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Numerical evaluation of mean values of topographical effects

Research Article

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Abstract:

The main problem treated in this paper is the determination of accurate mean values of the topographical effects from point values known on a regular geographical grid. Three kinds of topographical effects are studied: terrain correction, condensed terrain correction and direct topographical effect. The relation between the terrain roughness and optimal density of the points to be used in the computations is investigated in five morphologically different areas of Canada. The error of the geoid caused by the inaccuracy of the mean values computed from a variable number of points in a cell is estimated. These errors are then compared against the one centimetre target to give us the sufficient minimum number of points needed for the averaging. The mean terrain effects are computed from the point values as a simple average over a particular cell. Point values are assumed to be errorless so that the accuracy of the mean values is a function of the density of the point values only. The mentioned one centimetre criterion is applied in the sense of the Chebyshev norm. It has been observed that the relation between the number of points needed for the averaging and the terrain roughness as quantified by the terrain RMS is almost linear. After estimating the two parameters of this linear relation, seven minimally required grid densities are suggested for different intervals of terrain roughness. The results are applied to produce maps of a minimal density of points needed for sufficiently accurate determination of mean topographical effects for Canada.

Keywords:

Direct topographical effects • geoid • Helmert condensation • Stokes-Helmert method • terrain correction

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1. Introduction

In geodesy it is often necessary to integrate a function, given at a number of discrete points, over the surface of the earth or over some other surface. To evaluate such a surface integral a numerical integration technique must be applied. It is intuitively obvious that the larger the number of points, the better representation of a continuous function we get. On the other hand, with the growing number of point values the speed of computation goes down. Let's start by writing the following equation:

$$y = \oiint_{\Omega_0} x(\Omega)\, d\Omega, \tag{1}$$

where the integration area can be divided into sub-areas $C_i \equiv (\Delta\varphi_i. \Delta\lambda_i) \equiv \Delta\Omega_i$ belonging to the individual regular cells C_i and write the above integral as a sum of integrals over all these cells

$$y = \sum_{i=1}^{n} \oiint_{C_i} x(\Omega)\, d\Omega. \tag{2}$$

This is an exact operation if the function x is integrable, which we automatically assume here. We note that the integrals in this equation are nothing else but mean values of x in the cells C_i defined in an integral sense. Let us denote these integral averages by $\bar{x}_i$; in the above integrals they are multiplied by the area C_i.

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Substituting all this back into the Eq. (2) we get:

$$y = \sum_{i=1}^{n} \bar{x}_i C_i = \sum_{i=1}^{n} \bar{x}_i \Delta\Omega_i. \tag{3}$$

We note that this is again an exact equation. In our case, the only approximation is that the integral average would be replaced by an algebraic average. It is interesting that the last equation is nothing else but a prescription for numerical integration by means of finite elements.

Two questions arise: how to obtain the mean values and how to make sure they are accurate enough for our purpose? The usual way of determining the mean value is to apply the straight algebraic average of all point values within a particular cell C_i. The accuracy of so determined mean value can be estimated only by comparison with other mean values determined within the same cell C_i that use either a larger or smaller number of point values. Let us note that there are also other techniques that may estimate the mean value, e.g. Gauss-Legendre quadrature, however they are not treated in this paper.

It is a known fact that the terrain effects in flat areas are smooth functions with small amplitudes. In the mountains, however, they become more interesting as they somehow reflect the pattern of the terrain itself (see Janák et al., 2006). Let us introduce three terrain effects that we are focusing on in this paper: Terrain correction, condensed terrain correction and direct topographical effect. When doing so, we will follow the notation of Martinec (1998) and Novák (2000), where A^{tr} is the terrain correction (or negative attraction of the topographical roughness), A^{ctr} is the condensed terrain correction (or negative attraction of the condensed topographical roughness) and the δA^t is the direct topographical effect on gravity.

Terrain correction A^{tr} is well known and widely used in geodesy and geophysics. It is usually defined on the Earth surface as being the correction that removes the gravity effect of the surplus masses above and mass deficiency within the Bouguer shell. It is a part of the refined Bouguer reduction (Torge, 1989, Eq. (4.38)) or (Heiskanen and Moritz, 1967, Eq.(3-21)). For geophysical purposes it is usually computed over a spherical cap of radius 166.7 km, as introduced by Bullard (1936) and recommended also by, e.g., Pick et al. (1961). In geodesy, however, especially for geoid determination, we are interested in determining this and all the other topographical effects from all around the Earth.

From a practical point of view the integration domain is split into near-zone and far-zone. The near-zone integration involves Digital Elevation Models (DEMs), both detailed and global, while the far-zone effect can be estimated using global elevation coefficients, e.g., TUG87 (Wieser, 1987). For precise geoid computation it is suggested by Novák (2000) to choose the near-zone as a spherical cap with the radius ψ of three degrees. This is also the integration domain that is used for computing all the effects throughout this paper.

Condensed terrain correction A^{ctr} is mostly used in geodesy in the process called the regularisation of the geoid (Pick et al., 1973). In analogy to terrain correction it can be defined as a correction that removes the gravity effect of the excess mass above the Bouguer shell as well as the mass deficiency within the Bouguer shell both condensed into an infinitesimally thin layer. This correction can be computed in an arbitrary point on or above the "regularised geoid". According to the choice of the condensation method we can obtain different condensed terrain corrections. One particular choice is the second Helmert condensation method (Helmert, 1884) where the condensation layer is located directly on the geoid. Some discussions on how to optimally include this correction into the geoid determination process can be found in recent papers, (e.g., Vaníček and Martinec, 1994; Heck, 2003; Ellmann and Vaníček, 2005). In this paper we investigate the behaviour of A^{ctr} on the topographical surface generated by the mass of the terrain roughness condensed according to Helmert's second condensation method.

Direct topographical effect δA^t is simply a difference between the two previous corrections taken with a negative sign, hence the difference between effects and corrections,

$$\delta A^t = A^{tr} - A^{ctr}. \tag{4}$$

If we disregard the atmosphere, this effect can be viewed as a difference between the gravity generated by the real Earth and so called Helmert's gravity generated by the geoid regularised using Helmert's condensation method. The direct topographical effect is the largest effect in the compilation of the input gravity values for precise geoid computation and therefore it is important to compute it precisely and get rid of all possible systematic errors.

2. Design of the investigation

The basic aim of our investigation here is to find an empirical relation between the terrain roughness, expressed, e.g., by the root mean square (RMS) of elevations

$$RMS = \sqrt{\frac{1}{m} \sum_{i=1}^{m} \left(H_i - \bar{H} \right)^2} \tag{5}$$

and the minimally acceptable density of the point values to be used for the algebraic averaging for all the particular terrain effects. These point values of the terrain effects are computed by different numerical integrations and, naturally, the more values we compute the longer it takes to compute them. Thus in any cell, for which we want to determine the mean values of an effect, there should be as few point values as possible, to make the computations as fast as possible, yet there should be enough point values to make sure that the algebraic averaging is a good enough approximation of the integral averaging for the desired accuracy. The minimally acceptable number of points in a cell would generally be different for each of the three effects.

The investigation was designed to conform to the following algorithm:

1. Select several (L) areas of $2^\circ \times 2^\circ$, in our case in Canada, that have different morphologies and thus different elevation RMSs. As the three corrections/effects are on gravity, these areas have to be actually larger ($8^\circ \times 12^\circ$) as we want to see how the geoid is affected.

2. For each topographical effect generate in all L areas several (N) sets of point values on different regular geographical grids (with different grid steps, see Table 2) and the corresponding set of mean values using the simple (algebraic) average of point values in each geographical cell of specific dimensions, i.e., 5′ by 5′ in our case, in each of the L areas. A detailed DEM with a resolution of $3'' \times 3''$ and the global DEM GTOPO30 with resolutions of $30'' \times 30''$ and $5' \times 5'$ were used for the computation.

3. From N generated sets of mean values (for every effect in each area) compute the N-1 sets of differences between the sets referring to neighbouring grid steps.

4. Compile the N-1 contributions to the geoid caused by particular differences for all L chosen geographical areas.

5. In each of the L areas, order the N-1 results (contributions of differences to the geoid) in a descending order of step size of the grid and compute the Chebyshev norm from every resulting set, see Table 4. The Chebyshev norm of a certain function $u(x)$ in a domain D can be defined by the formula $\|u\|_C = \max\limits_{x \in \bar{D}} |u(x)|$, where $\bar{D}$ represents the domain D together with the boundary.

6. For all L areas, apply the criterion that requires that the minimally acceptable density of point values of topographical effects is the minimal density from which it is possible to generate the geoid which will differ from the geoid computed from the mean values obtained from a grid one step smaller by less than 1 cm at every point, i.e., in the sense of the Chebyshev norm and construct a graph showing dependency between the terrain RMS and optimal density of point values.

7. The empirical dependency can be approximated using a simple analytical function, the straight line in our case. This allows us to generalize the experimental results for entire world.

8. Once these minimally acceptable densities (computed specifically for 5′ by 5′ cells) are determined for all the L areas, with their attributed RMSs, these densities can be normalised for cells of any size.

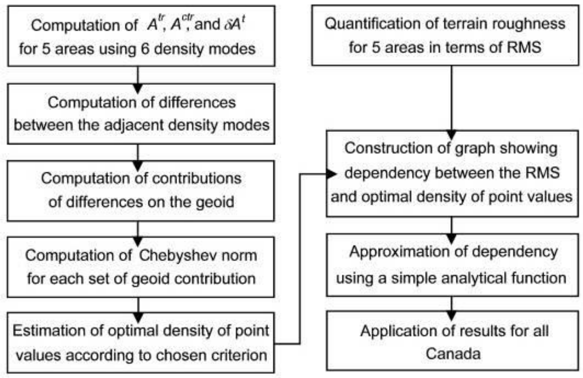

Figure 1. The computational scheme for estimation of optimal density of point values of the topographic effects for precise geoid determination.

The computational scheme of the experiment is shown in Figure 1. Let us now describe the experiment and its results in detail. We first chose the L = 5 testing areas ($2^\circ \times 2^\circ$ each), with different geo-morphological features, from flat to rugged high mountains (Figure 2). Their geographical extent and RMSs are given in Table 1.

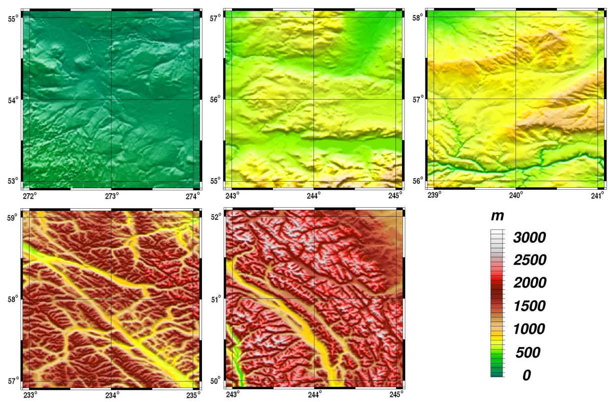

Figure 2. Topography of 5 testing areas ($2^\circ \times 2^\circ$) with different geomorphology chosen in Canada.

For every area the point values of A^{tr}, A^{ctr} and δA^t were computed on N = 6 geographical grids of different spacing, see Table 2. For all the 576 $5' \times 5'$ geographical cells within each of the 5 areas the mean values have been computed. Thus, for every effect we got 6 sets of 576 mean values; actually more (13824), as the geographical coverage of these 5 areas had to be extended to allow for the computation of the geoid. Consequently, the N-1=5 sets of differences were generated and all remaining work was done with these 75 sets, 5 sets for 3 topographical effects A^{tr}, A^{ctr} and δA^t in the 5 areas.

Subsequently, the contribution of every set to the geoid has been estimated using a numerical integration of Stokes's formula up to

Table 1. The geographical extent of the selected areas and elevation RMS computed from elevations for individual testing areas.

Area	Boundaries	RMS (m)	Description
1	$53°N \leqslant \varphi \leqslant 55°N$ $272°E \leqslant \lambda \leqslant 274°E$	33.9	almost ideally flat area near the Hudson Bay
2	$55°N \leqslant \varphi \leqslant 57°N$ $243°E \leqslant \lambda \leqslant 245°E$	97.3	eastern foothills of Rocky Mountains
3	$56°N \leqslant \varphi \leqslant 58°N$ $239°E \leqslant \lambda \leqslant 241°E$	119.7	eastern foothills of Rocky Mountains
4	$57°N \leqslant \varphi \leqslant 59°N$ $233°E \leqslant \lambda \leqslant 235°E$	363.2	northern part of Rocky Mountains
5	$50°N \leqslant \varphi \leqslant 52°N$ $243°E \leqslant \lambda \leqslant 245°E$	492.4	central part of Rocky Mountains

Table 2. Different grid densities for the 5' by 5' cells used in our investigations.

Mode	Number of point data to be averaged	Geographical grid step ($\Delta\varphi \times \Delta\lambda$)
1	1	5'×5'
2	4	2.5'×2.5'
3	10	1'×2.5'
4	25	1'×1'
5	50	0.5'×1'
6	100	0.5'×0.5'

spherical distance $\psi = 3°$. The Chebyshev norm of every resulting set of geoidal contribution was computed and compared against the 1 cm criterion. The computed 75 Chebyshev norms are listed in Table 3. The norms below 10 mm that passed the test are bolded.

Table 3. The estimated effect of the number of points on the geoid accuracy in terms of the Chebyshev norm. The values below the limit (10 mm) are bolded.

Area	Effect	Max. differences between the modes (mm)				
		1-2	2-3	3-4	4-5	5-6
1	A^{tr}	**1**	**1**	**1**	**1**	**1**
	A^{ctr}	**3**	**1**	**2**	**1**	**1**
	δA^{t}	**3**	**2**	**2**	**1**	**1**
2	A^{tr}	**4**	**2**	**1**	**1**	**1**
	A^{ctr}	26	13	**5**	**3**	**1**
	δA^{t}	23	11	**5**	**3**	**1**
3	A^{tr}	**8**	**2**	**1**	**1**	**1**
	A^{ctr}	43	28	**8**	**8**	**2**
	δA^{t}	46	29	**8**	**8**	**3**
4	A^{tr}	69	24	12	11	**4**
	A^{ctr}	498	160	60	49	**8**
	δA^{t}	477	148	56	40	**5**
5	A^{tr}	150	54	23	10	**7**
	A^{ctr}	627	131	79	47	16
	δA^{t}	604	109	72	40	13

3. Analysis of results

From Table 3, a minimally acceptable point density for each testing area can be estimated. Consider the contribution on the geoid caused by the differences between the n-th and $(n-1)$-th grid densities, for, $n = 2, 3, 4, 5, 6$. If the Chebyshev norm of these values in a particular testing area, see Table 3, is smaller than the prescribed limit (10 mm), then the $(n-1)$-th density is assumed to be the minimally acceptable one. Otherwise, n has to be increased by 1. If $n = 6$ and still the value in Table 3 is larger than the limit, the n-th grid density is recommended for use.

Based on the experimental results shown in Table 3, it is also possible to construct graphs where the Chebyshev norm of geoid differences for certain area is plotted as a function of number of points used for the averaging, see Figure 3.

Another interesting graph can be obtained when we plot the dependence between the RMS of the terrain for particular testing areas and the minimal acceptable number of points to be averaged, see Figure 4. This graph is plotted only for the direct topographical effect δA^{t}, because this effect is the most important for the geoid computation. The graph for A^{ctr} is exactly the same and the requirements for A^{tr} are less strict. We therefore suggest, in order to stay consistent, to use the equation derived for the direct topographic effect for the other two effects as well. Interestingly, this relationship seems to be approximately linear, although it is disputable because of very few testing areas, and it is therefore easy to quantify it by an equation of the straight line

$$y = ax + b, \tag{6}$$

where the coefficients a, b are determined using the least-squares linear regression $a = 4.6$ m/point and $b = 62.8$ m, see Figure 4. In Eq. (6) y represents the RMS of the terrain in metres and x represents the minimally acceptable number of points. The advantage of the analytical expression, Eq. (6), is that it allows an extension of the empirical relation obtained from 5 testing areas in Canada to any region of the world.

Also, in practice, one would want to be able to work with cells of sizes different from the 5' by 5' that we worked with. Then the equation can easily be transformed into another form that deals with point density rather than the number of points in a cell.

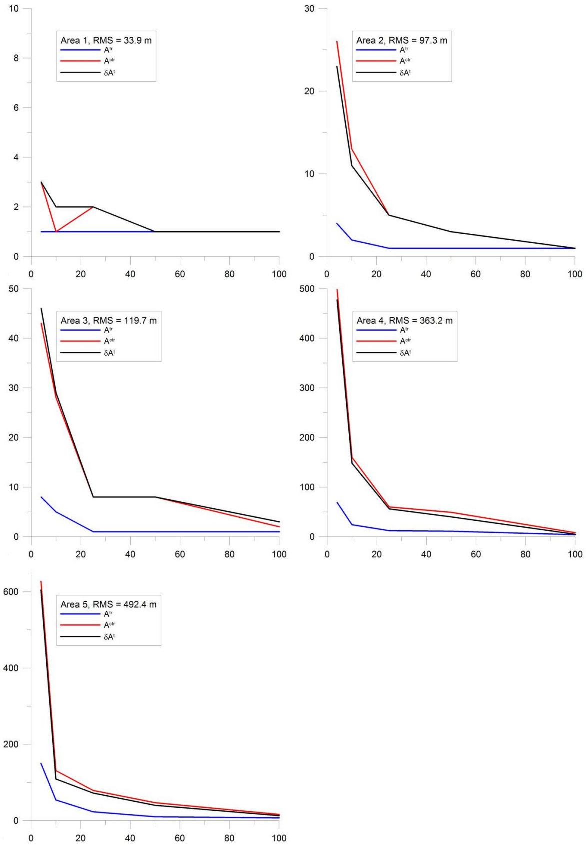

Figure 3. Dependency of the Chebyshev norm of the geoid differences (y-axis) on the number of points used for the averaging of topographic effects within the 5'×5' cells (x-axis). Units in y-axis: mm

Dividing a by 25, we obtain an equation from which the minimally acceptable point density x' per (arc-minute)^2 can be calculated

$$x' = 0.00870y - 0.54609. \tag{7}$$

Where x' is in the units of point density, i.e., in number of points per (arc-minute)^2 and, y the elevation RMS, is in metres.

On the basis of Eqs (6) or (7), a step function can be constructed which divides the terrain in 6 classes according to roughness represented by the RMS. These classes can now easily be assigned to the minimally acceptable number of points needed to construct the accurate mean value of certain topographical effect. The results for direct topographical effect δA^t are given in Table 4.

Table 4 can be used as a guide for the time consuming computations of the mean values of topographical effects. The whole Canada was divided in classes according to RMS of the terrain, see Figure 5. This was done in $1^\circ \times 1^\circ$ geographical rectangles.

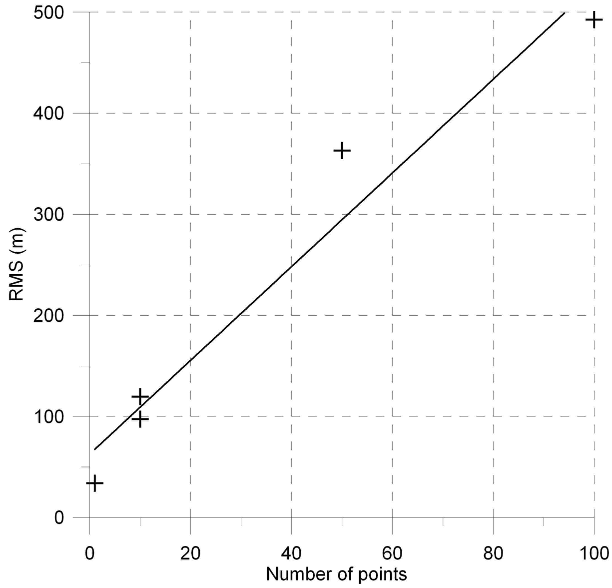

Figure 4. The empirical dependency between the RMS of particular testing areas (y-axis) and the optimal (minimal acceptable) number of points used for the averaging of topographic effects within the 5'×5' cells (x-axis). The black line shows the approximate analytical dependency obtained by least-squares linear regression.

Table 4. The optimal number of point values to be averaged within a 5'x5' cell and corresponding point densities per (arc-min)^2 according to the particular intervals of the RMS of the terrain.

RMS (m)	Optimal number of point values	Corresponding point density per (arc-min)^2
[0,67]	1	0.04
(67,81]	4	0.16
(81,109]	10	0.4
(109,178]	25	1
(178,293]	50	2
(293,523]	100	4
> 523	> 100	> 4

4. Conclusions and recommendations

The dependence between the terrain roughness, represented by the RMS of the terrain, and the optimal number of point values of topographical effects necessary to obtain a sufficiently accurate mean value within the $5' \times 5'$ cell was estimated. We found empirically that this relation is almost linear. The numerical and graphical tools were prepared in order to save the computation time during precise geoid computations. All Canada was divided into 7 zones according to the values of terrain roughness. Each zone corresponds to a specific number of point values to be averaged within the $5' \times 5'$ cell and this corresponds to the required point

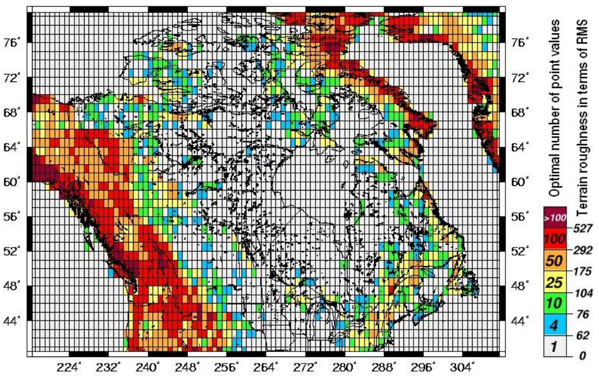

Figure 5. The area of Canada divided in classes with a 1°×1° step according to RMS of the terrain, indicating the optimal number of point values of topographic effects to be averaged within the 5'×5' cells.

density that can be used for computing the number of points needed in cells of other sizes than just 5′ by 5′.

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