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Reformulation of Stokes's Theory for Higher Than Second-Degree Reference Field and Modification of Integration Kernels

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An argument is put forward in favor of using a model gravity field of a degree and order higher than 2 as a reference in gravity field studies. Stokes's approach to the evaluation of the geoid from gravity anomalies is then generalized to be applicable to a higher than second-order reference spheroid. The effects of truncating Stokes's integration and of modifying the integration kernels are investigated in the context of the generalized approach. Several different modification schemes, starting with a Molodenskij-like modification and ending with the least squares modification, are studied. Particular attention is devoted to looking at both global and local biases and mean square errors of the individual schemes.

INTRODUCTION

After the famous Newton-Cassini argument about the basic shape of the Earth (oblate versus prolate) was settled by the French Academy's mid-eighteenth century expeditions, the oblate biaxial ellipsoid became the reference surface in geodesy. There is, of course, much to be said about the appropriateness of such a simple surface for the purpose of positioning. We wish to argue here, however, that for gravity field studies a higher than second-order surface (a spheroid of degree and order M) may now be used with considerable advantage.

This is not, of course, a new idea. Earth Gravity Models (EGM) expressed in terms of a series of zonal spherical harmonics have become part of the definition of geodetic reference systems (see, for instance, the definition of Geodetic Reference System (GRS) 80 [*Moritz*, 1980], or World Geodetic System (WGS) 1984 [*Smith*, 1988]). Low-degree and -order fields, mostly determined from satellite orbits, have been used by many researchers [e.g., *Nagy and Paul*, 1973], in the past few decades, mostly without acknowledging their reference role explicitly. We wish to point out that there is a definite gain in insight and thus a didactic advantage in the explicit acknowledgment of the reference field role played in effect by the EGMs.

To be sure, there is, and always will be, an error in any EGM to be adopted as a reference field. But this situation is no different from that we now face with the second-degree zonal (Somigliana-Pizzeti) field, a situation we have been living with for at least two centuries. There are ways of dealing with this problem, and we shall try to point them out as appropriate.

Throughout this paper, we shall be expressing the gravity field (the geoid in particular) or its components interchangeably in terms of convolution integrals of Green's type and in terms of finite or infinite series of spherical harmonics. We shall speak of those as integral or spectral representations, respectively, as has recently become the custom in geodesy.

In addition, it should be emphasized that the residual geoid obtained using a higher-degree and -order reference field may have particular advantages for regional geophysical interpretation [e.g., *Sjöberg*, 1984c; *Christou et al.*, 1989]. This fact alone should motivate a closer study of Stokes's formula for a higher-degree and -order reference field.

REFORMULATION OF STOKES'S CONVOLUTION INTEGRAL

Let us start by considering Stokes's original formula for geoid undulation N referred to the geodetic reference ellipsoid [*Heiskanen and Moritz*, 1967]:

$$N \doteq \kappa \oiint_{\mathcal{E}} S(\psi)\, \Delta g \,\mathrm{d}\mathcal{E} \ , \tag{1}$$

where $\kappa = R/(4\pi\gamma)$, R and γ are the mean surface radius and mean gravity of the Earth, Δg is the gravity anomaly defined as

$$\Delta g = g_{\mathcal{G}} - \gamma_{\mathcal{E}} = g - \gamma_o \tag{2}$$

($g_{\mathcal{G}} = g$ is the actual gravity on the geoid, and $\gamma_{\mathcal{E}} = \gamma_o$ is the normal gravity on the ellipsoid, both along the same normal to the ellipsoid), and the integration kernel S is called the Stokes function. The integration is carried over the whole reference ellipsoid $\mathcal{E}$ or, equivalently, over a unit sphere. We use the approximate equality sign because the expression is correct only to the order of e^2 (the square of eccentricity of the reference ellipsoid) [*Vaníček and Krakiwsky*, 1986]; this is known as the "spherical approximation." Stokes's function is usually written as a series of Legendre polynomials P_n,

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos\psi) \ , \tag{3}$$

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where ψ is the geocentric angle between the point of interest and the dummy point in the integration.

It has been shown by many authors (e.g., *Lachapelle* [1977] or *Vaníček and Krakiwsky* [1986]), that if a spheroid of degree M given by the first M degree spherical harmonic components (N_i) of the geoid

$$(N)_M = \sum_{i=2}^{M} N_i \quad , \tag{4}$$

where N_i contains ($2i$+1) spherical harmonics of appropriate orders, is taken as a reference surface, then the geoidal height N^M above that spheroid is given by the following equation, correct to the order of the eccentricity of the reference ellipsoid squared (e^2). We have

$$N^M \doteq \kappa \oiint_{\mathcal{E}} S^M(\psi)\, \Delta g^M \,\mathrm{d}\mathcal{E} \quad , \tag{5}$$

where

$$S^M(\psi) = \sum_{n=M+1}^{\infty} \frac{2n+1}{n-1} P_n(\cos\psi)$$
$$= S(\psi) - \sum_{n=2}^{M} \frac{2n+1}{n-1} P_n(\cos\psi) \quad , \tag{6}$$

$$\Delta g^M = g - \gamma^M \quad ; \quad \gamma^M = \sum_{n=2}^{M} \gamma_n \quad , \tag{7}$$

and γ_n is defined below. In the sequel, we shall call the spheroid of Mth degree and order simply "spheroid of Mth degree."

The following two notes are required. First, writing (5), we have neglected a term

$$\varepsilon_1 = \kappa \oiint_{\mathcal{E}} (S - S^M)\, \Delta g^M \,\mathrm{d}\mathcal{E} \quad , \tag{8}$$

which equals to zero when $\forall n \leq M$: $\Delta g_n^M = g_n - \gamma_n = 0$, i.e., in the absence of errors in the low-degree harmonic components g_n and γ_n of the observed gravity g and the model gravity γ. This condition will clearly not be satisfied generally, as the analogous condition for g_o and γ_o is not satisfied in the original Stokes formulation, and we will bring the neglected term into the picture later when we start discussing observational errors. Second, we note the nonstandard use of the symbols γ_n and γ^M. By γ_n we denote the spherical harmonic components of gravity generated by the EGM, i.e., the "model gravity" on the spheroid. The model gravity γ^M plays exactly the same role here as the normal gravity γ_o does in the original Stokes development. (It should be noted that γ_o does not figure in any of the new expressions as it should not. In practical computations, however, Δg around the world would be available rather than g in which case γ_o obviously has to be taken into account. But this step is irrelevant to the theory presented here.) The only place the reference ellipsoid is implied is in (4); by (4) the reference spheroid is presumed to be referred to the reference ellipsoid. Clearly, $(N)_M + N^M = N$ as required. As expected, as the degree M grows, Δg^M tends to zero. Moreover, for $n > M$: $\Delta g_n^M = g_n$, since, by definition, for $n > M$: $\gamma_n = 0$.

Equation (5) is an exact counterpart of the original Stokes formula (1) derived for the second-degree reference spheroid (i.e., the reference ellipsoid), and we shall be calling it the "generalized Stokes formula." It is accurate, like its original second-degree counterpart, to terms of the order of e^2, i.e., to the order of 0.3%. Since for a reasonable choice of M, say M = 20, $|N^M|$ is about one order of magnitude smaller than $|N|$, the effect of this inaccuracy is also one order of magnitude smaller, which amounts to a few centimeters.

We note that in (5), Δg^M can be replaced by Δg without any effect on the resultant N^M because S^M is "blind" to the first M harmonic components of Δg. This result follows from the orthogonality of sperical harmonics on the sphere. Nevertheless, we shall systematically use Δg^M, because we will want to perform certain operations on the generalized Stokes function S^M which may destroy its "blindness," and the retention of the low-order part of Δg in the convolution integral would give rise to unjustifiable terms.

We wish to point out that the adoption of the higher-degree reference spheroid in the Stokes theory as discussed above is responsible for the change in the shape of the Stokes kernel from S (really S^2) to S^M. The latter tapers off more rapidly than the former [e.g., *Jekeli*, 1980]; that is, the influence of distant gravity anomalies on local geoidal height is reduced. The reduction is proportional to the degree M of the reference spheroid. Thus, to evaluate the geoidal height N^M above the Mth degree reference spheroid, distant gravity anomalies may be treated in a more cavalier way than in the standard second-order Stokes theory.

As with the standard Stokes formula, the generalized formula is oblivious to the scale of the reference spheroid and to the geocentricity of the reference field. The question of "forbidden harmonics" [*Heiskanen and Moritz*, 1967] does not arise since we do not need to transform Δg_1 to N_1. We leave the topographical, indirect, and atmospheric effects in the generalized Stokes approach out of the discussion here. These effects were discussed exhaustively by *Vaníček and Kleusberg* [1987]. However, again we wish to point out that most errors, linear and nonlinear, coming from various sources, are reduced when transferring to a higher-degree reference field [see *Heck*, 1989].

NUMERICAL EVALUATION OF THE GENERALIZED STOKES CONVOLUTION INTEGRAL

Theoretically, the integration implied by (5) has to be carried out over the whole Earth. This is a nuisance, because gravity coverage of the Earth's surface is irregular and incomplete. Also the numerical effort involved would be huge. This is where the fast convergence of S^M to zero with growing ψ becomes very helpful. The integration does not have to be carried out all the way to $\psi = \pi$, because the contributions to N^M from distances ψ larger than a certain value ψ_o become manageably small. For practical evaluation of the convolution integral, we would welcome the critical distance ψ_o to be as small as possible. This would imply that as high a degree of M as possible should be used for the reference spheroid. On the other hand, the error in the reference field grows with growing M. Normally, therefore, a compromise value of M is used.

Now writing (5) as

$$N^M \doteq \kappa \iint_{\mathcal{C}_o} S^M(\psi)\, \Delta g^M \,\mathrm{d}\mathcal{E}$$
$$+ \kappa \iint_{\mathcal{E}-\mathcal{C}_o} S^M(\psi)\, \Delta g^M \,\mathrm{d}\mathcal{E} , \tag{9}$$

where $\mathcal{C}_o$ denotes a spherical cap of radius ψ_o, we can study the effect of M and ψ_o on the geoidal height N^M. The first term denotes the "truncated" convolution integral (for $\psi \leq \psi_o$), while the second term describes the "truncation correction," or the negative "truncation error" δN^M committed when the truncated integral is taken instead of the complete integral over the whole earth. Note that we still use the approximate equality because of the spherical approximation.

It was *Molodenskij et al.* [1960], who first introduced the idea of reducing the value of ψ_o further by allowing the kernel to be modified in such a way as to minimize the second term in (9), i.e., the truncation correction or truncation error. This idea has since been explored and developed in different directions by scores of researchers.

To explain the similarities and differences between the various possibilities, let us first explain how the Molodenskij-type modification works within the framework of generalized Stokes theory. Molodenskij's idea is to change (modify) the integration kernel by subtracting a modifying function M_S from it. We then get the modified kernel S^{M*} in the following form:

$$S^{M*}(\psi) = S^M(\psi) - M_S(\psi) \quad . \tag{10}$$

Substituting this into (9), we obtain

$$\begin{aligned} N^M \doteq \kappa & \iint_{\mathcal{C}_o} S^{M*} \, \Delta g^M \, d\mathcal{E} \\ & + \kappa \iint_{\mathcal{E}-\mathcal{C}_o} S^{M*} \, \Delta g^M \, d\mathcal{E} \\ & + \kappa \oiint_{\mathcal{E}} M_S \, \Delta g^M \, d\mathcal{E} , \end{aligned} \tag{11}$$

where the first term on the right-hand side represents a new approximation of N^M and the last two terms are the new truncation correction to be minimized.

To make things easier, the modifying function M_S is now chosen so that the last term disappears. Disregarding, once again, any errors in the EGM and in the low-degree components of Δg, all components Δg_n^M of degree lower than or equal to M disappear, and any polynomial in P_n of degree lower than or equal to M will satisfy the above requirement. (The effect of this disregarded term in the presence of long wavelength errors will be treated later.) We then choose

$$M_S(\psi) = \sum_{n=0}^{M} \frac{2n+1}{2} \, t_n \, P_n(\cos \psi) \ , \tag{12}$$

where the factors $(2n+1)/2$ are introduced for computational convenience, and t_n, called "Molodenskij's modification coefficients," are to be determined so that the new truncation error

$$\delta N^{M*} = - \kappa \iint_{\mathcal{E}-\mathcal{C}_o} S^{M*} \, \Delta g^M \, d\mathcal{E} \tag{13}$$

is minimized in one sense or another.

If the generalized Stokes formula is applied to a properly scaled EGM expressed in a geocentric coordinate system, then the summation (12) may begin with n=2. This is what we shall assume for simplicity from now on, so that we write

$$S^{M*}(\psi) = S^M(\psi) - \sum_{n=2}^{M} \frac{2n+1}{2} \, t_n \, P_n(\cos \psi) \ . \tag{14}$$

From Schwarz's inequality applied to (13) it follows that

$$(\delta N^{M*})^2 \leq \kappa^2 \, \|S^{M*}\|^2 \, \|\Delta g^M\|^2 \ , \tag{15}$$

where

$$\|\cdot\|^2 = \iint_{\mathcal{E}-\mathcal{C}_o} (\cdot)^2 \, d\mathcal{E} \ . \tag{16}$$

Now, Molodenskij required that the upper bound of $|\delta N^{M*}|$ (cf. equation (15)) be the minimum. For a given Δg^M (fixed reference field and location), the norm $\|\Delta g^M\|$ is constant, while $\|S^{M*}\|$ varies with the choice of t_n (n = 2,3,...,M). Minimizing the latter norm leads to the following system of normal equations:

$$\begin{aligned} \forall n \leq M: \ & \frac{\partial}{\partial t_n} \iint_{\mathcal{E}-\mathcal{C}_o} (S^{M*})^2 \, d\mathcal{E} \\ & = \frac{\partial}{\partial t_n} \int_{\psi=\psi_o}^{\pi} (S^{M*})^2 \sin \psi \, d\psi = 0) \ , \end{aligned}$$

or

$$\forall n \leq M: \ \int_{\psi=\psi_o}^{\pi} S^{M*} \, \frac{\partial M_S^*}{\partial t_n} \, \sin \psi \, d\psi = 0) \ . \tag{17}$$

Carrying out the differentiation, we obtain

$$\forall n \leq M: \quad \int_{\psi=\psi_o}^{\pi} (S^M - M_S) \, P_n(\cos \psi) \sin \psi \, d\psi = 0 \ . \tag{18}$$

Employing the usual notation,

$$\int_{\psi=\psi_o}^{\pi} P_i(\cos \psi) \, P_j(\cos \psi) \sin \psi \, d\psi = e_{ij}(\psi_o) \ , \tag{19}$$

$$\int_{\psi=\psi_o}^{\pi} S(\psi) \, P_i(\cos \psi) \sin \psi \, d\psi = Q_i(\psi_o) \ , \tag{20}$$

$$\int_{\psi=\psi_o}^{\pi} S^M(\psi) \, P_i(\cos \psi) \sin \psi \, d\psi = Q_i^M(\psi_o) \ , \tag{21}$$

we get finally

$$\forall n \leq M: \sum_{k=2}^{M} \frac{2k+1}{2} e_{nk} \, t_k = Q_n^M = Q_n - \sum_{k=2}^{M} \frac{2k+1}{2} e_{nk} \ . \tag{22}$$

This represents a system of $M-1$ linear equations for t_k which can be solved for any given ψ_o. These (Molodenskij-like) coefficients t_k are then substituted into (14) to give the Molodenskij-type modified kernel for the generalized Stokes formula. This approach was used in producing the "UNB Dec. '86" Canadian geoid [*Vaníček et al.*, 1986].

Equations (22) are slightly different from the original Molodenskij equations, and the resulting parameters t_n are also different from Molodenskij's. But it turns out that the generalized Stokes function S^M modified à la Molodenskij (S^{M*}_{Mol}) is exactly the same as Molodenskij's modified original Stokes's function (S^*_{Mol}). This is understandable because in both approaches we seek a function so modified as to have the minimal L_2-norm, and in both cases only the low frequencies (up to wave number M) are allowed to change. There is, however, a significant difference between the geoidal height N^*_{Mol} obtained by applying Molodenskij's modified kernel in the original Stokes theory and applying it in the generalized Stokes theory: the upper bound of the truncation error is, for the same radius of integration ψ_o, significantly smaller for the generalized theory. To show this, let us write

$$\left(\delta N^{M*}_{Mol}\right)^2 \leq \left(\|S^{M*}_{Mol}\| \, \|\Delta g^M\|\right)^2 = \left(\|S^*_{Mol}\| \, \|\Delta g^M\|\right)^2 \tag{23}$$

and, similarly, for the original truncation error:

$$\left(\delta N^*_{Mol}\right)^2 \leq \left(\|S^*_{Mol}\| \, \|\Delta g\|\right)^2 \ . \tag{24}$$

The expected value of $\|\Delta g^M\|$ is significantly smaller than the expected value of $\|\Delta g\|$, which proves the point.

SPECTRAL REPRESENTATION OF DIFFERENT KINDS OF GEOIDAL HEIGHTS

Before we discuss the modification issue further, let us derive the spectrum (harmonic series) representation of the individual kinds of geoidal heights. The simplest expression is obtained for the "exact" geoidal height N^M given by (5). Expressing both S^M and Δg^M in Legendre's polynomials, we get

$$N^M \doteq c \sum_{n=M+1}^{\infty} \frac{2}{n-1} \Delta g^M_n \ , \tag{25}$$

where $c = R/(2\gamma)$, and Δg^M_n can be replaced by Δg_n because for $n>M$ the two are identical. We use again the approximate equality symbol because the accuracy is only to the order of e^2. Bringing into the discussion also the term ε_1, given by (8) which was neglected originally, we get an additional term

$$\varepsilon_1 = c \sum_{n=2}^{M} \frac{2}{n-1} \Delta g^M_n \ . \tag{26}$$

The sum of (25) and (26) gives the complete expression:

$$N^M \doteq c \sum_{n=2}^{\infty} \frac{2}{n-1} \Delta g^M_n \ , \tag{27}$$

in the expected and correct form.

Let us now turn to the spectral representation of the geoidal height obtained from the truncated integration, i.e., from (9). First, to simplify the forthcoming equations, we introduce a new kernel $\bar{S}^M$ by the following expression:

$$\bar{S}^M = \begin{cases} S^M & \text{for} \quad \psi \leq \psi_o \\ 0 & \text{for} \quad \psi > \psi_o \end{cases} \tag{28}$$

and write it in a Legendre's polynomial series form as

$$\bar{S}^M = \sum_{n=2}^{\infty} \frac{2n+1}{2} s_n P_n(\cos\psi) \ , \tag{29}$$

where s_n are some coefficients to be determined. Disregarding the truncation error we have

$$\bar{N}^M \doteq \kappa \iint_{\mathcal{C}_o} S^M \Delta g^M \, d\mathcal{E}$$
$$= \kappa \oiint_{\mathcal{E}} \bar{S}^M \Delta g^M \, d\mathcal{E} \ . \tag{30}$$

Taking (29) into account, we can transfer this convolution integral into its spectral form as follows:

$$\bar{N}^M \doteq c \sum_{n=2}^{\infty} s_n \Delta g^M_n \ ,$$

or, equivalently,

$$\bar{N}^M \doteq c \sum_{n=2}^{M} s_n \Delta g^M_n + c \sum_{n=M+1}^{\infty} s_n \Delta g_n . \tag{31}$$

Here we recognize the first term to be again caused by long wavelength errors in γ^M and g; it would disappear if the EGM were errorless and if g were not contaminated by long wavelength errors.

The coefficients s_n can be easily determined from

$$\forall n: \quad s_n = \int_{\psi=0}^{\pi} \bar{S}^M(\psi) \, P_n(\cos\psi) \sin\psi \, d\psi ,$$

or, equivalently, from

$$\forall n: \quad s_n = \int_{\psi=0}^{\pi} S^M \, P_n \sin\psi \, d\psi$$
$$- \int_{\psi=0}^{\pi} S^M \, P_n \sin\psi \, d\psi \ . \tag{32}$$

Clearly,

$$s_n = \begin{cases} -Q^M_n & \text{for} \quad n \leq M \\ \dfrac{2}{n-1} - Q^M_n & \text{for} \quad n > M \end{cases} \ , \tag{33}$$

and (31) becomes

$$\bar{N}^M \doteq -c \sum_{n=2}^{M} Q^M_n \Delta g^M_n$$
$$+ c \sum_{n=M+1}^{\infty} \left(\frac{2}{n-1} - Q^M_n\right) \Delta g_n \ . \tag{34}$$

Here again, the first term will have a nonzero value only because of long wavelength errors in the EGM and in g.

It is interesting now to have a look also at the spectrum of the truncation error. The complete truncation error can be obtained from (9) and by considering the originally neglected term ε_1. We obtain

$$\delta\bar{N}^M = -\kappa \iint_{\mathcal{E}-\mathcal{C}_o} S^M \Delta g^M \, d\mathcal{E}$$
$$- \kappa \oiint_{\mathcal{E}} (S - S^M) \, \Delta g^M \, d\mathcal{E} \ . \tag{35}$$

This can be written in a spectral form as

$$\delta\bar{N}^M = -c \sum_{n=2}^{\infty} Q^M_n \Delta g^M_n - c \sum_{n=2}^{M} \frac{2}{n-1} \Delta g^M_n \ . \tag{36}$$

Subtracting $\delta\bar{N}^M$ from $\bar{N}^M$ (given by (34)) we obtain N^M (equation (27)) as we should.

Similarly, the geoidal height N^{M*} computed by means of a modified generalized Stokes function S^{M*} (equation (11)) has the following spectrum:

$$N^{M*} \doteq -c\sum_{n=2}^{M}\left(t_n + Q_n^{M*}\right)\Delta g_n^M + c\sum_{n=M+1}^{\infty}\left(\frac{2}{n-1} - Q_n^{M*}\right)\Delta g_n \,, \tag{37}$$

where we use the symbol Q_n^{M*} to denote

$$\forall n:\ Q_n^{M*} = Q_n^M - \sum_{i=2}^{M}\frac{2i+1}{2}\,e_{in}\,t_i \,. \tag{38}$$

Both the long and short wavelengths are affected by the modification. The difference between the geoidal heights computed from the truncated integration using the generalized Stokes kernel and the modified generalized Stokes kernel is given by

$$N^{M*} - \bar{N}^M = -c\sum_{n=2}^{M}\left(t_n - \sum_{i=2}^{M}\frac{2i+1}{2}e_{in}\,t_i\right)\Delta g_n^M + c\sum_{n=M+1}^{\infty}\sum_{i=2}^{M}\frac{2i+1}{2\,e_{in}\,t_i\,\Delta g_n} \,. \tag{39}$$

The critical role played by the Molodenskij modification coefficients t_n is clearly demonstrated in this equation.

The complete truncation error of modified geoidal height increased by the term ε_1 is

$$\delta N^{M*} = -\kappa\iint_{\mathcal{E}-\mathcal{C}_o} S^{M*}\,\Delta g^M\,\mathrm{d}\mathcal{E} - \kappa\oiint_{\mathcal{E}} M_S\,\Delta g^M\,\mathrm{d}\mathcal{E} - \varepsilon_1 \,. \tag{40}$$

In a spectral form it can be written as

$$\delta N^{M*} = -c\sum_{n=2}^{\infty} Q_n^{M*}\,\Delta g_n^M - c\sum_{n=2}^{M}\left(\frac{2}{n-1} + t_n\right)\Delta g_n^M \,, \tag{41}$$

which, once again, subtracted from N^{M*} (equation (37)) gives N^M.

We note that Molodenskij's modification (equation (22)) implies that

$$\forall n \le M:\ Q_n^{M*} = 0$$

rendering the following result:

$$\delta N_{Mol}^{M*} \doteq -c\sum_{n=2}^{M}\left(\frac{2}{n-1} + t_n\right)\Delta g_n^M - c\sum_{n=M+1}^{\infty} Q_n^{M*}\,\Delta g_n \,. \tag{42}$$

The Molodenskij modified geoidal height spectrum is then

$$N_{Mol}^{M*} \doteq -c\sum_{n=2}^{M} t_n\,\Delta g_n^M + c\sum_{n=M+1}^{\infty}\left(\frac{2}{n-1} - Q_n^{M*}\,\Delta g_n\right) \,. \tag{43}$$

Now comparing (37) and (43), we see that the Molodenskij modification affects not only the long wavelength geoidal heights, in the presence of long wavelength errors in Δg^M, but also the short wavelengths. The Q_n^{M*} factors in the two equations are different, as is apparent from (38): in the former case the modification coefficients are unspecified, in the latter case they are given by (22). This is true, of course, only for the modification which uses Legendre's polynomials up to the Mth degree. Other modification schemes can be used, and it will be interesting to investigate, for instance, modification schemes which use Legendre's polynomials up to a degree L, higher than the degree M of the reference spheroid.

To do so, we rewrite (5) in the following rather general way, which can be done due to the orthogonality of spherical harmonics on $\mathcal{E}$:

$$N^M = \kappa\oiint_{\mathcal{E}} S^L(\psi)\,\Delta g^M\,\mathrm{d}\mathcal{E} + c\sum_{n=M+1}^{L} s_n\,\Delta g_n \,, \tag{44}$$

or, equivalently,

$$N^M = \kappa\iint_{\mathcal{C}_o} S^L(\psi)\,\Delta g^M\,\mathrm{d}\mathcal{E} + \left\{\sum_{n=2}^{L}\left[Q_n^L(\psi_o) + s_n\right]\Delta g_n^M + \sum_{n=L+1}^{\infty} Q_n^L(\psi_o)\,\Delta g_n\right\} \,, \tag{45}$$

where

$$\forall L \ge M:\ S^L(\psi) = S(\psi) - \sum_{k=2}^{L}\frac{2k+1}{2}\,s_k\,P_k(\cos\psi) \,, \tag{46}$$

$$\forall n \le L:\ Q_n^L(\psi_o) = \int_{\psi_o}^{\pi} S^L(\psi)\,P_n(\cos\psi)\,\sin\psi\,\mathrm{d}\psi = Q_n(\psi_o) - \sum_{k=2}^{L}\frac{2k+1}{2}\,e_{nk}(\psi_o)\,s_k \,. \tag{47}$$

In (45) we have divided the integration area into a cap $\mathcal{C}_o$ of spherical angle ψ_o around the computation point and a remote zone area $\mathcal{E}$-$\mathcal{C}_o$. The latter integral becomes

$$\kappa\iint_{\mathcal{E}-\mathcal{C}_o} S^L(\psi)\,\Delta g^M\,\mathrm{d}\mathcal{E} = c\sum_{n=2}^{\infty} Q_n^L(\psi_o)\,\Delta g_n^M \,, \tag{48}$$

which when inserted into (44) yields (45). We have assumed that the maximum degree of modification (L) is at least as high as the degree of the reference field, i.e., $L \ge M$, and we will use this assumption throughout the rest of this paper. We also assume that potential coefficients, typically determined from satellite orbit analyses, are available to degree and order L. However, only the first M degrees and orders are used to define the reference spheroid.

Referring now to (45), it follows that $\forall n>M$: $\Delta g_n^M = \Delta g_n$. Furthermore, due to the orthogonality of the Laplace harmonics, it follows that (44) is equivalent to (5) for any choice of the parameters s_n (k=2,3,...,L). (For clarification, S^L is denoted by S^{L*}, when $s_n = (2/(n\text{-}1) + t_n$.) Equation (44) is essential in the error estimates to be derived below.

One thing is clear, however: compared to the standard Stokes approach, truncation of the convolution integral does much less damage in the generalized Stokes approach. The price one has to pay for this is the introduction of unmitigated errors in the EGM (really in $(N)_M$). We shall investigate the effect of these errors together with the effect of errors in Δg later.

The other source of errors, the discretization error in the numerical evaluation of the convolution integral for N^{M*}, is considered outside the scope of this paper. It represents a problem from the domain of numerical analysis and as such calls for development of techniques from that mathematical domain.

Two Generic Estimators

To be able to treat the case of different degrees (and orders) of the reference spheroid (M) and the modification (L), we shall directly introduce two kinds of generic estimators of N^M rather than deriving them from some desired properties. Later, we shall show that specific selections of free parameters s_n lead to specific properties of these estimators.

Consider first the following general estimator of N^M [cf. *Sjöberg*, 1987]:

$$\tilde{N}^{M'} = \kappa \iint_{\mathcal{C}_o} S^L(\psi)\, \Delta\hat{g}^M \, d\mathcal{E} + c \sum_{n=M+1}^{L} (Q_n^L + s_n)\, \Delta\hat{g}_n , \quad (49)$$

where $\Delta\hat{g}^M$ and $\Delta\hat{g}_n$ are observed values (estimates) of Δg^M and Δg_n, respectively. $\tilde{N}^{M'}$ suffers from errors in $\Delta\hat{g}^M$ and $\Delta\hat{g}_n$ as well as from the truncation error arising from the limited integration area $\mathcal{C}_o$.

A slightly different general estimator is given by [cf. *Sjöberg*, 1984a,b, 1987]

$$\tilde{N}^{M''} = \kappa \iint_{\mathcal{C}_o} S^L(\psi)\, \Delta\hat{g}^M \, d\mathcal{E} + c \sum_{n=M+1}^{L} s_n \, \Delta\hat{g}_n . \quad (50)$$

Note that the summation terms on the right-hand side of (49) and (50) reflect the assumptions that $\forall n \le M$: $\Delta\hat{g}_n^M = 0$; that is, that the reference field represents the first M degrees of the actual field perfectly, and, by definition, $\forall n > M$: $\Delta\hat{g}_n^M = \Delta\hat{g}_n$ (cf. discussion following (8)).

The errors of the two estimators (49) and (50) will be discussed in the next section.

Error Estimation

Let us denote the errors in terrestrial gravity and the reference field harmonics of anomalies by ε^T and ε^S, respectively. The nth Laplace harmonics of these errors are denoted by ε_n^T and ε_n^S. Then the estimator $\tilde{N}^{M'}$ can be rewritten as

$$\tilde{N}^{M'} = \kappa \iint_{\mathcal{C}_o} S^L(\Delta g^M + \varepsilon^T - \varepsilon^S)\, d\mathcal{E} + c \sum_{n=M+1}^{L} (Q_n^L + s_n)(\Delta g_n + \varepsilon_n^S) , \quad (51)$$

where we have taken, following (7),

$$\Delta\hat{g}^M = \Delta\hat{g}^T - \Delta\hat{g}^S = \Delta g + \varepsilon^T - (\Delta g_M + \varepsilon^S) = \Delta g^M + \varepsilon^T - \varepsilon^S , \quad (52)$$

and

$$\varepsilon^S = \sum_{n=2}^{M} \varepsilon_n^S . \quad (53)$$

By adding the stipulated (estimated) reference spheroid

$$\tilde{N}_M = c \sum_{n=2}^{M} \frac{2}{n-1} (\Delta g_n + \varepsilon_n^S) , \quad (54)$$

one obtains an estimator ($\tilde{N}'$) for the total undulation above the reference ellipsoid. Equations (51), (54), (4), and (45) yield the expression for the total geoid undulation error which includes both the truncation error and the errors in the satellite and terrestrial gravity anomalies:

$$\delta\tilde{N}' = \tilde{N}' - N = -\kappa \iint_{\mathcal{E}-\mathcal{C}} S^L \Delta g^M \, d\mathcal{E} + \kappa \iint_{\mathcal{C}_o} S^L(\varepsilon^T - \varepsilon^S)\, d\mathcal{E} + c \sum_{n=2}^{M} \frac{2}{n-1} \varepsilon_n^S + c \sum_{n=M+1}^{L} (Q_n^L + s_n)(\Delta g_n + \varepsilon_n^S) . \quad (55)$$

Note that (52) yields

$$\forall n \le M: \quad \Delta\hat{g}_n^M = \varepsilon_n^T - \varepsilon_n^S \quad (56)$$

and

$$\forall n > M: \quad \Delta\hat{g}_n^M = \Delta\hat{g}_n = \Delta g_n + \varepsilon_n^T . \quad (57)$$

Using the following relations between integral convolutions and Laplace expansions:

$$\kappa \iint_{\mathcal{E}-\mathcal{C}_o} S^L \Delta g^M \, d\mathcal{E} = \sum_{n=M+1}^{\infty} Q_n^L \Delta g_n \quad (58)$$

and

$$\kappa \iint_{\mathcal{C}_o} S^L(\varepsilon^T - \varepsilon^S)\, d\mathcal{E} = c \sum_{n=2}^{\infty} \left(\frac{2}{n-1} - Q_n^L - s_n^*\right)(\varepsilon_n^T - \varepsilon_n^S), \quad (59)$$

where

$$\forall n > L: \quad \varepsilon_n^S = 0 \quad (60)$$

and

$$s_n^* = \begin{cases} s_n & \text{for } 2 \le n \le L \\ 0 & \text{for } n > L \end{cases} , \quad (61)$$

one obtains, finally, the spectral form of the total error:

$$\delta\tilde{N}' = c \sum_{n=2}^{\infty} \left(\frac{2}{n-1} - s_n^* - Q_n^L\right) \varepsilon_n^T + c \sum_{n=2}^{L} \left(Q_n^L + s_n\right) \varepsilon_n^S - c \sum_{n=L+1}^{\infty} Q_n^L \Delta g_n . \quad (62)$$

Assuming that the observation errors have zero expectation, i.e.,

$$\forall n: \quad \mathrm{E}(\varepsilon_n^T) = \mathrm{E}(\varepsilon_n^S) = 0, \text{ and thus also } \mathrm{E}(\varepsilon^T) = \mathrm{E}(\varepsilon^S) = 0 , \quad (63)$$

it follows that the expected value of the total error is

$$\overline{\delta N}' = \mathrm{E}(\delta\tilde{N}') = -c \sum_{n=L+1}^{\infty} Q_n^L \Delta g_n . \quad (64)$$

Introducing the global average operator

$$\Gamma(\cdot) = \frac{1}{\mathcal{E}} \iint_{\mathcal{E}} \cdot \, d\mathcal{E} , \quad (65)$$

we note that

$$\Gamma(\overline{\delta N}') = -c \sum_{n=L+1}^{\infty} Q_n^L \, \Gamma(\Delta g_n) = 0 ; \quad (66)$$

that is, the global average of $\delta\tilde{N}'$ is unbiased. However, $\delta\tilde{N}'$ itself is locally biased, because for any given locality, the expected value $\overline{\delta N}'$ is not equal to zero. To study the local bias, we introduce

$$\Gamma[(\overline{\delta \tilde{N}'})^2] = c^2 \sum_{n=L+1}^{\infty} (Q_n^L)^2 \, c_n > 0 \ , \tag{67}$$

where c_n are the so-called "anomaly degree variances"

$$\forall n: \ \ c_n = \Gamma(\Delta g_n^2) \geq 0 \ , \tag{68}$$

which reflect the global behavior of the gravity field and have nothing to do with observing errors. Note also that the anomaly degree covariances vanish:

$$\forall n \neq k: \ \ \Gamma(\Delta g_n, \Delta g_k) = 0 \ . \tag{69}$$

This shows that the estimator $\tilde{N}'$ is locally biased from degree and order $L+1$ up. This is caused by the truncation of Stokes's integration to a cap $\mathcal{C}_o$ and by the use of a harmonic series to degree and order L in (49) (see also *Sjöberg* [1987]).

In a similar way, the error of the second general estimator $\tilde{N}''$ becomes (cf. equations (49) and (50))

$$\delta\tilde{N}'' = \tilde{N}'' - N = \delta\tilde{N}' - c \sum_{n=M+1}^{L} Q_n^L \, \Delta\hat{g}_n \tag{70}$$

with the expectation

$$\overline{\delta \tilde{N}''} = \mathrm{E}(\delta\tilde{N}'') = - c \sum_{n=M+1}^{\infty} Q_n^L \, \Delta g_n \ . \tag{71}$$

It follows that the estimator $\tilde{N}''$ is again globally unbiased but is locally biased from degree and order $M+1$ up. Clearly, if L is selected to equal to M, both estimators will have the same local bias.

Next we will determine the variances of the estimators $\tilde{N}'$ and $\tilde{N}''$. We will use the following notations for the gravity anomaly error covariances

$$\forall n,k: \ \ \mathrm{E}(\varepsilon_n^T \, \varepsilon_k^T) = \Lambda_{nk} \ \text{ and } \ \forall n,k \leq L: \ \ \mathrm{E}(\varepsilon_n^S \, \varepsilon_k^S) = \Omega_{nk} \ . \tag{72}$$

Note that Λ_{nk} and Ω_{nk} are position dependent! Furthermore, we assume that errors in $\Delta\hat{g}^T$ and $\Delta\hat{g}_n^S$ are uncorrelated, that is,

$$\forall n \neq k: \ \ \mathrm{E}(\varepsilon_n^T \, \varepsilon_k^S) = 0 \ . \tag{73}$$

This implies that terrestrial gravity should not have been used in the computation of the potential coefficients defining either the reference spheroid or the potential coefficients of degrees between M and L! The case of correlated ε^T and ε^S was treated by *Sjöberg* [1987]. Finally, we note that potential coefficient derived anomaly errors (ε_n^S) will contribute to the variances of $\tilde{N}'$ and $\tilde{N}''$ both through the reference field (see (54)) ($2 \leq n \leq M$) and through the potential coefficient representation for degrees between M and L.

Then the (expected) "local mean square error" of $\tilde{N}'$ becomes

$$\mathrm{MSE}(\tilde{N}') = \mathrm{E}[(\tilde{N}' - N)^2] = \mathrm{Var}(\tilde{N}') + \mathrm{Bias}^2(\tilde{N}') \ , \tag{74}$$

where

$$\mathrm{Var}(\tilde{N}') = \mathrm{E}\{[\tilde{N}' - \mathrm{E}(\tilde{N}')]^2\}$$

$$= c^2 \left[\sum_{n=2}^{\infty} \left(\frac{2}{n-1} - s_n^* - Q_n^L \right) \sum_{k=2}^{\infty} \left(\frac{2}{k-1} - s_k^* - Q_k^L \right) \Lambda_{nk} + \sum_{n=2}^{L} (Q_n^L + s_n) \sum_{k=2}^{L} (Q_k^L + s_k) \, \Omega_{nk} \right] \tag{75}$$

and

$$\mathrm{Bias}^2(\tilde{N}') = [\mathrm{E}(\tilde{N}') - N]^2 = c^2 \left(\sum_{n=L+1}^{\infty} Q_n^L \, \Delta g_n \right)^2 \ . \tag{76}$$

Similarly, one obtains the (expected) local mean square error of the estimator $\tilde{N}''$:

$$\mathrm{MSE}(\tilde{N}'') = \mathrm{Var}(\tilde{N}'') + \mathrm{Bias}^2(\tilde{N}'') \ , \tag{77}$$

where

$$\mathrm{Var}(\tilde{N}'') = c^2 \left[\sum_{n=2}^{\infty} \left(\frac{2}{n-1} - s_n^* - Q_n^L \right) \sum_{k=2}^{\infty} \left(\frac{2}{k-1} - s_k^* - Q_k^L \right) \Lambda_{nk} + \sum_{n=2}^{L} (P_n^L + s_n) \sum_{k=2}^{L} (P_k^L + s_k) \, \Omega_{nk} \right] , \tag{78}$$

where

$$P_n^L = \begin{cases} Q_n^L & \text{for} \quad 2 \leq n \leq M \\ 0 & \text{for} \quad M < n \leq L \end{cases} \tag{79}$$

and

$$\mathrm{Bias}^2(\tilde{N}'') = c^2 \left(\sum_{n=M+1}^{\infty} Q_n^L \, \Delta g_n \right)^2 \ . \tag{80}$$

Note that the above local mean square errors, variances, and biases are position dependent!

We now proceed to the global averages of the local mean square errors which we shall call "global mean square errors." For the estimator $\tilde{N}'$ we obtain

$$\Gamma[\mathrm{MSE}(\tilde{N}')] = \Gamma[\mathrm{Var}(\tilde{N}')] + \Gamma[\mathrm{Bias}^2(\tilde{N}')] \ , \tag{81}$$

where

$$\Gamma[\mathrm{Var}(\tilde{N}')] = c^2 \left[\sum_{n=2}^{\infty} \left(\frac{2}{n-1} - s_n^* - Q_n^L \right)^2 (\sigma_n^T)^2 + \sum_{n=2}^{L} (Q_n^L + s_n)^2 (\sigma_n^S)^2 \right] , \tag{82}$$

and

$$\Gamma[\mathrm{Bias}^2(\tilde{N}')] = c^2 \sum_{n=L+1}^{\infty} (Q_n^L)^2 \, c_n \ . \tag{83}$$

In these derivations, we have employed the following notation:

$$\Gamma(\Lambda_{nk}) = \begin{cases} (\sigma_n^T)^2 & \text{for} \quad n=k \\ 0 & \text{for} \quad n \neq k \end{cases} \tag{84}$$

and

$$\Gamma(\Omega_{nk}) = \begin{cases} (\sigma_n^S)^2 & \text{for} \quad n=k \\ 0 & \text{for} \quad n \neq k \end{cases} \ , \tag{85}$$

where $(\sigma_n^T)^2$ and $(\sigma_n^S)^2$ are called the "error degree variances" of the terrestrial and of the potential-coefficient-generated anomalies, respectively. In addition, (68) and (69) were used to derive the global averages of squared biases.

In the same way, the global mean square error of $\tilde{N}''$ is derived:

$$\Gamma[\mathrm{MSE}(\tilde{N}'')] = \Gamma[\mathrm{Var}(\tilde{N}'')] + \Gamma[\mathrm{Bias}^2(\tilde{N}'')]\,, \tag{86}$$

where

$$\Gamma[\mathrm{Var}(\tilde{N}'')] = \Gamma[\mathrm{Var}(\tilde{N}')] - c^2 \sum_{n=M+1}^{L} Q_n^L(Q_n^L + 2s_n)(\sigma_n^S)^2 \tag{87}$$

and

$$\Gamma[\mathrm{Bias}^2(\tilde{N}'')] = c^2 \sum_{n=M+1}^{\infty} (Q_n^L)^2\, c_n\,. \tag{88}$$

It is interesting to note that the mean square errors of $\tilde{N}'$ are independent of the choice of degree $M(\le L)$ of the reference field, while for $\tilde{N}''$ they depend on M (cf. (74) to (87)).

Some Special Modifications

We will now consider some special cases of modifying Stokes's formula, namely the "Molodenskij modification," the "strict separation modification," and the "least squares modification." The first method, limited to $L=M$, was formulated already above and will now be only restated for $L \ge M$.

Molodenskij's Modification

To begin with, we rewrite (45) as

$$N^M = \kappa\Big[\iint_{\mathcal{C}_o} S^L(\psi)\Delta g^M \mathrm{d}\mathcal{E} + \iint_{\mathcal{E}-\mathcal{C}_o} S^L(\psi)\Delta g^M \mathrm{d}\mathcal{E}\Big] + c\sum_{n=M+1}^{L} s_n\,\Delta g_n\ . \tag{89}$$

The truncation error is described by the second term (which is not contained in $\tilde{N}^{M''}$):

$$\delta N^L = \kappa \iint_{\mathcal{E}-\mathcal{C}_o} S^L(\psi)\,\Delta g^M\,\mathrm{d}\mathcal{E}\,. \tag{90}$$

In view of Molodenskij's choice of parameters s_n, this solution implies that

$$\forall n \le L:\ \ Q_n^L = 0\,. \tag{91}$$

Hence the two estimators (49) and (50) become identical, with

$$\hat{N}^M_{Mol} = \kappa \iint_{\mathcal{C}_o} S^L\,\Delta\hat{g}^M\,\mathrm{d}\mathcal{E} + c\sum_{n=2}^{L} s_n\,\Delta\hat{g}_n \tag{92}$$

with the local mean square error:

$$\mathrm{MSE}(\hat{N}^M_{Mol}) = c^2\sum_{n=2}^{\infty}\left(\frac{2}{n-1} - s_n^* - Q_n^L\right)\sum_{k=2}^{\infty}\left(\frac{2}{k-1} - s_k^* - Q_k^L\right)\Lambda_{nk} + c^2\sum_{n=2}^{L}\sum_{k=2}^{L} s_n s_k\,\Omega_{nk} + c^2\left(\sum_{n=L+1}^{\infty} Q_n^L\,\Delta g_n\right)^2. \tag{93}$$

The global mean square error becomes

$$\Gamma[\mathrm{MSE}(\hat{N}^M_{Mol})] = c^2\sum_{n=2}^{L}\left[\left(\frac{2}{n-1} - s_n\right)^2(\sigma_n^T)^2 + s_n^2(\sigma_n^S)^2\right] + c^2\sum_{n=L+1}^{\infty}\left[\left(\frac{2}{n-1} - Q_n^L\right)^2(\sigma_n^T)^2 + (Q_n^L)^2 c_n\right]. \tag{94}$$

Clearly, the errors of Molodenskij's modification of Stokes's formula are independent of the degree (M) of the reference field; rather they depend on the degree L of modification. Due to (69), the resulting truncation error becomes

$$\delta N^L_{Mol} = -c\sum_{n=L+1}^{\infty} Q_n^L\,\Delta g_n\,, \tag{95}$$

independent of the choice of degree M. This implies that the smaller truncation error bound for a higher-degree (M) reference field discussed earlier must be taken as applying rather to the degree L of modification for which M is the lowest bound.

The Strict Separation Modification

Frequently, the combined solution of the truncated (and modified) Stokes's formula and the reference field of degree M is said to be a merger of long-wavelength features given by the latter and short-wavelength features given by the former. This separation is generally not rigorous, and it is highly dependent on the type of modification of Stokes's function, i.e., the choice of the parameters s_n. For some applications, a strict wavelength separation might be advantageous. The derivation of such a solution, which we have not found in the open literature, is the intention of this section.

Reconsider the total error (62) of the general estimator $\tilde{N}^{M'}$. If s_n is selected in such a way that

$$\forall\ 2 \le n \le M \le L:\ \ Q_n^L + s_n = \frac{2}{n-1} \tag{96}$$

then the error becomes

$$\delta\hat{N}' = c\sum_{n=2}^{L}\frac{2}{n-1}\,\varepsilon_n^S + c\sum_{n=L+1}^{\infty}\left[\left(\frac{2}{n-1} - Q_n^L\right)\varepsilon_n^T - Q_n^L\,\Delta g_n\right]. \tag{97}$$

We have thus shown that for this choice of s_n, the estimator

$$\hat{N}' = \kappa \iint_{\mathcal{C}_o} S^L\,\Delta\hat{g}\,\mathrm{d}\mathcal{E} + c\sum_{n=2}^{L}\frac{2}{n-1}\Delta\hat{g}_n \tag{98}$$

is strictly determined from estimated Laplace harmonics of Δg up to degree L and from observed terrestrial gravity anomalies above degree L. Here it is completely irrelevant whether $\Delta\hat{g}^M$ or $\Delta\hat{g}$ is the argument under Stokes's integral of (98). The local and global mean square errors become

$$\mathrm{MSE}(\hat{N}') = c^2\Bigg[\sum_{n=2}^{L}\sum_{k=2}^{L}\frac{2}{n-1}\frac{2}{k-1}\,\Omega_{nk} + \sum_{n=L+1}^{\infty}\sum_{k=L+1}^{\infty}\left(\frac{2}{n-1} - Q_n^L\right)\left(\frac{2}{k-1} - Q_k^L\right)\Lambda_{nk} + \left(\sum_{n=L+1}^{\infty} Q_n^L\,\Delta g_n\right)^2\Bigg] \tag{99}$$

and

$$\Gamma[\mathrm{MSE}(\hat{N}')] = c^2 \left\{ \sum_{n=2}^{L} \left(\frac{2}{n-1}\right)^2 (\sigma_n^S)^2 + \sum_{n=L+1}^{\infty} \left[\left(\frac{2}{n-1} - Q_n^L\right)^2 (\sigma_n^T)^2 + (Q_n^L)^2 c_n \right] \right\} . \quad (100)$$

Similarly, we could get the estimator $\hat{N}''$ and its mean square errors from (96), (50), (74) to (76), and (77) to (80). $\hat{N}''$ and its errors are dependent on the degree (M) of the reference field, while this is not the case with $\hat{N}'$.

The Least-Squares Modifications

The least squares modification minimizes the mean square errors of the solution with respect to the choice of the parameters s_n $(k=2,3,\ldots,L)$. For each of the general estimators $\tilde{N}^{M\prime}$ and $\tilde{N}^{M\prime\prime}$, one obtains two least squares solutions: one locally best estimator, minimizing the (local) mean square error (equations (74) to (76)), and one globally best estimator, minimizing the global mean square error (equations (77) to (80)). In each case, the MSEs of the general solutions, i.e., equations (74) to (76) and (77) to (80), can be written in the following general form [*Sjöberg*, 1987]

$$\mathrm{MSE} = a + s^T A s - 2s^T h , \quad (101)$$

where a is the MSE without modification, that is, for all s_n set to zero, A is a symmetrical matrix, and h is a vector. For instance,

$$a = a_1 = c^2 \left[\sum_{n=2}^{\infty} \left(\frac{2}{n-1} - Q_n\right)^2 (\sigma_n^T)^2 + \sum_{n=2}^{L} Q_n^2 (\sigma_n^S)^2 + \sum_{n=L+1}^{\infty} Q_n^2 c_n \right] \quad (102)$$

and

$$a = a_2 = a_1 + c^2 \sum_{n=M+1}^{L} Q_n^2 [c_n - (\sigma_n^S)^2] , \quad (103)$$

where a_1 and a_2 refer to the Γ(MSE) of $\tilde{N}'$ and $\tilde{N}''$, respectively.

The minimum variance is obtained for

$$\frac{\partial \mathrm{MSE}}{\partial s} = As - h = 0 . \quad (104)$$

Thus the optimum set of parameters $\hat{s}$ are the solution of the system

$$A\hat{s} = h , \quad (105)$$

yielding the MSE:

$$\mathrm{MSE} = a - \hat{s}^T A \hat{s} = a - \hat{s}^T h . \quad (106)$$

The elements of A and h are given as follows:

1. Least squares estimator unbiased to degree and order L.

(1) The locally best estimator:

$\forall n,k=2,\ldots,L$:

$$A_{nk} = \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} (\delta_{ni} - E_{ni})(\delta_{kj} - E_{kj})(\Omega_{ij} + \Lambda_{ij}) + \sum_{i=L+1}^{\infty} \sum_{j=L+1}^{\infty} E_{ni} E_{kj} \Delta g_i \Delta g_j, \quad (107)$$

$\forall n=2,\ldots,L$:

$$h_n = \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} (\delta_{ni} - E_{ni}) \left[\Lambda_{ij} \left(\frac{2}{j-1} - Q_j\right) - \Omega_{ij} Q_j \right] + \sum_{i=L+1}^{\infty} E_{ni} \Delta g_i \sum_{j=L+1}^{\infty} Q_j \Delta g_j , \quad (108)$$

and

$$a = c^2 \left[\sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \left(\frac{2}{i-1} - Q_i\right)\left(\frac{2}{j-1} - Q_j\right) \Lambda_{ij} + \sum_{i=2}^{L} \sum_{j=2}^{L} Q_i Q_j \Omega_{ij} + \left(\sum_{i=L+1}^{\infty} Q_i \Delta g_i \right)^2 \right] , \quad (109)$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i=j \\ 0 & \text{for} \quad i \neq j \end{cases} \quad (110)$$

is the Kronecker symbol, and

$$E_{ni} = \frac{2n+1}{2} e_{ni} . \quad (111)$$

(2) The globally best estimator (see also *Sjöberg* [1984b, 1987]):

$$\forall k,r = 2,\ldots,L: \; A_{kr} = \chi_k \delta_{kr} - \frac{2r+1}{2} \chi_k e_{kr} - \frac{2k+1}{2} \chi_r e_{rk} + \frac{2k+1}{2} \frac{2r+1}{2} \sum_{n=2}^{\infty} e_{nk} e_{nr} \chi_n , \quad (112)$$

$$\forall k=2,\ldots,L: \; h_k = \frac{2(\sigma_k^T)^2}{k-1} - Q_k \chi_k + \frac{2k+1}{2} \sum_{n=2}^{\infty} \left[Q_n e_{nk} \chi_n - \frac{2}{n-1} e_{nk} (\sigma_n^T)^2 \right] , \quad (113)$$

and

$$a = c^2 \left[\sum_{i=2}^{\infty} \left(\frac{2}{i-1} - Q_i\right)^2 (\sigma_i^T)^2 + \sum_{i=2}^{L} Q_i^2 (\sigma_i^S)^2 + \left(\sum_{i=L+1}^{\infty} Q_i \Delta g_i \right)^2 \right] \quad (114)$$

where

$$\chi_n = (\sigma_n^T)^2 + \begin{cases} (\sigma_n^s)^2 & \text{for} \quad 2 \le n \le L \\ c_n & \text{for} \quad n > L \end{cases} \quad (115)$$

2. Least squares estimator unbiased to degree M (degree and order of the reference field.)

(1) The locally best estimator:

$\forall n,k=2,\ldots,M$:

$$A_{nk} = \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} (\delta_{ni} - E_{ni})(\delta_{kj} - E_{kj})(\Lambda_{ij} + \Omega_{ij}^{*}) + \sum_{i=M+1}^{\infty} \sum_{j=M+1}^{\infty} E_{ni} E_{kj} \Delta g_i \Delta g_j , \quad (116)$$

$\forall n=2,\ldots,M$:

$$h_n = \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} (\delta_{ni} - E_{ni}) \left[\left(\frac{2}{j-1} - Q_j\right) A_{ij} \text{-} \Omega^*_{ij} \right]$$

$$+ \sum_{i+M+1}^{\infty} E_{ni}\, \Delta g_i \sum_{j=M+1}^{\infty} Q_j\, \Delta g_j \tag{117}$$

and

$$a = c^2 \left\{ \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \left[\left(\frac{2}{i-1} - Q_i\right)\left(\frac{2}{j-1} - Q_j\right) A_{ij} \right.\right.$$

$$\left.\left. + Q_i\, Q_j\, \Omega^*_{ij} \right] + \left(\sum_{i=M+1}^{\infty} Q_i\, \Delta g_i \right)^2 \right\}, \tag{118}$$

where

$$\Omega^*_{ij} = \begin{cases} \Omega_{ij} & \text{for} \quad i \vee j \le M \\ 0 & \text{otherwise} \end{cases} \quad . \tag{119}$$

(2) The globally best estimator:

$\forall k,r=2,\ldots,M$:

$$A_{kr} = \delta_{kr}\, \chi_r - E_{rk}\, \chi^*_k - E_{kr} \chi^*_r + \sum_{n=2}^{\infty} E_{kn}\, E_{rn}\, \chi^{**}_n \tag{120}$$

$$\forall k=2,\ldots,M: \quad h_k = \left(\frac{2}{k-1} - Q_k\right) (\sigma^T_k)^2$$

$$+ \sum_{n=2}^{M} (E_{kn} - \delta_{kn})\, (\sigma^S_n)^2 + \sum_{n=M+1}^{\infty} E_{kn}\, Q_n\, c_n\,, \tag{121}$$

and

$$a = c^2 \left[\sum_{i=2}^{\infty} \left(\frac{2}{i-1} - Q_i\right)^2 (\sigma^T_i)^2 + \sum_{i=2}^{\infty} Q^2_i\, \chi^{**}_i \right], \tag{122}$$

where

$$\chi^*_n = (\sigma^T_n)^2 + \begin{cases} (\sigma^S_n)^2 & \text{for} \quad 2 \le n \le M \\ 0 & \text{for} \quad n > M \end{cases} \quad , \tag{123}$$

and

$$\chi^{**}_n = \begin{cases} (\sigma^S_n)^2 & \text{for} \quad 2 \le n \le M \\ c_n & \text{for} \quad n > M \end{cases} \quad . \tag{124}$$

FINAL REMARKS AND CONCLUSIONS

The generalized Stokes approach parallels the classical Stokes approach when the reference ellipsoid is replaced by an Mth degree spheroid, normal gravity γ_o on the reference ellipsoid is replaced by a model gravity γ^M on the spheroid, and the Stokes integration kernel S is replaced by the spheroidal kernel S^M. In standard geodetic practice, gravity values g are reduced to gravity anomalies Δg by subtracting normal gravity γ_o. In the context of the generalized Stokes approach, gravity values are reduced even more by subtracting the model gravity γ^M to obtain generalized gravity anomalies Δg^M. For growing M the Δg^M tends to zero, and for $n > M$ the harmonic components Δg^M_n become identically equal to the harmonic components g_n of the gravity itself.

In both the classical Stokes approach and the generalized Stokes approach, the spherical approximation causes an error in the evaluated geoidal height, but for growing M the error diminishes. For $M = 20$, the error is estimated to be within a few centimeters. In both approaches, the error in the reference field (second degree and Mth degree, respectively) has to be treated separately, and both approaches are oblivious to the scale of their respective reference surfaces and their geocentricity.

The main advantage of the generalized Stokes approach is that the integration kernel S^M converges rapidly to zero for the growing integration distance. Consequently, the effect of individual gravity anomalies vanishes more rapidly with distance from the point of interest, and the numerical evaluation of the convolution integral may thus be truncated much closer to the point of interest to achieve the same accuracy as with the classical Stokes approach. Thus the evaluation of the generalized Stokes convolution integral requires less computational effort as well as less extensive terrestrial gravity coverage.

Various modification schemes may also be used for practical evaluation of the generalized Stokes convolution integral. It turns out that if the classical Molodenskij modification is used with modification of degree M, the modified S^M has exactly the same shape as the original Molodenskij-modified S. However, the upper bound of the truncation error for the same radius of integration ψ_o is significantly smaller.

We have considered two generic models ($\tilde{N}^{M'}$ and $\tilde{N}^{M''}$) for the modification of Stokes's formula for a higher-degree reference field. For unbiased data, the first model is locally unbiased to the degree of modification (L) which equals to the maximum degree of harmonic coefficients, while the second model is locally unbiased to the degree ($M \le L$) of the reference field. The error of $\tilde{N}^{M'}$ is therefore independent of the choice of M, while the error of $\tilde{N}^{M''}$ is generally dependent on M. The choice between the two models, and the choice of degree M for $\tilde{N}^{M''}$ are still open questions. For Molodenskij's modification, however, the two estimators coincide. Examples 2 and 4 of *Sjöberg* [1987] show that the biased least squares estimator is superior to the unbiased estimator in the limiting case of a vanishing cap size. These examples also indicate that a low-degree reference field is to be preferred in this particular case.

We have derived a new type of least squares estimator, namely, the locally best one. Its application is restricted by the limited knowledge of the local error covariance functions of terrestrial gravity, the correlations among the potential coefficients, and the high degree spectrum of gravity. These limitations are considerably relaxed in the more modest global least squares modifications.

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