Comparison between planar and spherical models of topography

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Abstract

Topography plays an important role in solving many geodetic and geophysical problems. In the evaluation of a topographical effect of one kind or another, one can use either a planar model, or a spherical model, or, perhaps an even more sophisticated model still. In most applications, the planar model is considered appropriate enough for the purpose: recall the evaluation of gravity reductions of the free-air, Poicaré-Pray, or Bouguer kind.

For other applications, such as the evaluation of the direct topographical effect, or the primary indirect topographical effect in gravimetric geoid computations, it is necessary to use at least the spherical model of topography. In this contribution we will present the comparison of the two models and discuss the differences, including the apparent incongruencies.

Introduction

Periodically, people discover that planar and spherical models of topography give different results for Bouguer anomalies, as well as for direct and indirect topographical effects in the Stokes-Helmert technique for geoid computations. As examples, let us cite Karl [1971] and Véronneau [1998], who both question the compatibility of the "Bouguer plate" and "Bouguer spherical shell" topographical models. But there have been others: so what's going on?

When looking into it we discovered an interesting story which we shall try to illuminate here. To do so, we focus only on the "infinite plate" and "spherical shell" models, leaving out the terrain effects. The difference of the planar and spherical models of terrain has been discussed in [Novák and Vaníč ek, 1998] and will be further investigated in [Novák and Vaníč ek, 1999].

The story of Bouguer plate reduction

To show the pattern, let us show the gravitational potential, the gravitational attraction (negative first vertical derivative of the potential) and the vertical gradient of gravitational attraction (negative second derivative of the potential) of the topographical (Bouguer) plate and the topographical (Bouguer) shell side by side. And, to keep things simple, let's assume a constant density ρ (say, 2.67 g cm^{-3}) and the same thickness H for both the infinite plate and the shell of the inner radius R.

This is all done in Figure 1. The three quantities of interest are computed at two points:

Figure 1- Bouguer plate vs. Bouguer shell

on the top and at the bottom of the plate/shell. In addition, the second derivative which is discontinuous on the top (and also at the bottom) of the plate/shell is at this point computed both from above and from below. The expressions for the plate are derived from eqns. (3.5 and 3.7) found in [Heiskanen and Moritz, 1967] by simply extending the finite plate to infinity. The expressions for the shell are derived directly from eqns. (19, 24 and 25) for the potential of a sphericall shell found in [Wichiencharoen, 1982].

Now, examining this figure, how different really are the results for the planar and spherical models? Starting from the vertical gradient of attraction and neglecting the higher order terms (of the order of H/R and smaller) in the spherical model, the results are identical. The attraction of the plate at its top is only one half of that of the shell (at its top and neglecting the higher order terms), while the attraction at the bottom of the plate is exactly opposite to that at the top. The attraction at the downside of the shell is zero as it should be [Kellogg, 1929]. Note that the change in the attraction when vertically transiting the plate or shell is the same, except for higher order terms. The situation for the potential is naturally different: as the potential of the infinite plate is infinite, one cannot make any direct comparison between the two models. We can only note that in the spherical model, the difference between the potential at the top and on the downside of the shell differ only by the higher order terms.

What does it all mean? We wish to address here only the question of what this means in the context of the (incomplete, i.e., without the terrain correction) Bouguer gravity anomaly. The Bouguer anomaly is computed from the following formula

$$\Delta g = g + A + A^B - \gamma , \tag{1}$$

where g is the observed gravity on the earh surface (at altitude H), A is the "free-air reduction" (to the geoid) due to the earth masses within the spherical shell (including the latitude and altitude terms), A^B is the "Bouguer reduction" (to the geoid) due to the mass of the Bouguer plate and γ is the normal gravity (at the reference ellipsoid) [Heiskanen and Moritz, 1967, eqn.(3.19)]. Here, the Bouguer reduction is given by

$$A^B = 2\pi G\rho H. \qquad (2)$$

Clearly, A^B is **not** the difference between the gravity values at the top and the bottom of the infinite plate! It is **not** the difference between gravity values at the outside and downside of the spherical shell either! This difference, in both cases is equal to

$$A^{PP} = 4\pi G\rho H, \qquad (3)$$

known in geodesy as the Poincaré-Pray gravity reduction [Heiskanen and Moritz, 1967, eqn.(3.64)]. One arrives at this conclusion using either the attraction or the vertical gradient of attraction formulae as one should.

The inevitable conclusion is that the apparently incorrect Bouguer gravity reduction is **NOT** coming from the use of the (physically meaningless) planar model and that one should thus come up with the "correct" result by using the physically meaningful spherical model. As a matter of fact, the Bouguer reduction can be equally easily derived from the spherical model as well, once one realises where the reduction is coming from. The Bouguer reduction can be also written as

$$A^B = - \partial^2 V/\partial r^2 \; H \, , \qquad (4)$$

where the second derivative (negative attraction gradient) is evaluated at the top of the plate/shell. But, according to the figure, the gradient has at this point two values, one for the outside and the other for the inside of the plate/shell. Thus, strictly speaking, the gradient is at this point not defined! The "Bouguer gradient" is taken to be the average of the outside and inside values, i.e., a mathematically and physically meaningless quantity [Vaníč ek and Krakiwsky, 1986].

Our conclusion is thus that the Bouguer gravity anomaly, useful as it is in many applications, is a artificial construction. The Bouguer reduction numerically reduces the observed gravity from the surface of the earth to the mid-point of the infinite Bouguer plate, or mid-point of the spherical shell. This is a result of a particular selection of the value of the vertical gradient of gravity rather than the selection of planar model. The difference between using the planar and spherical model is of second order; it is the known spherical correction to Bouguer plate reduction:

$$\delta A^B{}_S = 4\pi G\rho H^2/R \qquad (5)$$

The direct and primary indirect topographical effects

Probably the most popular technique for solving the boundary value problem of geodesy (leading to geoid determination from observed gravity anomalies) is the one called Stokes-Helmert's technique. The essence of this technique is that topographical masses are replaced by a condensed mass layer on the geoid surface, resulting in an introduction of an abstract space (Helmert's space) in which the solution is sought. The main idea behind the introduction of the Helmert space is that the disturbing potential T^h sought in this space is harmonic everywhere above the geoid. It is related to the real disturbing potential T by the following equation:

$$T^h(r,\Omega) = T(r,\Omega) - V(r,\Omega), \tag{6}$$

where the residual topographical potential V is defined as

$$V(r,\Omega) = V^t(r,\Omega) - V^c(r,\Omega) , \tag{7}$$

where V^t denotes the potential of topographical masses and V^c stands for the potential of the (condensed) mass layer. The symbols r and Ω stand for geocentric distance and angle.

The transformation of observed gravity (at the surface of the earth) in the real space to its counterpart (Helmert's gravity) in the abstract space is achieved by subtracting from it the "Direct Topographical Effect" (DTE) given by the following formula

$$DTE(\Omega) = - \partial V(r,\Omega)/\partial r|_{r=rt} , \tag{8}$$

where the partial derivative (in the units of acceleration) is evaluated at the surface of the earth, i.e., on the topography, for $r(\Omega) = r_t(\Omega)$. The transformation of the resulting geoidal height in Helmert's space to the real geoidal height (geoidal height in the real space) is realized by adding to it the "Primary Indirect Topographical Effect" (PITE) given by the following formula

$$PITE(\Omega) = V(r_g,\Omega) / \gamma (r_g,\Omega) , \tag{9}$$

where γ is the normal gravity. We note that PITE is evaluated at the geoid (in Helmert's space), i.e., for $r(\Omega) = r_g(\Omega)$, and is in length units. There is also another, much smaller effect, called "Secondary Idirect Topographical Effect" (SITE), which we shall not discuss here as it can be neglected under most circumstances.

Let us now concentrate on these two terms, DTE and PITE. They can be evaluated by numerical integration over topography, considering the real topographical density $\rho(r,\Omega)$, and using one of many possible mass condensation schemes. In this contribution, we shall deal with only an average topgraphical density

$$\rho(r,\Omega) = \rho_0 = 2.67 \text{ g cm}^{-3} , \tag{10}$$

although a better density model has to be used in accurate geoid determination. Finally, we shall show the models for three different mass condensation schemes:
1) the <u>mean density</u> condensation, which gives the condensation layer density σ as

$$\sigma(\Omega) = \rho_0 \, H(\Omega) \quad , \tag{11}$$

where H is the orthometric height of the terrain;
2) the mass conservation condensation, which preserves the total mass of the earth when transforming from the real to Helmert's spaces:

$$\sigma(\Omega) = \rho_0 \, H(\Omega) \, [1 + H(\Omega)/R + H^2(\Omega)/(3 \, R^2)] \quad , \tag{12}$$

where R is the mean radius of the earth [Wichiencharoen, 1982];
3) the mass-centre conservation condensation, which preserves the position of the centre of mass of the earth in the transformation into the Helmert space:

$$\sigma(\Omega) = \rho_0 \, H(\Omega) \, [1 + 3 \, H(\Omega)/(2 \, R) + H^2(\Omega)/R^2 + H^3(\Omega)/(4 \, R^3) \,] \quad , \tag{13}$$

[Wichiencharoen, 1982]. We shall consider both planar and spherical models here as the comparison of the two is our main objective. Unfortunately, however, only the first condensation scheme can be used in conjunction with the planar model; the other two schemes do not make sense in their planar form - formulae (12) and (13) have been derived for a spherical model.

For any of the condensation schemes, both DTE and PITE can be expressed as a sum of the contribution of the Bouguer spherical shell (or an infinite Bouguer plate, in the case of the planar model) of thickness H plus the contribution of the real terrain on top of the shell/plate. It turns out that the terrain contribution (called the topographical roughness term by Martinec and Vaníč ek [1994a, 1994b]) is not too sensitive to the selection of the mass condensation scheme. The terrain contributions are going to be dealt with by [Novák and Vaníč ek, 1999], and we will concentrate here only on the Bouguer shell/plate contributions and denote them by $DTE^B(\Omega)$ and $PITE^B(\Omega)$. The following table gives the overview of the results for the three different condensation schemes and the two models:

Spherical model	**Planar model**
MEAN DENSITY CONDENSATION	
$DTE^B(\Omega) = - 4\prod G\rho_0 H^2(\Omega)/R$ [Martinec and Vaníč ek, 1994b]	$DTE^B(\Omega) = 0$ [Vaníč ek and Kleusberg,1987]
$PITE^B(\Omega) = 2\prod G\rho_0 H^2(\Omega)/\gamma(\Omega$ [Vaníč ek and Martinec, 1994]	$PITE^B(\Omega) = -\prod G\rho_0 H^2(\Omega)/\gamma(\Omega)$ [Vaníč ek and Kleusberg,1987]
MASS CONSERVATION CONDENSATION	
$DTE^B(\Omega) = 0$	Not defined

[Martinec, 1994]

$PITE^{B}(\Omega) = -\, 2\prod G\rho_0 H^2(\Omega)/\gamma(\Omega)$ Not defined
[Martinec and Vaníč ek, 1994a]

MASS-CENTRE CONSERVATION CONDENSATION

$DTE^{B}(\Omega) = -\, 2\prod G\rho_0 H^2(\Omega)/R$ Not defined
[Martinec, 1994]

$PITE^{B}(\Omega) = -\, 4\prod G\rho_0 H^2(\Omega)/\gamma(\Omega)$ Not defined
[Martinec , 1994]

Now, what can we say about the individual contributions? Is there any indication that one condensation scheme is better than the others? To answer this question, we should evaluate the total topographical effect for each of the condensation schemes and compare them to establish if the results are identical or not. To evaluate the total topographical effect, the DTE has to be first transferred from the earth surface to the geoid (downward continued, as it is usually referred to), then convolved with Stokes's kernel, and finally added to the PITE. Symbolically, we can write the following algorithm for the total topographical effect on the geoid, $\delta N^{B,total}(\Omega)$:

$$DTE^{B}(\Omega) = DTE^{B}(r_t,\Omega) \;\emptyset\; DTE^{B}(r_g,\Omega) \;\emptyset\; \delta N^{B}(r_g,\Omega) = \delta N^{B}(\Omega) \qquad (14)$$

$$\delta N^{B,total}(\Omega) = \delta N^{B}(\Omega) + PITE^{B}(\Omega) \;. \qquad (15)$$

The problem here is with the downward continuation $DTE^{B}(r_t,\Omega) \;\emptyset\; DTE^{B}(r_g,\Omega)$ of DTE. A harmonic function does have a uniquely defined downward continuation which can be obtained by means of solving a boundary value problem of a Dirichlet type, leading to the solution in the form of the Poisson integral. But the downward continuation of a non-harmonic function is not defined! It is easy to prove that the residual topographical potential V is not a harmonic function within the topography:
1) the disturbing potential T satisfies the following Poisson equation within the topography

$$\Delta\, T(r,\Omega) = -\, 4\prod G\rho(r,\Omega) \;, \quad \text{for } r_g \,\varepsilon\, r > r_t \;, \qquad (16)$$

where Δ stands for the Laplacian operator;
2) the Helmert disturbing potential T^h , on the other hand, satisfies the Laplace Laplace equation everywhere above the geoid

$$\Delta\, T^{h}(r,\Omega) = 0 \;, \quad \text{for } r_g > r \;; \qquad (17)$$

3) substituting for T^h in eqn.(17) from eqn.(6) and considering eqn.(16), we get finally

$$\Delta V(r,\Omega) = 4\prod G\rho(r,\Omega) \quad , \quad \text{for } r_g \, \varepsilon \; r > r_t \; , \tag{18}$$

which concludes the proof. As V is not harmonic, there is no reason to believe that V^B is harmonic and the downward continuation of V^B , and therefore even of the DTE^B is not defined.

We thus have to conclude that there is no way of theoretically comparing the performance of the three condensation schemes. All that can be ascertained is that the first scheme changes both the mass and the centre of mass, the second changes the centre of mass, while the third changes the mass of the earth in the Helmert space. Thus the resulting geoid in Helmert's space has to be corrected either for scale, by subtracting - 4.9 cm from all the geoidal heights [Martinec, 1994], or for the shift of the geoid with respect to the centre of mass (Hørmander's corrections), amounting to (- 0.6, - 1.5, 0.2) cm [ibid], or both (in the case of mean density condensation). From the numerical point of view, the scheme that preserves the mass of the earth should be recommended.

Is there any indication that the spherical model gives significantly better results than the planar model? Not from the results above! When investigating the relative performance of the planar and spherical models in the evaluation of DTE and PITE, we can no longer disregard the terrain effect. Our numerical experiments where we had taken both the plate/shell and the terrain into account have shown [Novák et al., 1998] that significant differences are encountered when spherical and planar models are used.

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