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Two different views of topographical and downward-continuation corrections in the Stokes–Helmert approach to geoid computation

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Abstract. Two different methods to handle the topographical and downward-continuation corrections in Stokes' formula are investigated. The first approach is to correct observed gravity anomalies for the the effect of topographic masses, then downward-continue the corrected gravity anomaly to the geoid using Poisson's integral. After Stokes' formula has been applied, the gravitational effects of the masses on the geoid are restored using the indirect primary effect. The second approach is to add all topographic effects as a total correction to the original Stokes formula, which includes a new method to estimate the effect of downward continuation. These two methods are compared at 23 global positioning system (GPS)-levelling stations in Sweden. The results of this comparison show that both methods work well, with the first method having better agreement with the GPS-levelling geoid. The standard deviation of fit in the first method is computed to be ±1.1 cm, while it is ±2.1 cm for the second method after a four-parameter fit.

Key words: Geoid – Direct effect – Indirect effect – Downward continuation – Topographical correction

1 Introduction

Geoid determination by Stokes' formula requires that there are no masses outside the geoid. This can be achieved by removing the effects of external masses or reducing them inside the geoid (direct effect). The effects of masses are then restored after applying Stokes' integral (indirect effect). Stokes' formula also requires that gravity anomalies Δg must refer to the geoid. To satisfy this second condition, a reduction of the observed gravity anomalies from the topography to the geoid is necessary. This reduction is called downward continuation.

The corrections mentioned above, combined with the idea of Stokes–Helmert integration, are realized by the formula (see e.g. Heiskanen and Moritz 1967, p. 324)

$$N = \frac{R}{4\pi\gamma}\iint_{\sigma} S(\psi)\Delta g^{\mathrm{H}^*}\,\mathrm{d}\sigma + \delta N_{\mathrm{I}} \tag{1}$$

where N is the geoid height, Δg^{H^*} is the gravity anomaly that has been corrected for the direct topographic correction (Helmert anomaly Δg^{H}) and reduced to the geoid (i.e. downward-continued to the geoid), γ is normal gravity, $S(\psi)$ is Stokes' function, ψ is the spherical distance between the surface and running points, σ is the unit sphere, R is the mean radius of the Earth, and δN_{I} is the indirect topographic effect on the geoid.

In this study, the condensation method that preserves the Earth's mass is used, for which the Helmert-model Earth has the same mass as the real Earth. For more details on Helmert's second condensation method, see e.g. Wichiencharoen (1982), Sjöberg (1994, 1997, 2000), Vaníček and Martinec (1994) and Nahavandchi and Sjöberg (1998).

The Helmert gravity anomaly Δg^{H} at the ground level can be expressed as (see e.g. Vaníček and Martinec 1994; Vaníček et al. 1996; Nahavandchi 2000a; Sjöberg 2000)

$$\Delta g^{\mathrm{H}} = \Delta g + \delta\Delta g_{\mathrm{dir}} \tag{2}$$

where Δg is the surface free-air gravity anomaly and $\delta\Delta g_{\mathrm{dir}}$ is the direct effect on gravity determined at the Earth's surface. The surface free-air anomaly is calculated from the surface gravity observation, which is reduced by normal gravity and the free-air correction (0.3086H mGal), where H is the height of the gravity station in metres. The notation Δg^{H^*} is the analytically downward-continued Δg^{H} from the Earth's surface to the geoid. Poisson's integral is used for this downward-continuation process. Equation (1) is defined as the first method in this study.

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The regularized Stokes' formula using Helmert's second condensation method can be expressed as (see e.g. Sjöberg 2000)

$$N = \frac{R}{4\pi\gamma} \iint_{\sigma} S(\psi)(\Delta g + \delta\Delta g^{*}_{\mathrm{dir}} + \delta\Delta g_{\mathrm{dc}})\mathrm{d}\sigma + \delta N_{\mathrm{I}} \qquad (3)$$

where $\delta\Delta g^{*}_{\mathrm{dir}}$ is the direct effect on the gravity anomaly downward-continued to the geoid, and $\delta\Delta g_{\mathrm{dc}}$ is the correction due to the downward continuation of the free-air anomaly Δg. Again, the Helmert-model Earth has the same mass as the real Earth. Equation (3) will be called the traditional method. The new method for handling the topographical and downward-continuation corrections is another form of Eq. (3), which can be written as (Sjöberg 2000)

$$N = \frac{R}{4\pi\gamma} \iint_{\sigma} S(\psi)\Delta g \, \mathrm{d}\sigma + \delta N_{\mathrm{total}} \qquad (4)$$

where

$$\delta N_{\mathrm{total}} = \delta N_{\mathrm{comb}} + \delta N_{\mathrm{dc}} \qquad (5)$$

In Eq. (5), δN_{comb} is the sum of direct and indirect topographical effects, and δN_{dc} is the effect on the geoid of downward-continuing the gravity anomaly to the geoid. Equation (4) is considered as the second approach in this study. In this method, a new way of downward-continuing of gravity anomalies to the geoid has been tested.

Two classical formulas of the remove–compute–restore problem were presented by Moritz (1980) and Vaníček and Kleusberg (1987), which are based on a planar approximation. The former referred the direct topographical effect to the geoid, while the latter referred it to the Earth's surface. A recent description of the Stokes–Helmert method for geoid determination is given by, e.g. Vaníček and Martinec (1994), Nahavandchi (1998a) and Sjöberg (2000). These authors point out that the classical formulas may be severely biased because of the planar approximation. In addition, the classical formulas suffer from other approximations. The most important one is that the slope of the topography must be within 45°. This limitation has been pointed out by e.g. Heck (1992), Martinec and Vaníček (1994a) and Sjöberg and Nahavandchi (1999).

In this study, we will compare the gravimetric geoid at 23 GPS-levelling stations with the two different methods of topographic and downward-continuation corrections. The result of these comparisons will be used to empirically test the efficiency of the second approach compared to that of the first approach.

2 Topographic and downward-continuation corrections

2.1 The first method

In the first method [Eq. (1)], the masses above the geoid are first reduced to (or shifted inside) the geoid, which is simply called the direct effect, using Helmert's second condensation method. This preserves the mass of the Earth, but changes the gravitational potential of the topography. The direct gravity effect of the topography at a topographic surface point P to the second power of H is derived by Nahavandchi (2000a) and Sjöberg (2000) as

$$\delta\Delta g_{\mathrm{dir}}(H_P) = -\frac{4\pi\mu}{R}H_p^2 - \frac{3\mu}{8}\iint_{\sigma}\frac{H^2 - H_P^2}{\ell_0}\,\mathrm{d}\sigma + \frac{\mu R^2}{2}\iint_{\sigma}\frac{H_P^2 - H^2}{\ell^3}\left(1 - \frac{3H_P^2}{\ell^2}\right)\mathrm{d}\sigma \qquad (6)$$

or

$$\delta\Delta g_{\mathrm{dir}}(H_P) = -\frac{5\pi\mu}{2R}H_p^2 - \frac{3\pi\mu}{2R}\overline{H_P^2} + \frac{\mu R^2}{2}\iint_{\sigma}\frac{H_P^2 - H^2}{\ell^3}\left(1 - \frac{3H_P^2}{\ell^2}\right)\mathrm{d}\sigma \qquad (7)$$

where

$$(H^v)_{nm} = \frac{1}{4\pi}\iint_{\sigma} H_P^v Y_{nm}\,\mathrm{d}\sigma; \quad v = 1, 2, 3, \ldots \qquad (8)$$

$$H_P^v = \sum_{n,m}(H^v)_{nm} Y_{nm} \qquad (9)$$

$$\overline{H_P^v} = \sum_{n,m}\frac{1}{2n+1}(H^v)_{nm} Y_{nm}(P) \qquad (10)$$

Furthermore, $\ell = \sqrt{r_P^2 + r^2 - 2r_P r\cos\psi}$; $\ell_0 = 2R\sin\psi/2$; $r_P = R + H_P$; H_P and H are the topographic heights of the surface point P and running point, respectively; ℓ and ψ are the spatial and spherical distances between the surface point P and the surface element of the terrestrial sphere $R = r$; $\mu = G\rho_0$, G being the universal gravitational constant and ρ_0 the density of topography (assumed to be constant); and Y_{nm} are the fully normalized spherical harmonics, obeying the following rule:

$$\frac{1}{4\pi}\iint_{\sigma} Y_{nm}Y_{n'm'}\,\mathrm{d}\sigma = \begin{cases} 1 & \text{if } n = n' \text{ and } m = m' \\ 0 & \text{otherwise} \end{cases} \qquad (11)$$

Martinec and Vaníček (1994a) have also developed a formula to account for the direct topographical effect at the Earth's surface. The differences between Eq. (7) and their formula were evaluated in a test area in the northwest of Sweden with a height variation between 354 and 1147 m. The maximum difference reaches 2.31 μGal (Nahavandchi 2000a).

Notice that the traditional definition for the Helmert anomaly [Eq. (2)] is strictly inconsistent because the direct effect $\delta\Delta g_{\mathrm{dir}}$ corresponds to gravity and not the gravity anomaly. This problem is usually solved by adding the so-called secondary indirect effect to gravity. However, as this term is usually small, it is neglected in this method. Nahavandchi (1998a) computed this effect at 23 GPS-levelling stations in Sweden with a mean value of less than 0.7 cm. This effect is usually two orders of magnitude smaller than the direct topographic effect. It should be noted that other corrections to Helmert's anomalies on the Earth's surface were derived, e.g. by Vaníček et al. (1999), and have not been studied here.

In the next step, the downward-continuation process of topography-corrected [with Eq. (7)] gravity anomalies Δg^{H} (at the surface of the topography) to the gravity anomalies Δg^{H^*} at the geoid (analytically downward-continued Δg^{H}) is applied. The application of the direct effect makes the gravity anomalies smoother and then better suited to downward continuation. These two anomalies can be related by Poisson's formula (including the spherical harmonics of degrees zero and one) (see e.g. Kellogg 1929; MacMillan 1930)

$$\Delta g^{\mathrm{H}} = \frac{R}{4\pi} \iint_{\sigma} \Delta g^{\mathrm{H}^*} K(r, \psi, R) \mathrm{d}\sigma \tag{12}$$

where $K(r,\psi,R)$ is Poisson's kernel, described by

$$K(r, \psi, R) = \sum_{n=0}^{\infty} (2n+1) \left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi) = R \frac{r^2 - R^2}{\ell^3} \tag{13}$$

where $H = r - R$ and $P_n(\cos\psi)$ are the Legendre polynomials. In Eq. (12), a spherical approximation is used. The gravity anomalies Δg^{H} at the topographic surface are known, and the gravity anomalies Δg^{H^*} at the geoid are to be determined. Equation (12) can be solved in different ways – for example by a linear approximation. In this study, an iterative process is implemented by transforming Poisson's integral [Eq. (12)] to a system of 24 000 linear algebraic equations (see Vaníček et al. 1996; Nahavandchi 1998b).

The Poisson kernel, which dominates the behaviour of Poisson's integral, tapers off rapidly with increasing distance from the computation point. Therefore, it only needs to be integrated over a small spherical cap ψ_0 instead of over the whole Earth. Rewriting the integral in Eq. (12) gives

$$\Delta g^{\mathrm{H}} = \frac{R}{4\pi} \iint_{\sigma_0} \Delta g^{\mathrm{H}^*} K(r, \psi, R) \mathrm{d}\sigma + \frac{R}{4\pi} \int\int_{\sigma-\sigma_0} \Delta g^{\mathrm{H}^*} K(r, \psi, R) \mathrm{d}\sigma \tag{14}$$

where σ_0 denotes the spherical area within a spherical cap of radius ψ_0. This truncation error is reduced using Molodenskij's truncation modification technique (Molodenskij et al. 1962; Sjöberg 1984). A spherical cap with radius equal to 1° assures us that the contribution from the rest of the world is small (see Vaníček et al. 1996; Nahavandchi 1998b). The low-degree spherical harmonics of the Helmert anomaly Δg_M^{H} are also subtracted from the gravity anomalies Δg^{H} at the surface of the Earth (also see Vaníček et al. 1996). Then Eq. (14) can be rewritten as

$$\Delta g^{\mathrm{H}} = \frac{R}{4\pi} \iint_{\sigma_0} \Delta g^{\mathrm{H}^*} K^M(r, \psi, R, \psi_0) \mathrm{d}\sigma + \mathrm{d}g \tag{15}$$

where

$$\mathrm{d}g = \delta g_{\mathrm{T}} + \Delta g_M^{\mathrm{H}} \tag{16}$$

where δg_{T} is the truncation error and Δg_M^{H} are the low-degree spherical harmonics of the gravity anomaly. The truncation error has to be minimized following the Molodenskij technique to minimize potential errors coming from the employed global gravity model. The minimization is carried out in the sense of minimizing the upper bound of the absolute value of the truncation error by subtracting from the Poisson's kernel an appropriately selected linear combination of spherical harmonic functions taken to degree and order M. Hence, the modified Poisson's kernel can be evaluated from

$$K^M(r, \psi, R, \psi_0) = K(r, \psi, R) - \sum_{n=0}^{M} \frac{2n+1}{2} s_n(r, R, \psi_0) P_n(\cos\psi) \tag{17}$$

where $s_n(r,R,\psi_0)$ are the unknown coefficients to be computed from the following system of equations (cf. Molodenskij et al. 1962)

$$\sum_{n=0}^{M} \frac{2n+1}{2} s_n(r, R, \psi_0) e_{in}(\psi_0) = Q_i(r, R, \psi_0); \quad i = 0, 1, \ldots, M \tag{18}$$

where

$$e_{in}(\psi_0) = \int_{\psi_0}^{\pi} P_i(\cos\psi) P_n(\cos\psi) \sin\psi \, \mathrm{d}\psi \tag{19}$$

and

$$Q_n(r, R, \psi_0) = \int_{\psi_0}^{\pi} K(r, \psi, R) P_n(\cos\psi) \sin\psi \, \mathrm{d}\psi \tag{20}$$

The value of $M = 20$ is selected in this study, referring to a relatively low-degree reference field and to a set of solely satellite-determined potential coefficients. The contribution of the rest of the world is quite small (see Vaníček et al. 1996; Nahavandchi 1998b) and the truncation error δg_{T} can be evaluated using a global gravity model as (see e.g. Vaníček et al. 1996)

$$\delta g_{\mathrm{T}} = \frac{R}{2r} \sum_{n=2}^{\infty} \overline{Q_n}(r, R, \psi_0) \Delta g_n^{\mathrm{H}^*}(P) \tag{21}$$

where

$$\overline{Q_n}(r, R, \psi_0) = \int_{\psi_0}^{\pi} K^M(r, \psi, R, \psi_0) P_n(\cos\psi) \sin\psi \, \mathrm{d}\psi \tag{22}$$

and the modified Poisson's kernel in a spectral form is

$$K^M(r,\psi,R,\psi_0)=\sum_{n=0}^{\infty}\frac{2n+1}{2}\overline{Q_n}(r,R,\psi_0)P_n(\cos\psi) \quad (23)$$

In the above, $\Delta g_n^{\mathrm{H}^*}$ are the Laplace harmonics of the Helmert gravity anomaly, and Δg_M^{H} can be computed from a global gravity model as

$$\Delta g_M^{\mathrm{H}}=\gamma\sum_{j=2}^{M}(j-1)\left(\frac{R}{r}\right)^{j+2}\sum_{m=-j}^{j}A_{jm}^{*}Y_{jm}(P) \quad (24)$$

where A_{jm}^* are the potential coefficients taken from a global gravity model and corrected for the direct topographic effect. The direct topographic correction to potential coefficients, limited to second power of H, is derived by Nahavandchi and Sjöberg (1998) as

$$\Delta A_{\mathrm{Geop}}=-2\pi\mu\sum_{n=0}^{M'}\sum_{m=-n}^{n}\frac{n+2}{2n+1}(H^2)_{nm}Y_{nm}(P) \quad (25)$$

where M' is the maximum degree of height coefficients. Finally, after the Stokes integration [Eq. (1)], the effect of the removed masses is restored, which is the indirect effect δN_{I}. Again, the most common reduction method (i.e. Helmert's second condensation method) is used. The indirect effect is determined by (Sjöberg and Nahavandchi 1999)

$$\delta N_{\mathrm{I}}=\delta N_{\mathrm{I}}^{\mathrm{classic}}(P)-\Delta\delta N_{\mathrm{I}}(P) \quad (26)$$

where

$$\delta N_{\mathrm{I}}(P)^{\mathrm{classic}}=\frac{-\pi\mu H_P^2}{\gamma}-\frac{\mu R^2}{6\gamma}\iint_{\sigma}\frac{H^3-H_P^3}{\ell_0^3}\,\mathrm{d}\sigma \quad (27)$$

is the classical indirect topographical effect, derived by Moritz (1980) and Vaníček and Kleusberg (1987), and

$$\Delta\delta N_{\mathrm{I}}(P)=-\frac{3\pi\mu}{\gamma}\overline{H_P^2}+\frac{\pi\mu}{2R\gamma}\left(H_P^3-\overline{H_P^3}\right) \quad (28)$$

$\overline{H_P^2}$ and $\overline{H_P^3}$ follow from spherical presentation in Eq. (10).

Martinec and Vaníček (1994b) also derive a formula for the indirect effect, which is different from Eq. (26). Their formula was compared with Eq. (26) in a test area in Sweden and the results were found to be the same.

2.2 The second method

The second approach in this study is based on the idea that all direct and indirect effects and downward-continuation corrections can be computed separately, and then be added as corrections to the uncorrected geoid height computed from the original Stokes formula [see Eqs. (4) and (5)]. This study is limited to H^2, as the contribution of the higher powers of H is less than 1 cm in most parts of the world (Nahavandchi and Sjöberg 1998; Nahavandchi 1999). It was shown by Sjöberg (1997, 2000) and Nahavandchi and Sjöberg (1998) that

$$\delta N_{\mathrm{comb}}=-\frac{2\pi\mu}{\gamma}\tilde{H}^2 \quad (29)$$

where $\tilde{H}$ is the orthometric height excluding zero- and first-degree terms. The effect of the downward continuation on the geoid was derived by Sjöberg (1999) as

$$\delta N_{\mathrm{dc}}=\frac{2\zeta_P H_P}{R}-2\frac{\overline{\zeta H}}{R}-3R\sum_{m=-1}^{1}f_{1m}Y_{1m}(P)-\frac{R^4}{2\pi}\iint_{\sigma}\frac{f-f_P}{\ell_0^3}\,\mathrm{d}\sigma \quad (30)$$

where

$$f(\Delta g,\zeta,H)=\frac{\Delta g}{2\gamma}\left(\frac{H}{R}\right)^2+\frac{\zeta H}{R^2} \quad (31)$$

$$f_{nm}=\frac{1}{4\pi}\iint_{\sigma}fY_{nm}\,\mathrm{d}\sigma \quad (32)$$

where ζ is the height anomaly and $\overline{\zeta H}$ is the global average of ζH.

For practical purposes, with the selection of a sufficiently high degree M and a large near-zone cap σ_0, Eq. (30) is rewritten as (see Sjöberg 1999)

$$\delta N_{\mathrm{dc}}=\frac{2\zeta_P H_P}{R}-2\frac{\overline{\zeta H}}{R}-4Rf_1(P)-R\sum_{n=2}^{M}nf_n(P)-\frac{R^4}{2\pi}\iint_{\sigma_0}\frac{f^M-f_P^M}{\ell_0^3}\,\mathrm{d}\sigma \quad (33)$$

where

$$f^M(P)=f(P)-\sum_{n=0}^{M}f_n(P) \quad (34)$$

$$f_n(P)=\sum_{m=-n}^{n}f_{nm}Y_{nm}(P) \quad (35)$$

Summing Eqs. (29) and (33), the total effect $\delta N_{\mathrm{total}}$ in Eq. (5) is obtained. This total term should be added to the geoid height derived from the original Stokes integral [Eq. (4)].

3 Numerical investigations

For the numerical investigation of these two methods of handling topographical and downward-continuation corrections of the gravity anomaly in Stokes' formula, the gravimetric geoid heights at 23 GPS stations were computed using both approaches, and the gravimetric results compared with the GPS-levelling geoid heights. The GPS stations belong to the Swedish Permanent GPS Network (SWEPOS) and are distributed over Sweden. These stations were established by the Onsala space observatory, Chalmers University of Technology and the National Land Survey of Sweden. The accuracy of the ellipsoidal heights of these 23 GPS stations is of the order of a few centimetres (Ekman, pers. commun. 1998). The height system in Sweden is the RH70 normal system. Therefore, a correction for the separation between the normal heights and orthometric heights was needed. We used the correction mentioned by Nahavandchi [1998a, Eqs. (2.20) and (2.22)]. The normal height of the GPS stations varies from 7.024 to 469.271 m.

In order to compute the topographical effects according to the first method, first the direct topographical effect on gravity was determined by Eq. (7). The integration area was limited to 6° from the computation point. Equation (7) accounts for both near and outer zones. Nahavandchi (1998a, 2000a) has shown that the near-zone area to be truncated at 6°. The height data used in this inner zone were from the Geophysical Exploration Technology (GETECH) $2.5' \times 2.5'$ digital terrain model (DTM) (GETECH 1995a). Note that the $2.5' \times 2.5'$ DTM is definitely not dense enough for accurate direct topographical effect evaluation. The harmonic coefficients of heights $(H^2)_{nm}$ were determined from Eqs. (8) and (9). For this, a $30' \times 30'$ DTM was generated using the GETECH $5' \times 5'$ DTM (GETECH 1995b). This $30' \times 30'$ DTM was averaged using area weighting. Since the interest was in continental elevation coefficients and we were trying to evaluate the effect of the masses above the geoid, the heights below sea level were all set to zero. The coefficients were computed to spherical harmonic degree and order 360. The parameter $\mu = G\rho_0$ was computed using $G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2} and $\rho_0 = 2670$ kg/m^3. The values of $R = 6371$ km and $\gamma = 981$ Gal were also used in the computations. Next, the surface free-air gravity anomaly was corrected with the direct effect determined by Eq. (7), resulting in the Helmert anomaly Δg^{H}. Thereafter, the downward continuation of the surface Helmert anomaly Δg^{H} to the geoid (Δg^{H^*}) was implemented by Eq. (15). An iterative process was used for this correction. Poisson's integral was transformed to a system of 24 000 linear algebraic equations. The truncation error [Eq. (21)] and long-wavelength contribution [Eq. (24)] were also considered in the computations. The equations in Hagawara (1976) and Paul (1973) were used to compute the truncation error. The low-degree harmonics Δg_M^{H} were determined using the potential coefficients, which were taken from global gravity model EGM96 (Lemoine et al. 1997) to degree and order 20. For more details on the choice of this value, see Vaníček et al. (1996) and Nahavandchi (1998a, b). Finally, the indirect effect in Eqs. (26)–(28) was applied to the geoid heights estimated by Stokes' integral [Eq. (1)]. A 6° integration area was selected. Equation (28) requires that the height coefficients $(H^3)_{nm}$ are also determined from Eqs. (8) and (9). In this way, the corrections due to the topographical effects and downward continuation of gravity anomalies to the geoid could be completed according to the first method. The statistics of corrections to the geoid are summarized in Table 1. This method was realized at the 23 GPS-levelling stations.

Table 1. Statistics of topographical and downward-continuation corrections at 23 GPS-levelling stations with the first method. Units in metres

	Direct effect + downward continuation	Indirect effect
Min	0.071	0.041
Max	0.212	0.102
Mean	0.162	0.062
Standard deviation	0.035	0.021

The second approach computes the combined topographical effect (the sum of direct and indirect effects) using Eq. (29). This effect was added separately as a correction to the geoid heights estimated by Stokes' formula [Eq. (4)]. Thereafter, it remains to compute the correction due to the downward continuation of gravity anomalies to the geoid. This was computed according to Eq. (33). To determine this correction, the parameter f was computed according to Eq. (31). This parameter was determined in $6' \times 10'$ cells in an area of $22° \times 30°$ over Scandinavia, considered as the inner-zone area. In this area, Δg was derived from Nahavandchi (2000b); outside the inner-zone area Δg was taken from global gravity model EGM96. H in the inner-zone area was taken from the GETECH $2.5' \times 2.5'$ DTM (GETECH 1995a); further out, the $30' \times 30'$ DTM was used. The height anomaly ζ in the inner zone was taken from Nahavandchi (1998a). For this, the $N - \zeta$ correction was determined [see Nahavandchi 1998a, Eq. (2.20)] and applied to geoid heights N (see Nahavandchi 1998a, Fig. 5.10), resulting in ζ. Outside the inner-zone area, EGM96 was used to determine the height anomaly. Thereafter, f^M and f_n were computed from Eqs. (34) and (35). For this, f_{nm} values were computed to degree and order 360 according to Eq. (32). These computations complete the corrections using the second method. The statistics of corrections to the geoid with the second method at the 23 GPS-levelling stations are listed in Table 2.

Comparison of the results of Tables 1 and 2 shows the differences between the two approaches. These differences were expected. In the first method, the direct and downward-continuation corrections were kept together in a convolution integral with Stokes' kernel, while in the second method, the correction due to these effects was derived outside Stokes' integral as a correction to the original Stokes formula. The main idea of

Table 2. Statistics of topographical and downward-continuation corrections at 23 GPS stations with the second method. Units in metres

	Direct effect + indirect effect	Downward-continuation correction
Min	0.057	0.006
Max	0.062	0.198
Mean	0.060	0.082
Standard deviation	0.001	0.047

this study was to test if the second approach (which is very easy to compute) works as well as the first method, which is the most precise procedure to date for topographic corrections in geoid determination (but which involves very severe computation).

For further comparison, the corrected gravimetric geoid height, applying the topographical and downward-continuation corrections, was computed according to the two methods. The statistics of differences between the gravimetric and the GPS-levelling geoid height, including and excluding these corrections, are shown in Table 3.

Table 3 shows that the corrections according to both methods improve the fit of the gravimetric geoid to the GPS-levelling stations, as expected. Table 3 also shows that both methods are mostly in good agreement with each other.

In addition, a fitting process between the gravimetric and the GPS-levelling geoid was conducted. The geoid change ΔN corresponding to a general seven-parameter datum-shift transformation will be independent of the rotations, and in geographical coordinates will be of the form (Heiskanen and Moritz 1967)

$$N_{\mathrm{Grav}} - N_{\mathrm{GPS}} = \Delta N = \cos\phi\cos\lambda\Delta X + \cos\phi\sin\lambda\Delta Y + \sin\phi\Delta Z + kR \qquad (36)$$

where ϕ and λ are geographical coordinates, ΔX, ΔY, ΔZ are the three translations, and k is the scale factor. Equation (36) represents a very useful regression formula, which may be used for transforming a regional gravimetric geoid to a GPS-levelling geoid. However, it should be noted that some long-wavelength geoid, GPS and vertical datum errors will be absorbed by the parameters. Therefore, the datum-shift parameters derived from the regression are not to be used for any kind of coordinate transformation. Table 4 shows the statistics of differences, after fitting, between the gravimetric and the GPS-levelling geoid.

Table 3. Statistics of differences between gravimetric and 23 GPS-levelling stations' geoid height, with and without topographical and downward-continuation corrections. Units in metres

	Without corrections	With corrections according to first method	With corrections according to second method
Min	0.201	0.044	0.067
Max	0.402	0.260	0.258
Mean	0.287	0.101	0.142
Standard deviation	0.105	0.055	0.047

Table 4. Statistics of differences between GPS-levelling and gravimetric geoid models with two different methods after fitting to 23 GPS stations. Units in metres

	First method	Second method
Min	−0.031	−0.120
Max	0.052	0.114
Mean	0.000	0.000
Standard deviation	0.011	0.051

Table 4 shows that, after regression, the results improve significantly, i.e. the use of the four-parameter datum-shift fit eliminates the possible tilt of the gravimetric geoid. In addition, the topographic corrections applied with the first method agree slightly better with GPS-levelling geoid. The standard deviation of fit after regression is computed to be ±1.1 cm using first method while it is equal to ±2.1 cm according to the second method.

The first formula for topographical and downward-continuation corrections is more sophisticated and yields more accurate results, while the second one is simpler to compute. However, surprisingly, the second approach gives results which are very close to those of the first one in this study. The inner-zone area of integration in the first method was derived with the spherical approximation, contrary to the classical method (in the classical approach, a planar approximation and smooth topography are used). In addition, an outer-zone effect (determined with a global terrain model) was applied. The topographical corrections in the second method was derived with a very simple formula [Eq. (29)], and the results show that this simple formula largely coincides with the sophisticated formulas of the first method [Eqs. (7) and (26)] (see Table 3). Also, the effect due to downward continuation of gravity anomalies to the geoid was derived by Poisson's integral in the first method, while in the second approach an explicit formula proposed by Sjöberg (1999) was used. However, the comparison of results shows that the downward continuation by the second method also coincides with that of the first one (see Table 3). It should be noted that the downward-continuation correction very carefully considers both the inner- and outer-zone effects.

For further comparison, the NKG96 geoid model (Forsberg et al. 1996), the best model for the region so far, was fitted to the same GPS-levelling stations using Eq. (36). The results yield a standard deviation of ±6.1 cm, which shows the superiority of topographical and downward-continuation correction with the two methods in the GPS-levelling stations of this study.

4 Conclusions

The effect of external masses on the geoid, including direct and indirect effects and downward-continuation correction of gravity anomalies to the geoid, have been treated using two different methods. The first method uses the correction of the surface free-air gravity anomaly for the direct effect of topographical masses, resulting in the Helmert anomaly, and the downward continuation of it to the geoid. The indirect effect of terrain masses is then added after Stokes' integration.

In the second method, direct and indirect effects are added together as a combined effect and an explicit formula is used for the correction to the geoid height due to the downward continuation of gravity anomalies. This means that, in contrast to the first method, the direct topographical effect and downward-continuation correction is added as a separate correction directly to geoid heights determined by the original Stokes integration.

The aim of this study was to test the precision of the second method of corrections in comparison with the most precise procedure of handling the topographical and downward-continuation corrections of the first approach. These two methods were realized on 23 GPS-levelling stations in Sweden. The results of gravimetric geoid height corrected with the first method agree slightly better with the GPS-levelling geoid. The standard deviation of fit is determined to be equal to ±1.1 cm with the first method, and ±2.1 cm with the second method (after the regression procedures). The results of Tables 3 and 4 for the first method were expected, while the good fit of the geoid height corrected using the second method to the GPS-levelling geoid was surprising. Thus it is shown that the very simple computation of the corrections in the second approach could be an alternative method of treatment for topographical and downward-continuation corrections. For verification of the method, these computations must be implemented in other test areas with different types of GPS-levelling data.

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