THE TOTAL TERRAIN EFFECTS IN GEOID AND QUASIGEOID DETERMINATIONS USING HELMERT'S SECOND CONDENSATION METHOD

by

Lars E. Sjöberg
Royal Institute of Technology
Department of Geodesy and Photogrammetry
S-100 44 STOCKHOLM
SWEDEN

ABSTRACT

In precise geoid determination by Stokes' formula direct, primary and secondary terrain effects are applied for the reduction and restoring of the terrain masses. We use Helmert's second condensation method to derive the sum of the above effects, called the total terrain effect, both for the geoid and the height anomaly. The total terrain effect can be added as a correction to the original Stokes' formula. For the geoid and height anomaly computations the corrections are within 8.1 metres and 9 decimetres, respectively. Assuming that the error of the estimated density of the terrain is within 5%, the error of the geoid terrain correction is generally within 1 cm. However, for the highest mountains it may reach 40 cm.

1. INTRODUCTION

In precise geoid determination the effect of the Earth's topography is essential. The application of Stokes' formula requires that there are no masses outside the geoid. The classical way of handling this problem is to reduce for the terrain in one way or another. The reduction yields **direct effects** on gravity and the geoid determination by Stokes' integral formula. Also a small, secondary indirect effect may be applied. Finally a **primary indirect effect** is considered to correct for the height from the cogeoid to the geoid.

A question one may ask is to what accuracy the geoid may be determined due to the uncertainty of density of the terrain. The answer is also dependent on the reduction/restore procedure that is used. In an application of the Helmert's second condensation technique Vanicek and Martinec (1994) seem convinced that the accuracy 1 cm is attainable for most areas of the Earth. Below we will look into this problem and try to answer the question.

General descriptions of Helmert's second method of condensation are given in Helmert (1884), Lambert (1930), Heck (1993), Martinec and Vanicek (1993 a,b), Martinec et al. (1993), Vanicek and Martinec (1994) and Sjöberg (1993). In this paper we will use formulas for the direct and indirect effects derived in Sjöberg (1993) to power two of an expansion of the terrain elevation H. See also Sjöberg (1994).

2. DERIVATION OF TERRAIN EFFECTS

Following Heiskanen and Moritz (1967, p. 324) the geoidal undulation (N) and the height anomaly (ζ) can be determined from the following terrain corrected Stokes' formulas:

$$N_P = \frac{R}{4\pi\gamma} \iint_\sigma S(\psi_{PQ}) \Delta g_Q^* d\sigma_Q + \delta N_{II}(P) \tag{1a}$$

and

$$\zeta_P = \frac{R}{4\pi\gamma} \iint_\sigma S(r_P, \psi_{PQ}) \Delta g_Q^* d\sigma_Q + \delta\zeta_{II}(P) + \frac{2H_P}{R}\zeta_P \tag{1b}$$

where

R = mean Earth surface radius

γ = mean surface gravity

$$S(\psi_{PQ}) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(t) \tag{1c}$$

$$S(r_P, \psi_{PQ}) = \sum_{n=2}^{\infty} \left(\frac{R}{r_P}\right)^{n+1} \frac{2n+1}{n-1} P_n(t) \tag{1d}$$

$t \quad = \cos \psi_{PQ}$

$\psi_{PQ} \quad =$ geometric angle between the points P and Q

$$\Delta g_Q^{\bullet} = \Delta g_{P'}^F - H_{P'}\left(\frac{\partial \Delta g^F}{\partial H}\right)_{P'} + \delta A(H_{P'}). \tag{2}$$

In formula (2) Δg^F is the free-air anomaly and δA is the direct terrain effect; see section 2.1. The point P' is located at the Earth's surface along the normal to the reference ellipsoid through the current point Q.

In all derivations below we assume that the density of the terrain (μ) is constant.

2.1 The direct effect

The direct gravity effect (correction) $\delta A(H_P)$ was derived to order $(H_P / R)^2$ in Sjöberg (1993):

$$\delta A(H_P) = -\frac{\pi\mu}{2R}\left\{5H_P^2 + 3(\overline{H}_P)^2 + 2\sum_{n,m} n(H^2)_{nm} Y_{nm}(P)\right\} \tag{3a}$$

where

$\mu = G\rho =$ gravitational constant x density of terrain (assumed constant)

$$\overline{H}_P = \sum_{n=0}^{\infty} \frac{1}{2n+1} H_{nm} Y_{nm}(P) \tag{3b}$$

$$(H^2)_{nm} = \frac{1}{4\pi}\iint_\sigma H^2 Y_{nm} d\sigma \tag{3c}$$

$Y_{nm}(P) =$ fully normalized spherical harmonic, obeying

$$\frac{1}{4\pi}\iint_\sigma Y_{nm} Y_{n'm'} d\sigma = \begin{cases} 1 & \text{if } n = n' \text{ and } m = m' \\ 0 & \text{otherwise} \end{cases}$$

Formula (3a) may also be expressed

$$\delta A(H_P) = 12\pi\mu\frac{H_P^2}{R} - \frac{2\pi\mu}{R}\sum_{n,m}\frac{n(n-1)}{2n+1}(H^2)_{nm}Y_{nm}(P) \tag{4}$$

Inserting (4) into Stokes' formula we obtain the **direct geoid effect**

$$\delta N_{dir} = \frac{R}{4\pi\gamma}\iint_\sigma S(\psi)\delta A(H_Q)d\sigma_Q = \frac{R}{\gamma}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\frac{1}{n-1}\delta A_{nm}Y_{nm}(P), \tag{5a}$$

where

$$\delta A_{nm} = \frac{2\pi\mu}{R}\left\{6 - \frac{n(n-1)}{2n+1}\right\}(H_{nm}^2). \tag{5b}$$

Thus we obtain

$$\begin{aligned}\delta N_{dir} &= \frac{2\pi\mu}{\gamma}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\left\{\frac{6}{n-1} - \frac{n}{2n+1}\right\}(H^2)_{nm}Y_{nm} = \\ &= -\frac{\pi\mu}{\gamma}\tilde{H}_P^2 + \frac{\pi\mu}{\gamma}\overline{H}_P^2 + \frac{12\pi\mu}{\gamma}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\frac{(H^2)_{nm}}{n-1}Y_{nm}(P),\end{aligned} \tag{6}$$

(?)

where

$(H^2)_{0,0} + 5(H^2)_{1,0}$

$$\tilde{H}_P^2 = H_P^2 - \frac{1}{4\pi}\iint_\sigma H^2(1+5t)d\sigma$$

$$\overline{H}^2 = \sum_{n=2}^{\infty}\sum_{m}\frac{(H^2)_{nm}}{2n+1}Y_{nm}(P).$$

From formulas (1b) we get also to the same power of H/R:

$$\delta\zeta_{dir} = \delta N_{dir}. \tag{7}$$

The direct gravity effect (3a) includes the zero and first degree harmonics. These harmonics are filtered out in Stokes' formula (5) as $S(\psi)$ contains no zero and first degree harmonics. However, the Helmert reduction of the terrain affects the zero and first degree harmonics of the geoid as the negative of these harmonics of the primary indirect effects given below in formula (8a), i.e. by

$$\frac{-\mu}{2\gamma}\iint_\sigma H^2 d\sigma$$

and 0 , respectively.

The corresponding direct effect harmonics for the height anomaly are provided by formula (9b):

$$(\delta\zeta_{dir})_0 = -\frac{\mu}{\gamma}\iint_\sigma H^2 d\sigma \tag{8a}$$

and

$$(\delta\zeta_{dir})_1 = -\frac{\mu}{2\gamma}\sum_{m=-1}^{1}(H^2)_{1m}Y_{1m} \tag{8b}$$

2.2 The primary indirect effect

The **primary indirect effect** is the correction for restoring the terrain. The following formulas were derived in Sjöberg (1993):

$$\delta N_{I1} = \frac{\pi\mu}{\gamma}(3\overline{H}_P^2 - H_P^2) = -\frac{2\pi\mu}{\gamma}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{n-1}{2n+1}(H^2)_{nm}Y_{nm}(P) \tag{9a}$$

and

$$\delta\zeta_{I1} = \frac{\pi\mu}{\gamma}(3\overline{H}_P^2 + H_P^2) = \frac{2\pi\mu}{\gamma}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{n+2}{2n+1}(H^2)_{nm}Y_{nm}(P) \tag{9b}$$

2.3 The secondary indirect effect

The secondary indirect effect is a free air correction of gravity from geoid to cogeoid, i.e. $2\gamma\delta N_{11}/R$. This yields the following correction to the geoid determination

$$\delta N_{12}=\frac{1}{2\pi}\iint_{\sigma} S(\psi)\delta N_{11}d\sigma=-\frac{4\pi\mu}{\gamma}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\frac{1}{2n+1}(H^2)_{nm}Y_{nm}(P) \tag{10a}$$

For the height anomaly the corresponding correction becomes

$$\delta\zeta_{12}=\frac{4\pi\mu}{\gamma}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\frac{(n+2)}{(2n+1)(n-1)}(H^2)_{nm}Y_{nm}(P) \tag{10b}$$

3. THE TOTAL TERRAIN EFFECTS

Let us define the total terrain effect as the sum of indirect effect, primary and secondary indirect effects. Thus we obtain for the geoid

$$\delta N_{total}=\frac{2\pi\mu}{\gamma}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\left\{\frac{6}{n-1}-\frac{n}{2n+1}-\frac{n-1}{2n+1}-\frac{2}{2n+1}\right\}(H^2)_{nm}Y_{nm}=$$

$$=\frac{2\pi\mu}{\gamma}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\left\{\frac{6}{n-1}-1\right\}(H^2)_{nm}Y_{nm}= \tag{11a}$$

$$=-\frac{2\pi\mu}{\gamma}\tilde{H}_P^2+\frac{12\pi\mu}{\gamma}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\frac{1}{n-1}(H^2)_{nm}Y_{nm}$$

where

$$\tilde{H}^2=H^2-(H^2)_{00}-\sum_{m=-1}^{1}H_{1m}^2Y_{1m}\,. \tag{11b}$$

4. CONCLUDING REMARKS

We have shown that the direct terrain effects of the geoid and the height anomaly are equal and dominated by the term $-c_1 = -\pi\mu H_P^2/\gamma$. The primary indirect effect of the height anomaly has a major term c_1, which compensates for the direct effect. The total terrain effect is therefore small. This is not the case for the geoid computation, where the major term of the primary indirect effect $(-c_1)$ adds to the direct effect, yielding a major term $-2\pi\mu H^2/\gamma$, which reaches almost -9 metres for M. Everest.

As an alternative to the traditional application of Stokes' formula with corrections for each of the direct, primary and secondary indirect effects, we propose that the total terrain effect is computed by formula (11a) or (13) and (14) as a correction to Stokes' formula. This simplifies the computational labour.

If we assume that the terrain density in formula (13) is in 5% error, the propagated geoid error is within 40 cm. For all areas of the Earth's surface below 1300 m elevation the error is within 1 cm. Thus,. in general, the concluding statement by Vanicek and Martinec (1994), that "the Stokes-Helmert approach is good enough to give geoidal heights to a centimetre accuracy everywhere except in the highest mountains", holds. However, for the highest mountains we have indicated a somewhat higher error bound than the 10-20 cm suggested by Vanicek and Martinec.

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