$$\delta N_D^M = \frac{1}{\gamma}\sum_{n=0}^{M}\iint_\sigma \mu\, I(r_P, r_S)P_n(t)\,d\sigma \tag{11}$$

3. THE HELMERT CONDENSATION APPROACH

In this section we will use the second condensation method of Helmert (1884) to reduce the terrain. The reduction implies that the exterior harmonic series of the geopotential is valid all the way down to sea-level. However, the reduction of the terrain necessitates a direct, a first and second indirect effect be applied to correctly estimate the geoid.

3.1 The direct effect

At satellite altitude the exterior series (2) of the terrain potential is convergent. After integration with respect to r we get from (2)

$$V^t(P) = \mu \sum_{n=0}^{\infty} \frac{1}{n+3} \iint_{\sigma} \frac{r_S^{n+3} - R^{n+3}}{r_P^{n+1}} P_n(t)\, d\sigma \,, \tag{12}$$

or, with $r_S = H + R$ and omitting terms of power $(H/R)^3$ and higher

$$V^t(P) = \mu R \sum_{n=0}^{\infty} \left(\frac{R}{r_P}\right)^{n+1} \iint_{\sigma} \left\{ H + \frac{n+2}{2} \frac{H^2}{R} \right\} P_n(t) d\sigma . \tag{13}$$

This is the total terrain potential. The corresponding Helmert condensation potential for any point $r_P \geq R$ becomes

$$V^H(P) = \mu R^2 \iint_{\sigma} \frac{H}{\ell} d\sigma = \mu R \sum_{n=0}^{\infty} \left(\frac{R}{r_P}\right)^{n-1} \iint_{\sigma} H\, P_n(t) d\sigma \tag{14}$$

Hence the direct effect of the Helmert condensation to the application of the external geopotential series to degree and order M becomes

$$\delta N_{dir}^{M} = \frac{1}{\gamma}\left(-V^t_{r_P=R} + V^H_{r_P=R}\right) = -\frac{\mu}{2\gamma} \sum_{h=0}^{M} (n+2) \iint_{\sigma} H^2 P_n(t) d\sigma = -\frac{2\pi\mu}{\gamma} \sum_{n=0}^{M} \frac{n+2}{2n+1} (H^2)_{nm} Y_{nm}(P) \tag{15}$$

3.2 The primary indirect effect

For $r_P \leq R$ the convergent expansion of (1) is of the interior type:

$$V^t(P) = \frac{\mu}{r_P} \sum_{n=0}^{\infty} \iint_{\sigma} \int_{R}^{r_S} \left(\frac{r_P}{r}\right)^{n+1} r^2 dr\, P_n(t)\, d\sigma \tag{16}$$

or, after integration

$$V^{1}(P)=\mu\iint_{\sigma}\left\{\sum_{n=2}\frac{1}{n-2}r_P^{n}\left(R^{-n+2}-r_s^{-n+2}\right)P_n(t)-r_P^{2}P_2(t)\ell n\left(r_s/R\right)\right\}d\sigma. \quad (17)$$

Inserting

$$r_S = R + H$$

we obtain also {after omitting terms of power higher than $(H/R)^2$}:

$$V^{1}(P)=\mu R\sum_{n=0}^{\infty}\left(\frac{r_P}{R}\right)^{n}\iint_{\sigma}\left\{H-\frac{n-1}{2}\frac{H^2}{R}\right\}P_n(t)\,d\sigma\,. \quad (18)$$

Subtracting the Helmert potential (14) we obtain the limiting primary indirect geoid effect (for $r_P = R$).

$$\delta N_{I1}=-\frac{\mu}{2\gamma}\sum_{n=0}^{\infty}(n-1)\iint_{\sigma}H^2P_n(t)d\sigma=-\frac{2\pi\mu}{\gamma}\sum_{n,m}\frac{n-1}{2n+1}(H^2)_{nm}Y_{nm}(P). \quad (19)$$

3.3 The secondary indirect effect

Before applying Stokes' formula, the Helmert corrected gravity anomalies must be reduced from the geoid to the cogeoid (Heiskanen and Moritz 1967, p. 142). This is done by free-air reduction, i.e. by adding the correction $c^{-1}\delta N_{I1}$, where $c = R/(2\gamma)$, to each gravity anomaly. This yields the **secondary indirect effect** on the geoid

$$\delta N_{I2}=\frac{c}{2\pi}\iint_{\sigma}S(\psi)\,c^{-1}\delta N_{I1}\,d\sigma\,, \quad (20a)$$

where

$$S(\psi)=\sum_{n=2}^{\infty}\frac{2n+1}{n-1}P_n(t). \qquad \text{(Stokes' function)} \quad (20b)$$

Using formulas (6) and (20) we get

$$\delta N_{I2} = -\frac{4\pi\mu}{\gamma}\sum_{n,m}\frac{1}{2n+1}(H^2)_{nm}Y_{nm}(P) \tag{21}$$

This formula shows that the secondary indirect effect is significantly reduced for short wavelength compared with the primary indirect effect.

3.4 The total effect

Disregarding the secondary indirect effect we get the following total correction to the geoid for the Helmert reduction:

$$\delta N^M_{tot.} = \delta N^M_{dir} + \delta N^M_{I1} = -\frac{2\pi\mu}{\gamma}\sum_{n=0}^{M}\sum_{m=-n}^{n}(H^2)_{nm}Y_{nm}(P), \tag{22}$$

which is the minus of the downward continuation error δN^M_D derived in section 2.

4. CONCLUSIONS

We have derived the error in the downward continuation of satellite derived harmonic coefficients of the gravitational potential in two different ways. The first result, eqn. (5), is a volume integral over all terrain masses located above the computation point. Expanding the integrand of (5) as a power series in terrain elevation H over sea-level radius R, the first no-zero term $(H/R)^2$ contributes by $2\pi\mu(H^2)_{nm}/\gamma$ to the spectral geoid error. This implies that the total geoid error $2\pi\mu H^2/\gamma$ may reach 9 metres. This shows that the error is most significant for high mountain areas, being of the same order of magnitude as the indirect effect in geoid computation. The minus of the error is useful for correcting geoid estimates from satellite derived harmonic coefficients.