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Direct topographical effect of Helmert's condensation for a spherical approximation of the geoid

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Abstract

The direct topographical effect that arises when the Stokes problem is treated by means of Helmert's 2nd condensation technique has been a subject of several studies in the recent past. In this paper, we use a spherical rather than planar model of the geoid, to derive more accurate expressions for the effect. Also our expressions are formulated so that the influence of lateral variations of topographical density can be taken into account. As a by-product of our investigation, we show that the integrals figuring in the direct topographical effect are only weakly singular and we proceed to remove the singularities by introducing spherical "Bouguer" shells.

Introduction

The effect of terrain on the computation of precise geoid undulations has been discussed by many geodesists (Helmert, 1884; Heiskanen and Moritz, 1967; Wichiencharoen, 1982; Vaníček et al., 1987; Vaníček and Kleusberg, 1987; Wang and Rapp, 1990, etc.). The problem is that Stokes' formula for gravimetric determination of the geoid requires that there be no masses outside the geoid and the gravity be referred to the geoid. These assumptions require the real Earth's topography to be disposed of.

Another complication is that the topographical masses, i.e., masses between the geoid and the topographical surface, generate a strong gravitational field capable of displacing equipotential surfaces, such as the geoid, by as much as 10^3 m. Geoidal heights of this magnitude are not observed in reality because the gravitational effect of the topographical masses is compensated very efficiently by isostatically distributed masses within the earth's crust and lithosphere. Since the knowledge of the topographical mass density is rather poor, the above fact points out that the gravitational potential of the topographical masses cannot be evaluated directly (e.g., by means of the Newton integral) to a sufficient accuracy if the geoidal height determination accuracy is required to be of the order of centimeters.

It is much more convenient to approximate the actual potential of the topographical masses by the potential of a material single layer. This can be done provided that the shape and surface density of the single layer are chosen in such a way that the layer potential approximates the basic features of the topographical potential. One way of doing it is by means of Helmert's second condensation technique (Heiskanen and Moritz, 1967, sects. 3-7, 4-3), whereby the topographical masses are condensed as a surface material layer on the geoid with the surface density given as the product of the average column density of topographical masses with the height of the topographical surface. Then the actual topographical potential V^t may be written as

$$V^t = V^c + \delta V \; , \tag{1}$$

where V^c is the gravitational potential of the material layer on the geoid and δV is the residual gravitational potential. The potential V^c has the desired properties: it is harmonic outside the geoid as required in the Stokes integration and it approximates the basic features of the topographical potential. Thus V^c can be added to the potential generated by masses contained within the geoid without destroying its harmonicity outside the geoid. It is then the sum of these two potentials that is solved for as the solution of the boundary value problem of geodesy (Heiskanen and Moritz, 1967, Chapter 2.) bypassing the necessity to evaluate V^c from topographical mass density.

Compared with the topographical potential V^t, the residual potential δV is a very small quantity generating geoidal undulation changes of the order of 2 m. That is why, δV may be evaluated to a sufficient accuracy

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directly by the Newton integration using only a rough information about the topographical density. By introducing the Helmert condensation layer the requirement on topographical density accuracy is thus reduced by about 2 orders of magnitude.

The gravitational attraction δA, defined as the attraction difference between the topographical masses and condensed topography,

$$\delta A \equiv \frac{\partial\, \delta V}{\partial r} = \frac{\partial V^t}{\partial r} - \frac{\partial V^c}{\partial r} , \qquad (2)$$

is sometimes called the "direct topographical effect" (Heiskanen and Moritz, 1967, p.145) and denoted by $-A_T + A_C$, sometimes the "attraction change effect" (Wichiencharoen, 1982), or the "topographical attraction effect" δg_T (Vaníček and Kleusberg, 1987). Throughout the paper we will refer to δA as the **direct topographical effect on gravity**. For the purpose of geoid calculation the attraction of both the topography and the condensed topography in eqn.(2) must be considered on the topographical surface (Martinec et al., 1993). Note that for quasigeoid determination the attraction of the condensed topography must be considered on the quasigeoid (Wang and Rapp, 1990; Heck, 1993).

This paper deals, once again, with the direct topographical effect, but the planar approximation of the geoid is replaced by a spherical approximation. Moreover, the usual assumption of constant density of the topographical masses (Moritz, 1968; Wichiencharoen, 1982; Vaníček and Kleusberg, 1987) is no longer needed. We also find the formulae for the direct topographical effect which enable us to integrate over the topographical masses in the immediate neighbourhood of the computation point as well as in the far-zone. Thus the presented theoretical results are more general and more accurate than those derived by Vaníček and Kleusberg (1987).

The external gravitational potential of topographical masses

Let the topographical masses be bounded by the the geoid with the geocentric radius $r_g(\Omega)$ and by the outer topographical surface with the geocentric radius $r_g(\Omega)+H(\Omega)$. This means that $H(\Omega)$ is the height of the topographical surface above the geoid reckoned along the geocentric radius; this height, to a relative accuracy better than 5×10^{-6}, is equal to the ordinary orthometric height. We shall assume throughout the paper that $H > 0$. The argument Ω stands for a horizontal position given by co-latitude ϑ and longitude λ. The gravitational potential induced by the topographical masses at a point (r,Ω) is given by the Newton volume integral

$$V^t(r,\Omega) = G \int_{\Omega'} \int_{r'=r_g(\Omega')}^{r_g(\Omega')+H(\Omega')} \frac{\rho(r',\Omega')}{|\mathbf{r}'-\mathbf{r}|}\, r'^2 dr' d\Omega' , \qquad (3)$$

where $\rho(r',\Omega')$ is the density of the topographical masses, $|\mathbf{r}'-\mathbf{r}|$ is the distance between the integration and the computation points, and Ω' is the full solid angle.

To decompose the potential V^t according to eqn.(1), we can proceed in at least two ways. Taking a suitable analytical model for density ρ, the integral over r' in eqn.(3) may be evaluated analytically. Then the result can be expanded into a series expansion with respect to H in such a way that the first term is the potential V^c. We will follow the other possibility: we first expand the reciprocal distance $1/|\mathbf{r}'-\mathbf{r}|$ in eqn.(3) by means of a Taylor series expansion at a point on the geoid. This ensures that the condensation layer will be on the geoid. Integration over r' will then show that the potential V^t may be decomposed according to eqn.(1).

In the formula for the direct topographical effect, the gravitational potential $V_t(r,\Omega)$ and its radial derivative must be evaluated at a point on the topographical surface (Martinec at al., 1993). Since the reciprocal distance becomes infinite at this point, we have to proceed the following way: we first derive the gravitational field induced by the topographical masses at an arbitrary point above the topographical surface. Then the point will be brought down to the topographical surface. We will show that for this case the integration kernel of the potential V^t is weakly singular. The singularity can be removed by subtracting and adding the gravitational potential of a spherical Bouguer shell.

Let us emphasize that in this paper we will treat only the **external** gravitational field generated by topographical masses. The reason is evident: as mentioned in the Introduction, the formulae are intended to be used only at the earth's surface. When the potential $V^t(r,\Omega)$ is evaluated outside the topographical masses, i.e., for points $r > r_g(\Omega)+H(\Omega)$, the distance $|\mathbf{r}'-\mathbf{r}|$ never goes to zero, and the reciprocal distance is thus always bounded. The function $1/|\mathbf{r}'-\mathbf{r}|$ may be therefore expanded into a Taylor series. Denoting the kernel of the Newton integral (3) by

$$J(r,\Omega,r',\Omega') = \frac{r'^2}{|\mathbf{r}'-\mathbf{r}|} , \qquad (4)$$

we have

$$J(r,\Omega,r',\Omega') = \sum_{i=0}^{n} \frac{1}{i!} \left.\frac{\partial^i J(r,\Omega,r',\Omega')}{\partial r'^i}\right|_{r'=r_g(\Omega')} [r'-r_g(\Omega')]^i + R_{n+1}(r,\Omega,r',\Omega';\eta) . \qquad (5)$$

The remainder of the Taylor series, R_{n+1}, may be written in the Lagrange form

$$R_{n+1}(r,\Omega,r',\Omega';\eta) =$$

$$= \frac{1}{(n+1)!} \frac{\partial^{n+1} J(r,\Omega,r',\Omega')}{\partial r'^{n+1}}\bigg|_{r'=r_g(\Omega')+\eta(r'-r_g(\Omega'))} \times$$

$$\times \left[r' - r_g(\Omega')\right]^{n+1} , \qquad (6)$$

where the parameter η takes a value from the interval $(0,1)$. The $(n+1)$-th derivative of the kernel $J(r,\Omega,r',\Omega')$ with respect to r' may be expressed as a finite power series of the reciprocal distance $1/|\mathbf{r}'-\mathbf{r}|$ (cf., eqn. (A11) in the first Appendix). As explained above, the reciprocal distance is always bounded, and so is the $(n+1)$-st derivative of the kernel $J(r,\Omega,r',\Omega')$. Mathematically, for each $r' \in \langle r_g(\Omega'), r_g(\Omega')+H(\Omega')\rangle$ and $r > r_g(\Omega)+H(\Omega)$, it holds

$$\left| \frac{\partial^{n+1} J(r,\Omega,r',\Omega')}{\partial r'^{n+1}}\bigg|_{r'=r_g(\Omega')+\eta(r'-r_g(\Omega'))} \right| < C , \quad (7)$$

where C is a finite number equal to

$$C = \max\left(\frac{\partial^{n+1} J(r,\Omega,r',\Omega')}{\partial r'^{n+1}}\bigg|_{r'=r_g(\Omega')} ; \right.$$

$$\left. \frac{\partial^{n+1} J(r,\Omega,r',\Omega')}{\partial r'^{n+1}}\bigg|_{r'=r_g(\Omega')+H(\Omega')} \right) . \quad (8)$$

Because of

$$\lim_{n\to\infty} \frac{\left[r'-r_g(\Omega')\right]^{n+1}}{(n+1)!} = 0 \qquad (9)$$

for each $r' \in \langle r_g(\Omega'), r_g(\Omega')+H(\Omega')\rangle$, the limit of eqn. (6) for $n \to \infty$ becomes

$$\lim_{n\to\infty} R_{n+1}(r,\Omega,r',\Omega';\eta) = 0 . \qquad (10)$$

This means that the Taylor series (5) converges and its sum from $i=0$ to $i=\infty$ yields $r'^2/|\mathbf{r}'-\mathbf{r}|$.

Inserting series (5) for $n\to\infty$ into (3), and interchanging the summation over i and integration over r' and Ω' (which is admissible because of the uniform convergence of the Taylor series (5)), we get

$$V^t(r,\Omega) = \sum_{i=0}^{\infty} V_i(r,\Omega) , \qquad (11)$$

where for all i:

$$V_i(r,\Omega) = \frac{G}{i!}\int_{\Omega'} \frac{\partial^i J(r,\Omega,r',\Omega')}{\partial r'^i}\bigg|_{r'=r_g(\Omega')} \times$$

$$\times \int_{r'=r_g(\Omega')}^{r_g(\Omega')+H(\Omega')} \rho(r',\Omega')\left[r'-r_g(\Omega')\right]^i dr' d\Omega' . \quad (12)$$

Let us have a look first at the potential V_0. Taking into account the definition (4), eqn.(12) for $i=0$ reads:

$$V_0(r,\Omega) =$$

$$= G\int_{\Omega'} \frac{r_g^2(\Omega')}{|\mathbf{r}_g(\Omega')-\mathbf{r}|}\int_{r'=r_g(\Omega')}^{r_g(\Omega')+H(\Omega')} \rho(r',\Omega')dr'd\Omega' . \qquad (13)$$

Denoting the average density along the topographical column of height $H(\Omega')$ as

$$\rho(\Omega') = \frac{1}{H(\Omega')}\int_{r'=r_g(\Omega')}^{r_g(\Omega')+H(\Omega')} \rho(r',\Omega')dr' , \qquad (14)$$

the potential V_0 takes the following form

$$V_0(r,\Omega) = G\int_{\Omega'} \rho(\Omega')\, H(\Omega')\, \frac{r_g^2(\Omega')}{|\mathbf{r}_g(\Omega')-\mathbf{r}|}\, d\Omega' . \quad (15)$$

The potential V_0 is the gravitational potential of a material surface on the geoid with the surface density $\sigma(\Omega') = \rho(\Omega')\,H(\Omega')$ and radius $r_g(\Omega)$. In the context of Stokes-Helmert approach, this surface is the condensation layer, and V_0 is thus the potential V^c defined in the Introduction. We have thus shown that if the Helmert condensation layer has the density $\rho(\Omega')\,H(\Omega')$, where $\rho(\Omega')$ represents the mean actual value of topographical density along the column of height $H(\Omega')$ in the sense of eqn.(14), the **external** gravitational potential of topographical masses V^t can be written according to eqn.(1). The residual potential δV is given by a sum of gravitational potentials V_i, $i = 1, 2, \dots$:

$$\delta V(r,\Omega) = \sum_{i=1}^{\infty} V_i(r,\Omega) . \qquad (16)$$

As shown by Wichiencharoen (1982), Wang and Rapp (1990), or Martinec and Vaníček (1994), the equipotential surface undulations generated by the residual potential δV for a constant density of 2.67 g/cm^3 are of the order of 2 m. That is why, for the purpose of computing the residual potential δV, the geoid may be modelled by a sphere of radius R,

$$r_g(\Omega) = R , \qquad (17)$$

where R is the mean radius of the Earth. This approximation is justifiable because the error introduced by this approximation is at most 3×10^{-3} (Heiskanen and Moritz, 1967, sect. 2-14.) which then causes an error of at most 6 mm in the geoidal height.

For the same reason, the density of topographical masses in the residual potential δV may be modelled by a mean value $\rho(\Omega')$ of the actual density $\rho(r',\Omega')$ along the topographical column of height $H(\Omega')$,

$$\rho(r',\Omega') = \rho(\Omega') . \qquad (18)$$

Note that this assumption still allows us to consider lateral density variations in topographical masses which is a more general model than the usual assumption of constant density (Wichiencharoen, 1982; Vaníček and Kleusberg,1987).

Under the assumptions (17) and (18), the potentials V_i, $i = 1, 2, ...$, acquire the following form

$$V_i(r,\Omega) = \frac{G}{i!}\int_{\Omega'} \rho(\Omega')\frac{\partial^i}{\partial r'^i}\left(\frac{r'^2}{|\mathbf{r}'-\mathbf{r}|}\right)\Bigg|_{r'=R} \times$$

$$\times \int_{r'=R}^{R+H(\Omega')} [r'-R]^i \, dr' d\Omega' . \qquad (19)$$

Performing the integration over r', we get

$$V_i(r,\Omega) =$$

$$= \frac{G}{(i+1)!}\int_{\Omega'} \rho(\Omega')H^{i+1}(\Omega')\frac{\partial^i}{\partial r'^i}\left(\frac{r'^2}{|\mathbf{r}'-\mathbf{r}|}\right)\Bigg|_{r'=R} d\Omega' . \qquad (20)$$

It is advantageous to re-arrange eqn.(20) in a different form. Multiplying and dividing it by R^{i-2}, we get

$$V_i(r,\Omega) = \frac{GR^{-i+2}}{(i+1)!}\int_{\Omega'} \rho(\Omega')H^{i+1}(\Omega')M_i(r,\Omega,R,\Omega')d\Omega' , \qquad (21)$$

where the integration kernel M_i, $i > 0$, is isotropic and can be written as

$$M_i(r,\Omega,R,\Omega') \equiv M_i(r,\psi) = R^{i-2}\frac{\partial^i}{\partial r'^i}\left(\frac{r'^2}{|\mathbf{r}'-\mathbf{r}|}\right)\Bigg|_{r'=R} . \qquad (22)$$

As usual, ψ is the geocentrical angular distance between the points Ω and Ω'.

As shown in the Appendix A, the integration kernel $M_i(r,\psi)$, $i = 1, 2, ...$, may be expressed in two different ways: (i) in a spectral form as an infinite series of Legendre polynomials in the variable $\cos\psi$, cf. eqn.(A3),

$$M_i(r,\psi) = \frac{i!}{R}\sum_{j=0}^{\infty}\binom{j+2}{i}\left(\frac{R}{r}\right)^{j+1} P_j(\cos\psi) , \quad (23)$$

or (ii) in a spatial form as a finite power series of the reciprocal spatial distance $1/\ell$, cf. eqns.(A12)-(A13),

$$M_i(r,\psi) = \frac{1}{\ell}\tilde{M}_i(r,\psi) , \qquad (24)$$

where, for $i > 1$,:

$$\tilde{M}_i(r,\psi) = \sum_{s=1}^{i-1}\frac{i!(i-2)!}{(i-s-1)!(s-1)!}\left(\frac{r}{\ell}\right)^{i+1-s} \times$$

$$\times \sum_{t=0}^{i+1-s}(-1)^{\frac{3i+1-s+t}{2}}\frac{(i+2-s-t)!!(i-s+t)!!}{(i+2-s-t)!\,t!}\times$$

$$\times\left(\frac{r-R\cos\psi}{\ell}\right)^t . \qquad (25)$$

The summation in eqn.(25) must be taken over such t's for which $i-s+t+1$ is an even number. The symbol ℓ denotes the spatial distance between points (r,Ω) and (R,Ω'), i.e.,

$$\ell = \sqrt{r^2+R^2-2rR\cos\psi} . \qquad (26)$$

It is instructive to evaluate the first two integration kernels M_i explicitly. Performing the required differentiation with respect to r' and putting $r' = R$, eqn.(22) (for $i = 1$) reads

$$M_1(r,\psi) = \frac{1}{\ell} + \frac{r}{\ell^3}(r - R\cos\psi) . \qquad (27)$$

Putting $i = 2$ in eqns.(24) and (25), we get

$$M_2(r,\psi) = -\frac{r^2}{\ell^3} + \frac{3r^2}{\ell^5}(r-R\cos\psi)^2 . \qquad (28)$$

The direct topographical effect on the potential

Until now we have assumed that the point of interest was outside the topographical surface. Let us now constrain the computation point to the earth surface so that for any Ω, $r = r(\Omega) = R + H(\Omega)$. We note that under this constraint we have

$$\lim_{\psi\to 0} H(\Omega') = H(\Omega) = H . \qquad (29)$$

The gravitational potential $\delta V(r,\Omega)$ at a point on the topographical surface $(r = R + H)$ will be called the **direct topographical effect on potential**.

In the vicinity of a computation point at the sea level, i.e., when ψ and H go to 0 simultaneously, i.e., also $\ell \to 0$, the integration kernels M_i grow beyond all limits. To decide what kind of singularity we would be faced with in eqn.(21), let us investigate the behaviour of the product $H^{i+1}(\Omega')M_i(r,\psi)$ at the point $\psi = 0$. First we transform the integration over spherical coordinates Ω' (eqn.(21)) to an integration over polar coordinates (ψ,α):

$$V_i(r,\Omega) = \frac{GR^{-i+2}}{(i+1)!}\times$$

$$\times\int_{\alpha=0}^{2\pi}\int_{\psi=0}^{\pi}\rho(\psi,\alpha)H^{i+1}(\psi,\alpha)\tilde{M}_i(r,\psi)\frac{\sin\psi}{\ell}d\psi d\alpha , \qquad (30)$$

where the integration kernel M_i has been expressed by means of the reduced kernel $\tilde{M}_i$ (cf. eqn.(24)).

Eqns.(A15) and (A16) in the Appendix A show that

$$\lim_{\psi\to 0}\tilde{M}_1(r,\psi) = 1 + \frac{r}{H} , \qquad (31)$$

and

$$\forall i > 1: \qquad \lim_{\psi\to 0}\tilde{M}_i(r,\psi) = \sum_{s=1}^{i-1} a_{is}\left(\frac{r}{H}\right)^{i+1-s} , \quad (32)$$

where H is the height of the computation point above the geoid and the coefficients a_{is} are finite (given by eqn.(A17)). Since the computation point is on the topographical surface, then for $\psi \to 0$: $H(\psi,\alpha) \to H$. Thus we have

$$\lim_{\psi\to 0} H^2(\psi,\alpha)\tilde{M}_1(r,\psi) = H^2 + rH \ , \tag{33}$$

and

$$\forall i>1: \quad \lim_{\psi\to 0} H^{i+1}(\psi,\alpha)\tilde{M}_i(r,\psi) = r^{i+1}\sum_{s=1}^{i-1} a_{is}\left(\frac{H}{r}\right)^s . \tag{34}$$

Since the height H is bounded, eqns.(33) and (34) show that the function $H^{i+1}\tilde{M}_i$ is also bounded at the point $\psi = 0$. Moreover, it vanishes if $H \to 0$. The ratio $\sin\psi/\ell$ remains bounded if $\psi \to 0$. Thus the whole integrand in the integral (30) remains bounded at the point $\psi = 0$. The integrals in eqn.(30) or (21) are ordinary improper integrals: they are only weakly singular.

To remove the (removable) singularity at the point $\psi = 0$, let us transform the integral (21) to the following form, by a simple algebraic operation:

$$\forall i>0: \quad V_i(r,\Omega) = \frac{GR^{-i+2}}{(i+1)!}\left\{\int_{\Omega'}\left[\rho(\Omega')H^{i+1}(\Omega') - \right.\right.$$
$$\left.\left. -\rho(\Omega)H^{i+1}\right] M_i(r,\psi)d\Omega' + \rho(\Omega)H^{i+1}\int_{\Omega'} M_i(r,\psi)d\Omega'\right\} . \tag{35}$$

As shown in the Appendix A, for i=1,2 the integral over $M_i(r,\psi)$ is equal to $8\pi/r$, (eqns.(A22) and (A24)) and vanishes for $i > 2$. By summing up the potentials V_i, $i = 1, 2, ...$, we get the direct topographical effect on the potential in the following form:

$$\delta V(r,\Omega) = \delta V_B(r,\Omega) + \delta V_R(r,\Omega) \ , \tag{36}$$

where

$$\delta V_B(r,\Omega) = 4\pi G\rho(\Omega)H^2\left(1+\frac{H}{3R}\right)\frac{R}{r} \tag{37}$$

$$\delta V_R(r,\Omega) = G\sum_{i=1}^{\infty}\frac{R^{-i+2}}{(i+1)!}\int_{\Omega'}\left[\rho(\Omega')H^{i+1}(\Omega') - \right.$$
$$\left. -\rho(\Omega)H^{i+1}\right] M_i(r,\psi)d\Omega' . \tag{38}$$

The "Bouguer term" δV_B represents the potential of the spherical Bouguer shell of density equal to topographical density $\rho(\Omega)$ at the computation point and thickness equal to the height $H(\Omega)$ of the computation point (cf., Wichiencharoen, 1982), and the "terrain roughness term" δV_R shows the contribution of the varying topographical heights and the effect of lateral variations in topographical density.

The direct topographical effect on gravity

Differentiating $\delta V(r,\Omega)$ with respect to r and evaluating the result at the point on the topography, we obtain the change of the gravitational attraction caused by the direct topographical effect (c.f. eqn.(2)):

$$\delta A(r,\Omega) = \left.\frac{\partial\,\delta V}{\partial r}\right|_{r=R+H} . \tag{39}$$

Substituting for δV from eqn.(36), the attraction change δA may be split into two terms as

$$\delta A(r,\Omega) = \delta A_B(r,\Omega) + \delta A_R(r,\Omega) \ , \tag{40}$$

where the Bouguer term δA_B and the terrain roughness term δA_R are:

$$\delta A_B(r,\Omega) = -4\pi G\rho(\Omega)H^2\left(1+\frac{H}{3R}\right)\frac{R}{r^2} \ , \tag{41}$$

$$\delta A_R(r,\Omega) = G\sum_{i=1}^{\infty}\frac{R^{-i+2}}{(i+1)!}\int_{\Omega'}\left[\rho(\Omega')H^{i+1}(\Omega') - \right.$$
$$\left. -\rho(\Omega)H^{i+1}\right] N_i(r,\psi)d\Omega' . \tag{42}$$

Here, the integration kernels $N_i(r,\psi)$ are defined as the radial derivatives of the $M_i(r,\psi)$ kernels:

$$\forall i>0: \quad N_i(r,\psi) = \frac{\partial M_i(r,\psi)}{\partial r} . \tag{43}$$

Using the recurrence relation (A8), the kernels $N_i(r,\psi)$ may be expressed as

$$\forall i>0: \quad N_i(r,\psi) = -\frac{1}{r}\left[M_{i+1}(r,\psi) + (i-1)M_i(r,\psi)\right] . \tag{44}$$

The interpretation of the terms in eqn.(40) is similar to that for the direct topographical effect on potential.

Let us explore now the behaviour of the product $H^{i+1}(\Omega')N_i(r,\psi)$ in the vicinity of the computation point, i.e., when $\psi \to 0$. We will proceed the same way as we did in the case of the direct effect on the potential. We first introduce the reduced kernels $\tilde{N}_i(r,\psi)$ by the prescription

$$\forall i>0: \quad \tilde{N}_i(r,\psi) = \ell\, N_i(r,\psi) \tag{45}$$

and express them in terms of the reduced kernels $\tilde{M}_i(r,\psi)$ using eqn.(44):

$$\forall i>0: \quad \tilde{N}_i(r,\psi) = -\frac{1}{r}\left[\tilde{M}_{i+1}(r,\psi) + (i-1)\tilde{M}_i(r,\psi)\right] . \tag{46}$$

We can now evaluate the limit of the kernel $\tilde{N}_i(r,\psi)$ for $\psi \to 0$ by means of eqns.(31) and (46),

$$\forall i>0: \quad \lim_{\psi\to 0}\tilde{N}_i(r,\psi) = -\frac{1}{r}\left[\sum_{s=1}^{i} a_{i+1,s}\left(\frac{r}{H}\right)^{i+2-s} + \right.$$

$$+(i-1)\sum_{s=1}^{i-1} a_{is}\left(\frac{r}{H}\right)^{i+1-s}\Bigg] , \tag{47}$$

where the coefficients a_{is} are given by eqn.(A17). Eqn. (47) shows that if the height of the computational point goes to zero, the integral kernel $\tilde{N}_i(r,\psi)$ goes to infinity as $1/H^{i+1}$.

Since the computation point is considered to be on the topographical surface, then as $\psi \to 0$: $H(\Omega') \to H$, and we can write for all $i > 0$:

$$\lim_{\psi\to 0} H^{i+1}(\Omega')\tilde{N}_i(r,\psi) = -r^i\left[\sum_{s=1}^{i} a_{i+1,s}\left(\frac{H}{r}\right)^{s-1} + \right.$$

$$\left. +(i-1)\sum_{s=1}^{i-1} a_{is}\left(\frac{H}{r}\right)^{s}\right] . \tag{48}$$

This relation shows that the product $H^{i+1}(\Omega')\tilde{N}_i(r,\psi)$ is bounded at the point $\psi = 0$. Then also the product $H^{i+1}(\Omega')N_i(r,\psi)\sin\psi$ is bounded at $\psi = 0$ and thus the integral (42) is regular.

The planar approximation

The general formulae (40)-(42), accurate to a spherical approximation and using the mean radial topographical density, can be further simplified by accepting certain approximations permissible from the accuracy point of view. One such approximation, useful for the regional determination of the geoid is the planar approximation based on the fact that the ratio H/R never exceeds the value of 1.4×10^{-3}. The planar approximation (not to be confused with a planar model of the geoid) is acceptable because it produces an error of the same order as the relative error 3×10^{-3} of spherical approximation used throughout the paper. Employing this approximation, quantities of the order of magnitude of H/R are neglected with respect to 1. For instance, the planar approximation of the spatial distance ℓ between the points $(R+H,\Omega)$ and (R,Ω') is found to be (Moritz, 1966):

$$\ell^2 \approx \ell_0^2 + H^2 , \tag{49}$$

where

$$\ell_0 = 2R\sin\frac{\psi}{2} \tag{50}$$

is the (horizontal) spatial distance between points (R,Ω) and (R,Ω').

Let us look now for the planar approximation of the direct topographical effect on gravity. The direct topographical effect on gravity δA_B caused by a Bouguer shell, cf. eqn.(41), reads in planar approximation:

$$\delta A_B \doteq -4\pi G\rho(\Omega)\frac{H^2}{R} . \tag{51}$$

To compare our result with those derived by other authors, we shall take only the first term in the summation (eqn.(42)) to represent δA_R and then carry out its planar approximation. We denote this term by δA_{R1} and write

$$\delta A_{R1}(r,\Omega) =$$

$$= \frac{1}{2}GR\int_{\Omega'}\left[\rho(\Omega')H^2(\Omega') - \rho(\Omega)H^2\right] N_1(r,\psi)d\Omega' , \tag{52}$$

where the kernel $N_1(r,\psi)$ is given by eqn.(43) (for $i=1$) as

$$N_1(r,\psi) = \frac{\partial M_1(r,\psi)}{\partial r} . \tag{53}$$

Differentiating M_1 (eqn.(27)) with respect to r, we get

$$N_1(r,\psi) = \frac{r}{\ell^3}\left[1 - 3\left(\frac{r - R\cos\psi}{\ell}\right)^2\right] . \tag{54}$$

Using the relation

$$r - R\cos\psi = H + \frac{\ell_0^2}{2R} , \tag{55}$$

which follows from eqns.(26) and (50), and substituting for $r = R+H$, the kernel $N_1(r,\psi)$ may be expressed as

$$N_1(r,\psi) = \frac{R}{\ell^3}\left(1+\frac{H}{R}\right)\left(1 - 3\frac{H^2}{\ell^2} - 3\frac{H\ell_0^2}{R\ell^2} - \frac{3}{4}\frac{\ell_0^4}{R^2\ell^2}\right) . \tag{56}$$

Let us now take the planar approximation of eqn.(56), i.e., we will neglect all the terms whose magnitude is of the order of H/R or smaller with respect to 1. Under this approximation, the second term in the first bracket and the third term in the second bracket (because $\ell_0/\ell \leq 1$ for all ψ's) on the right-hand side of eqn.(56) may be omitted. We obtain

$$N_1(r,\psi) \approx \frac{R}{\ell^3}\left(1 - 3\frac{H^2}{\ell^2} - \frac{3}{4}\frac{\ell_0^4}{R^2\ell^2}\right) . \tag{57}$$

Since the relative magnitude of the individual terms in eqn.(57) depends on the distance ℓ_0 between the computation point and the integration point, we divide the integration domain into two zones according to the magnitude of the variables ℓ_0 and H that affect the spatial distance ℓ (cf. eqn.(49)).

(1) The **near-zone** covers the immediate neighbourhood of the computation point. Substituting for ℓ from eqn.(49), the kernel N_1 (eqn.(57)) reads

$$N_1(r,\psi) \approx \frac{R}{\ell^3}\left(1 - 3\frac{H^2}{\ell^2} - \frac{3}{4}\frac{\ell_0^2}{R^2}\frac{\ell_0^2}{\ell_0^2+H^2}\right) . \tag{58}$$

To maintain the accuracy of the planar approximation, the last term on the right-hand side may be neglected when its magnitude becomes as small as H/R, i.e., when

$$\frac{\ell_0^2}{R^2} < \frac{H}{R} . \tag{59}$$

Substituting for ℓ_0 from eqn.(50), this condition reads

$$\sin\frac{\psi}{2} < \frac{1}{2}\sqrt{\frac{H}{R}} \,. \qquad (60)$$

Since $H/R \ll 1$, we may put $\sin\psi/2 \doteq \psi/2$, and the condition (60) reduces to:

$$\psi < \sqrt{\frac{H}{R}} \,. \qquad (61)$$

The near-zone is then defined by those ψ's that satisfy the above inequality. Under this condition, the last term in eqn.(58) can be also neglected, and we have

$$N_1(r,\psi) \approx \frac{R}{\ell^3}\left(1 - 3\frac{H^2}{\ell^2}\right) \,. \qquad (62)$$

(2) The **far-zone** covers the area farther from the computation point, for which $\ell_0 \gg H$. In this zone, the spatial distance ℓ is approximately equal to ℓ_0, and the integration kernel (57) reduces to

$$N_1(r,\psi) \approx \frac{R}{\ell_0^3}\left(1 - 3\frac{H^2}{\ell_0^2} - \frac{3}{4}\frac{\ell_0^2}{R^2}\right) \,. \qquad (63)$$

Let us look for a condition under which the second term may be also neglected. To maintain the accuracy of planar approximation, this term may be neglected if it is as small as H/R, i.e., if

$$\frac{H^2}{\ell_0^2} \leq \frac{H}{R} \,. \qquad (64)$$

Substituting for ℓ_0 from eqn.(50), this condition reads

$$\sin\frac{\psi}{2} \geq \frac{1}{2}\sqrt{\frac{H}{R}} \,, \qquad (65)$$

or, putting $\sin\psi/2 \doteq \psi/2$, we get:

$$\psi > \sqrt{\frac{H}{R}} \,. \qquad (66)$$

Under this condition, the second term in eqn.(63) can be also neglected, and we have

$$N_1(r,\psi) \approx \frac{R}{\ell_0^3}\left(1 - \frac{3}{4}\frac{\ell_0^2}{R^2}\right) \,. \qquad (67)$$

Considering eqn.(50) we may also write

$$N_1(r,\psi) \approx \frac{R}{\ell_0^3}\left(1 - 3\sin^2\frac{\psi}{2}\right) \,; \qquad (68)$$

the integration kernel N_1 in the far-zone is homogeneous (as well as isotropic) because it does not depend on the vertical position of the computational point.

The boundary between the near- and far-zones is defined by the spherical distance ψ_0,

$$\psi_0 = \sqrt{\frac{H}{R}} \,. \qquad (69)$$

The smaller the height of the computation point, the smaller the near zone and the larger is the far-zone. In the extreme case when the computational point is on Mount Everest, the near zone extends to the angular distance of about $\psi_0 = 2^o$, and the far-zone from 2^o to 180^o. When $H = 0$, there is no near zone.

Moreover, if on top of the planar approximation of the integration kernel, the density of the topographical masses is assumed to be constant, $\rho(\Omega) = \rho_0$, the planar approximation of the direct topographical effect δA becomes

$$\delta A(r,\Omega) \doteq \delta A_B(r,\Omega) + \delta A_{R1}(r,\Omega) = -4\pi G\rho_0\frac{H^2}{R} +$$

$$+\frac{1}{2}GR^2\rho_0\left[\int_{\Omega'_1}\frac{H^2(\Omega') - H^2}{\ell^3}\left(1 - 3\frac{H^2}{\ell^2}\right)d\Omega' + \right.$$

$$\left. + \int_{\Omega'_2}\frac{H^2(\Omega') - H^2}{\ell_0^3}\left(1 - 3\sin^2\frac{\psi}{2}\right)d\Omega'\right] \,, \qquad (70)$$

where Ω'_1 denotes the near-zone and Ω'_2 the far-zone. It is not known if eqn.(70) is a sufficient approximation of the rigorous equation arising from the use of formule (54) or (56). Thus, it should be pointed out that the formula (52) based on (54) or (56) should be preferred in practical computations.

Vaníček and Kleusberg (1987), for example, determined that the change of the gravitational attraction due to the direct topographical effect is equal to (ibid., eq.(14))

$$\delta g_T = \frac{1}{2}GR^2\rho_0\int_{\Omega'}\frac{H^2(\Omega') - H^2}{\ell_0^3}d\Omega' \,. \qquad (71)$$

This formula was derived taking into consideration the grid size of the then available gridded topography and was really meant to apply only to $\ell_0 \gg H$. This implies that only the effect of the far-zone has been considered by the authors, as already noted by Heck (1993), and eqn.(71) can thus be compared only to the 3rd term in eqn.(70). Even then the correction term $3\sin^2\psi/2$ revealed by the spherical model is missing in eqn.(71). This, perhaps, is not a crucial omission since the correction term starts affecting the resulting geoid in the 0.5% range only for $\psi > 4.7^o$ at which distance the effect is well subdued by the $1/\ell_0^3$ factor.

The effects of the near-zone and the Bouguer shell are completely missing in eqn.(71). The first term, once again, cannot be derived from a planar model because it represents a correction for the sphericity of the geoid. Yet, its magnitude is significant; in Canada, for example, it may reach as much as 0.4 mGal, which cannot be neglected when a precise geoid is to be determined. We have not investigated numerically the magnitude of the near-zone effect but from the shape of the formula it is evident that it cannot be neglected either.

Comparison of this result with the "standard" expression, e.g., Wang and Rapp, (1990), Sideris and Forsberg (1990),

$$\delta A(r,\Omega) = \frac{1}{2} G R^2 \rho_0 \int_{\Omega'} \frac{[H(\Omega') - H]^2}{\ell_0^3} d\Omega' \qquad (72)$$

reveals even more serious problems. The derivation of this formula necessitates that the Pellinen (1962) credo of linear dependence of free-air anomalies on heights be invoked. This makes the validity of eqn.(72) for computing the direct topographical effect in the context of geoid computations questionable (Martinec et al., 1993).

Conclusions

This theoretical study was motivated by one of the fundamental assumptions of the Stokes technique for determination of geoidal heights which requires that there be no masses outside the geoid. We have discussed the reason why the gravitational field of the topographical masses cannot be evaluated directly (by, e.g., Newton integration), and why it is advantageous to approximate the gravitational field of the topographical masses by that of the condensation layer - as originally suggested by Helmert. We have shown that the Helmert condensation layer may be introduced even if the density of the topographical masses is modelled as laterally varying. The condensation layer is placed on the geoid and its surface density is equal to the average column density of topographical masses multiplied by topographical heights.

The external gravitational field of the topographical masses is described by the potential of condensation layer and by a series of potentials that can be viewed as potentials of multi-layered material surfaces with different densities. We have found the spectral as well as spatial representation of the integration kernels figuring in these multi-layer gravitational potentials. It is still unclear under which circumstances the series of these approximating potentials converges, and how many terms of the potential series are to be taken to approximate the external gravitational field of topographical masses with a sufficient accuracy. In contrast to the Molodensky integral solution (Moritz, 1980, sect.43), the gravitational attraction induced by the residual potential δV at the point on the topography is represented by weakly singular integrals. The singularities can be easily removed by adding and subtracting the Bouguer shell to topographical masses.

Throughout the paper we have used the spherical approximation of the geoid. This approximation enabled us to derive the direct topographical effect with high enough accuracy to cause an error in geoidal heights of at most 6 mm. We have discussed the planar approximation of the general formula and concluded that the standard expressions used for computing the topographical effect are of questionable accuracy because of the planar model used for their derivation. As a result, they are uniformly 'blind' to all spherical effects.

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Appendix A:

Integration kernels $M_i(r,\psi)$

In this section we shall derive some useful properties of the integration kernels $M_i(r,\psi)$, $i = 0,1,2,...$, introduced in eqn.(22).

Spectral form. Let us start by expressing the integration kernel $M_i(r,\psi)$ as a series of Legendre polynomials in $\cos\psi$. Because $r > r'(= R)$, the Newton kernel may be expanded into a uniformly convergent series as

$$\frac{1}{|\mathbf{r}'-\mathbf{r}|} = \frac{1}{r}\sum_{j=0}^{\infty}\left(\frac{r'}{r}\right)^{j} P_j(\cos\psi) \ . \qquad (A1)$$

Multiplying eqn.(A1) by ${r'}^2$ and differentiating the result i-times with respect to r', we get

$$\frac{\partial^i}{\partial {r'}^i}\left(\frac{{r'}^2}{|\mathbf{r}'-\mathbf{r}|}\right) = (r')^{-i+1}\times$$

$$\times\sum_{j=0}^{\infty}(j+2)(j+1)...(j+2-i+1)\left(\frac{r'}{r}\right)^{j+1} P_j(\cos\psi) \ . \qquad (A2)$$

It is possible to change the order of summation and differentiation because the series on the right-hand side is uniformly convergent. Inserting eqn.(A2) into (22), we obtain for all $i > 0$:

$$M_i(r,\psi) =$$

$$= \frac{1}{R}\sum_{j=0}^{\infty}(j+2)(j+1)...(j+2-i+1)\left(\frac{R}{r}\right)^{j+1} P_j(\cos\psi) =$$

$$= \frac{i!}{R}\sum_{j=0}^{\infty}\binom{j+2}{i}\left(\frac{R}{r}\right)^{j+1} P_j(\cos\psi) \ . \qquad (A3)$$

Recursive formula. The integration kernel $M_i(r,\psi)$ may be obtained by means of a recursive formula. Multiplying eqn.(A3) by r^{i-1} and differentiating with respect to r, we get

$$\frac{\partial}{\partial r}\left(r^{i-1}M_i(r,\psi)\right) =$$

$$= -\frac{i!}{R}r^{i-2}\sum_{j=0}^{\infty}\binom{j+2}{i}(j+2-i)\left(\frac{R}{r}\right)^{j+1} P_j(\cos\psi) \ . \qquad (A4)$$

But, it can be shown that:

$$i!\binom{j+2}{i}(j+2-i) = (i+1)!\binom{j+2}{i+1} \ , \qquad (A5)$$

and eqn.(A4) becomes

$$\frac{\partial}{\partial r}\left(r^{i-1}M_i(r,\psi)\right) =$$

$$= -r^{i-2}\frac{(i+1)!}{R}\sum_{j=0}^{\infty}\binom{j+2}{i+1}\left(\frac{R}{r}\right)^{j+1} P_j(\cos\psi) \ . \qquad (A6)$$

Taking into account eqn.(A3) (for $i+1$), we get

$$M_{i+1}(r,\psi) = -\frac{1}{r^{i-2}}\frac{\partial}{\partial r}\left(r^{i-1}M_i(r,\psi)\right) \ , \qquad (A7)$$

or

$$M_{i+1}(r,\psi) = -r\frac{\partial M_i(r,\psi)}{\partial r} - (i-1)M_i(r,\psi) \ . \qquad (A8)$$

The inital value for the recursive process is $M_0(r,\psi)$. Putting $i = 0$ in eqn.(22), we have:

$$M_0(r,\psi) = \frac{1}{\ell} \ , \qquad (A9)$$

where ℓ is given by eqn.(26).

Spatial form. The integration kernels $M_i(r,\psi)$ may be further expressed as power series of the reciprocal distance $1/\ell$. This representation is helpful for understanding the behaviour of the kernel in the vicinity of the computation point ($\psi = 0$). Applying the relation (A8) i-times recursively, we get (for details see Appendix B)

$$M_i(r,\psi) = (-1)^i\sum_{s=1}^{i-1}\frac{i!(i-2)!}{(i-s+1)!(i-s-1)!(s-1)!}\times$$

$$\times r^{i+1-s}\frac{\partial^{i+1-s}M_0(r,\psi)}{\partial r^{i+1-s}} \qquad (A10)$$

We note that this expression is valid only for $i \geq 2$. Taking into account (A9), the series in eqn.(A10) consists of higher-order radial derivatives of the reciprocal distance $1/\ell$. These may be expressed as

$$\frac{\partial^k}{\partial r^k}\left(\frac{1}{\ell}\right) =$$

$$= \sum_{t=0}^{k} (-1)^{\frac{k+t}{2}} \frac{(k-t+1)!!(k+t-1)!!}{(k-t+1)!} \frac{k!}{t!} \frac{(r-R\cos\psi)^t}{\ell^{k+t+1}} , \qquad (A11)$$

where the summation must be taken over such t's for which $k+t$ is an even number. Inserting (A11) into (A10), we get

$$M_i(r,\psi) = \frac{1}{\ell}\tilde{M}_i(r,\psi) , \qquad (A12)$$

with

$$\forall i > 1: \quad \tilde{M}_i(r,\psi) = \sum_{s=1}^{i-1} \frac{i!(i-2)!}{(i-s-1)!(s-1)!} \left(\frac{r}{\ell}\right)^{i+1-s} \times$$

$$\times \sum_{t=0}^{i+1-s} (-1)^{\frac{3i+1-s+t}{2}} \frac{(i+2-s-t)!!(i-s+t)!!}{(i+2-s-t)!\,t!} \times$$

$$\times \left(\frac{r-R\cos\psi}{\ell}\right)^t , \qquad (A13)$$

and $i-s+t+1$ must be an even number. The last two equations represent the integration kernels $M_i(r,\psi)$ as finite power series of the reciprocal distance $1/\ell$.

Singularity. An important property of the reduced integration kernel $\tilde{M}_i(r,\psi)$ is its behaviour in the vicinity of the point $\psi = 0$. At this point, the distance ℓ is equal to the height H of the computation point above the geoid,

$$\ell|_{\psi=0} = r - R = H . \qquad (A14)$$

The integration kernel $\tilde{M}_i(r,\psi)$ may then be written as

$$\forall i > 1: \qquad \tilde{M}_i(r,0) = \sum_{s=1}^{i-1} a_{is} \left(\frac{r}{H}\right)^{i+1-s} , \qquad (A15)$$

and

$$\tilde{M}_1(r,0) = 1 + \frac{r}{H} , \qquad (A16)$$

where the coefficients a_{is} are readily obtained from eqn. (A13) as

$$a_{is} = \frac{i!(i-2)!}{(i-s-1)!(s-1)!} \times$$

$$\times \sum_{t=0}^{i+1-s} (-1)^{\frac{3i+1-s+t}{2}} \frac{(i+2-s-t)!!(i-s+t)!!}{(i+2-s-t)!\,t!} , \qquad (A17)$$

The summation is again taken over those t's for which $i-s+t+1$ is an even number. Eqns.(A15) and (A16) show that if the height of the computation point goes to zero, the integral kernel $\tilde{M}_i(r,\psi)$ goes to infinity as $1/H^i$.

Integrals. To remove the singularity of the integration kernels $M_i(r,\psi)$ at the point $\psi = 0$, the angular integrals

$$\int_{\Omega'} M_i(r,\psi)d\Omega' \qquad (A18)$$

are needed. Let us begin with $i = 1$. Applying Poisson's integral (Heiskanen and Moritz, 1967, eq.(1-88)) to a function R/r, we get

$$\frac{1}{4\pi}\int_{\Omega'} \sum_{j=0}^{\infty} (2j+1)\left(\frac{R}{r}\right)^{j+1} P_j(\cos\psi)d\Omega' = \frac{R}{r} . \qquad (A19)$$

The angular integral (A18) may then be expressed in terms of (A19) as

$$\int_{\Omega'} M_1(r,\psi)d\Omega' =$$

$$= \frac{1}{2R}\int_{\Omega'} \sum_{j=0}^{\infty} (2j+1)\left(\frac{R}{r}\right)^{j+1} P_j(\cos\psi)d\Omega' +$$

$$+ \frac{3}{2R}\int_{\Omega'} \sum_{j=0}^{\infty} \left(\frac{R}{r}\right)^{j+1} P_j(\cos\psi)d\Omega' . \qquad (A20)$$

Since the series over j in the last integral is uniformly convergent, we may change the order of integration and summation. Then using the orthogonality relation for the Legendre polynomials,

$$\frac{1}{4\pi}\int_{\Omega'} P_j(\cos\psi)d\Omega' = \delta_{j0} , \qquad (A21)$$

we get

$$\int_{\Omega'} M_1(r,\psi)d\Omega' = \frac{8\pi}{r} . \qquad (A22)$$

It is possible to derive the angular integrals (A18) for the other kernels $M_i(r,\psi)$ by means of the recursive relation (A7). We obtain

$$\int_{\Omega'} M_{i+1}(r,\psi)\,d\Omega' = -\int_{\Omega'} \frac{1}{r^{i-2}}\frac{\partial}{\partial r}\left(r^{i-1}M_i(r,\psi)\right)d\Omega' =$$

$$= -\frac{1}{r^{i-2}}\frac{\partial}{\partial r}\left(r^{i-1}\int_{\Omega'} M_i(r,\psi)\,d\Omega'\right) . \qquad (A23)$$

For $i = 1, 2$ eqn.(A23) reads

$$\int_{\Omega'} M_2(r,\psi)\,d\Omega' = -r\frac{\partial}{\partial r}\left(\int_{\Omega'} M_1(r,\psi)\,d\Omega'\right) = \frac{8\pi}{r} , \qquad (A24)$$

$$\int_{\Omega'} M_3(r,\psi)\,d\Omega' = -\frac{\partial}{\partial r}\left(r\int_{\Omega'} M_2(r,\psi)\,d\Omega'\right) = 0 . \qquad (A25)$$

Using successively eqn.(A23) we can see that the angular integral (A18) vanishes for all $i > 3$ and we have

$$\int_{\Omega'} M_i(r,\psi)\, d\Omega' = 0 \,, \qquad i \geq 3 \,. \qquad (A26)$$

Appendix B:

Proofs of eqns.(A10) and (A11)

In this section we show how eqns.(A10) and (A11) are derived. To do so, we employ mathematical induction, an efficient and simple tool for demonstrating the validity of these relations.

Let us start with eqn.(A10): in the first step of mathematical induction, we show that eqn.(A10) holds for $i = 2$. For $i = 2$, eqn.(A10) reads

$$M_2(r,\psi) = r^2 \frac{\partial^2 M_0(r,\psi)}{\partial r^2} \,. \qquad (B1)$$

Substituting for M_0 from eqn.(A9), we have

$$M_2(r,\psi) = -r^2 \frac{\partial}{\partial r}\left[\frac{r - R\cos\psi}{\ell^3}\right] =$$

$$= -\frac{r^2}{\ell^3} + \frac{3r^2}{\ell^5}(r - R\cos\psi)^2 \,. \qquad (B2)$$

On the other hand, the kernel $M_2(r,\psi)$ may be evaluated from the recursive formula (A8):

$$M_2(r,\psi) = -r \frac{\partial M_1(r,\psi)}{\partial r} \,. \qquad (B3)$$

Taking the radial derivative of eqn.(27) and substituting it into eqn.(B3), we again obtain eqn.(B2). Thus we have demonstrated that eqn.(A10) holds for $i = 2$.

In the next step, we assume that formula (A10) holds for a particular value of i; we wish to prove that it is also valid for $i + 1$. Assuming that eqn.(A10) is valid for M_i, we can substitute it into eqn.(A7) obtaining

$$M_{i+1}(r,\psi) =$$

$$= \frac{(-1)^{i+1}}{r^{i-2}} \sum_{s=1}^{i-1} \frac{i!(i-2)!}{(i+1-s)!(i-s-1)!(s-1)!} \times$$

$$\times \frac{\partial}{\partial r}\left[r^{2i-s} \frac{\partial^{i+1-s} M_0(r,\psi)}{\partial r^{i+1-s}}\right] \,. \qquad (B4)$$

Taking the radial derivative, we have

$$M_{i+1}(r,\psi) =$$

$$= (-1)^{i+1} \sum_{s=1}^{i-1} \frac{i!(i-2)!}{(i+1-s)!(i-s-1)!(s-1)!} \times$$

$$\times \left[(2i-s) r^{i+1-s} \frac{\partial^{i+1-s} M_0(r,\psi)}{\partial r^{i+1-s}} + \right.$$

$$\left. + r^{i+2-s} \frac{\partial^{i+2-s} M_0(r,\psi)}{\partial r^{i+2-s}}\right] \,. \qquad (B5)$$

Let us now shift the summation index s in the first term to $s + 1$. We get:

$$M_{i+1}(r,\psi) =$$

$$= (-1)^{i+1} \sum_{s=2}^{i} \frac{i!(i-2)!(2i-s+1)}{(i+2-s)!(i-s)!(s-2)!} \times$$

$$\times r^{i+2-s} \frac{\partial^{i+2-s} M_0(r,\psi)}{\partial r^{i+2-s}} +$$

$$+ (-1)^{i+1} \sum_{s=1}^{i-1} \frac{i!(i-2)!}{(i+1-s)!(i-s-1)!(s-1)!} \times$$

$$\times r^{i+2-s} \frac{\partial^{i+2-s} M_0(r,\psi)}{\partial r^{i+2-s}} \,. \qquad (B6)$$

Writing separately the contribution with $s = i$ in the first summation and the contribution with $s = 1$ in the second summation, and summing the rest, we get

$$M_{i+1}(r,\psi) =$$

$$= (-1)^{i+1} \left[\frac{(i+1)!}{2!} r^2 \frac{\partial^2 M_0(r,\psi)}{\partial r^2} + r^{i+1} \frac{\partial^{i+1} M_0(r,\psi)}{\partial r^{i+1}}\right] +$$

$$+ (-1)^{i+1} \sum_{s=2}^{i-1} \frac{i!(i-2)!}{(i+2-s)!(i-s)!(s-1)!} \times$$

$$\times [(s-1)(2i-s+1) + (i-s+2)(i-s)] \times$$

$$\times r^{i+2-s} \frac{\partial^{i+2-s} M_0(r,\psi)}{\partial r^{i+2-s}} \,. \qquad (B7)$$

The last expression may be arranged further as follows:

$$M_{i+1}(r,\psi) =$$

$$= (-1)^{i+1} \left[\frac{(i+1)!}{2!} r^2 \frac{\partial^2 M_0(r,\psi)}{\partial r^2} + r^{i+1} \frac{\partial^{i+1} M_0(r,\psi)}{\partial r^{i+1}}\right] +$$

$$+ (-1)^{i+1} \sum_{s=2}^{i-1} \frac{(i+1)!(i-1)!}{(i+2-s)!(i-s)!(s-1)!} \times$$

$$\times r^{i+2-s} \frac{\partial^{i+2-s} M_0(r,\psi)}{\partial r^{i+2-s}} =$$

$$= (-1)^{i+1} \sum_{s=1}^{i} \frac{(i+1)!(i-1)!}{(i+2-s)!(i-s)!(s-1)!} \times$$

$$\times r^{i+2-s} \frac{\partial^{i+2-s} M_0(r,\psi)}{\partial r^{i+2-s}} \,. \qquad (B8)$$

Finally, comparing eqns.(B8) and (A10) we can see that eqn.(B8) is equal to eqn.(A10) for subscript $i+1$, and we have thus proved that eqn.(A10) holds also for $M_{i+1}(r,\psi)$. Concluding, eqn.(A10) is valid for an arbitrary integer subscript i, Q.E.D.

Now, let us continue with proving eqn.(A11). We will again use mathematical induction. In the first step, we should prove that eqn.(A11) is valid for $k = 1$, but since this is simple, we leave it to the reader. In the next step, we assume that eqn.(A11) holds for subscript k and we show that it is also valid for subscript $k+1$. Assuming that eqn.(A11) is valid for the k-th radial derivative of the reciprocal distance, we may take another radial derivative of eqn.(A11) getting

$$\frac{\partial^{k+1}}{\partial r^{k+1}}\left(\frac{1}{\ell}\right) =$$

$$= \sum_{t=1}^{k}(-1)^{\frac{k+t}{2}} \frac{(k-t+1)!!(k+t-1)!!k!}{(k-t+1)!(t-1)!} \frac{(r-R\cos\psi)^{t-1}}{\ell^{k+t+1}} -$$

$$- \sum_{t=0}^{k}(-1)^{\frac{k+t}{2}} \frac{(k-t+1)!!(k+t+1)!!k!}{(k-t+1)!t!} \frac{(r-R\cos\psi)^{t+1}}{\ell^{k+t+3}} . \tag{B9}$$

Let us change the summation index t to $t-1$ in the first sum, and to $t+1$ in the second summation. We obtain

$$\frac{\partial^{k+1}}{\partial r^{k+1}}\left(\frac{1}{\ell}\right) =$$

$$= \sum_{t=0}^{k-1}(-1)^{\frac{k+t+1}{2}} \frac{(k-t)!!(k+t)!!}{(k-t)!} \frac{k!}{t!} \frac{(r-R\cos\psi)^{t}}{\ell^{k+t+2}} -$$

$$- \sum_{t=1}^{k+1}(-1)^{\frac{k+t-1}{2}} \frac{(k-t+2)!!(k+t)!!}{(k-t+2)!} \frac{k!}{t-1!} \frac{(r-R\cos\psi)^{t}}{\ell^{k+t+2}} , \tag{B10}$$

where $k+t+1$ must be an even number. Writing separately the term with $t = 0$ in the first summation and the term with $t = k+1$ in the second summation, and summing the rest, we have

$$\frac{\partial^{k+1}}{\partial r^{k+1}}\left(\frac{1}{\ell}\right) =$$

$$= (-1)^{\frac{k+1}{2}} \frac{(k!!)^2}{\ell^{k+2}} - (-1)^{k}(2k+1)!! \frac{(r-R\cos\psi)^{k+1}}{\ell^{2k+3}} +$$

$$+ \sum_{t=1}^{k-1}(-1)^{\frac{k+t+1}{2}} \frac{(k-t+2)!!(k+t)!!}{(k-t+2)!} \frac{k!}{t!} \times$$

$$\times \left[(k-t+1)+t\right] \frac{(r-R\cos\psi)^{t}}{\ell^{k+t+2}} . \tag{B11}$$

By simple manipulations, we finally get

$$\frac{\partial^{k+1}}{\partial r^{k+1}}\left(\frac{1}{\ell}\right) =$$

$$= \sum_{t=0}^{k+1}(-1)^{\frac{k+t+1}{2}} \frac{(k-t+2)!!(k+t)!!}{(k-t+2)!} \frac{(k+1)!}{t!} \times$$

$$\times \frac{(r-R\cos\psi)^{t}}{\ell^{k+t+2}} . \tag{B12}$$

Comparing the last equation with eqn.(A11), we can see that we have proved that eqn.(A11) holds also for subscript $k+1$. Therefore, eqn.(A11) is valid for an arbitrary integer number k, Q.E.D.