Stokes-Helmert's Scheme for Precise Geoid Determination

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Resumen

El esquema Stokes-Helmert para la determinación precisa del geoide es un trabajo desarrollado en la Universidad de New Brunswick, Canadá, por más de diez años de investigación sin interrupción, en la que han participado destacados científicos. Este trabajo, presenta en forma sintética los pasos para obtener un geoide centimétrico, así como la formulación matemática en la que se fundamenta este esquema.

Abstract

The Stokes-Helmert's scheme for the precise geoid determination is a work developed in the University of New Brunswick, Canada, by more than ten years of investigation without interruption, in which outstanding scientists have participated. This work, presents in synthetic form the steps to obtain a centimeter geoide, as well as the mathematical formulation on which east scheme is based.

Stokes-Helmert's geoid software

Stokes-Helmert's geoid software (SHGEO) is a scientific software for precise geoid determination based on the Stokes-Helmert theory of determination of the gravimetric geoid. The software has been developed during more then 10 years period under leadership of professor Petr Vaníček at the Department of Geodesy and Geomatics Engineering, University of New Brunswick. Authors of particular programs are: M. Najafi, P. Novák, J. Huang, J. Janák and R. Tenzer. We also have to mention Z. Martinec, A. Kleusberg, L.E. Sjöberg, W.E. Featherstone, W. Sun whose research presented in their papers was incorporated into the SHGEO software. SHGEO soft-

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ware uses various global models (e.g. TUG87, GRIM4-S4, EGM96). These global models play an important role in the geoid computation scheme. Therefore we acknowledge the contribution of all research teams that have developed these or other global models. Reference manual was compiled by R. Tenzer and J. Janák.

Stokes-Helmert's scheme for precise geoid determination

Introduction

This part of the manual gives a brief theoretical overview of the precise geoid determination process. The details can be found in references.

The Stokes-Helmert scheme for determination of the precise geoid can be summarized to the following steps:

1. Formulation of the boundary value problem on the Earth surface
2. Evaluation of the Helmert gravity anomalies on the Earth surface
3. Downward continuation of the Helmert gravity anomalies onto the geoid
4. Stokes's integration (solution to the Stokes's boundary value problem)
5. Transformation of geoidal heights from the Helmert space to the real space.

Formulation of the boundary value problem

The quantity to be solved is the earth's gravity potential $W(r,\Omega)$ on and outside the geoid and the geoid itself. The geoid is the equipotential surface that approximates the mean sea level most closely. The gravity potential on the geoid is denoted by $W_o(r,\Omega) = const.$ In order to solve this problem a normal gravity potential $U(r,\Omega)$ generated by the mean geocentric ellipsoid of revolution is introduced. The normal gravity potential U_o on the mean geocentric ellipsoid is chosen to be equal to the earth's potential on the geoid: $U_o = W_o$.

The difference of the gravity potential $W(r,\Omega)$ and the normal gravity potential $U(r,\Omega)$ defines the disturbing potential $T(r,\Omega)$,

$$T(r,\Omega) = W(r,\Omega) - U(r,\Omega) \qquad (1.1)$$

When atmospheric attraction is neglected, $T(r,\Omega)$ is harmonic outside the Earth and it satisfies the Laplace equation

$$\nabla^2 T(r,\Omega) = 0 \qquad (1.2)$$

Once $T(r,\Omega)$ has been solved, the gravity potential $W(r,\Omega)$ can be obtained at any point by adding $U(r,\Omega)$, which can be computed from existing models. Also when $T(r,\Omega)$ is known on the geoid, the vertical separation between the reference ellipsoid and the geoid can be obtained by the Bruns formula

$$N(\Omega)=\frac{T(r_g(\Omega))}{\gamma_o(\Omega)} \tag{1.3}$$

where $T(r_g(\Omega))$ is the disturbing potential on the geoid, and $\gamma_o(\Omega)$ is the normal gravity on the mean geocentric ellipsoid. The problem is now reduced to the determination of $T(r,\Omega)$ on and outside the geoid.

However, the disturbing potential $T(r,\Omega)$ does not satisfy the Laplace equation inside of topographical masses where the geoid is often located. Therefore in order to satisfy Laplace's equation, all atmospheric and topographical masses have to be removed or condensed on or beneath the geoid. In Helmert's second condensation method the atmospheric and topographical masses are condensed directly onto the geoid.

When atmospheric and topographical masses are condensed as a single layer that is located on the geoid, the Earth gravity field will slightly change. The space obtained after such a condensation is the Helmert space. The quantities given in the Helmert space are denoted by superscript H. Helmert's gravity potential is defined as follows

$$W^H(r,\Omega)=W(r,\Omega)-\delta V^t(r,\Omega)-\delta V^a(r,\Omega) \tag{1.4}$$

The residual topographical potential $\delta V^t(r,\Omega)$ is defined as a difference of the gravitational potential $V^t(r,\Omega)$ of topographical masses and the gravitational potential $V^{ct}(r,\Omega)$ of condensed topographical masses

$$\delta V^t(r,\Omega)=V^t(r,\Omega)-V^{ct}(r,\Omega) \tag{1.5}$$

Similarly, the residual atmospheric potential $\delta V^a(r,\Omega)$ is defined as a difference of the gravitational potential $V^a(r,\Omega)$ of atmospheric masses and the gravitational potential $V^{ca}(r,\Omega)$ of condensed atmospheric masses

$$\delta V^{a}(r,\Omega) = V^{a}(r,\Omega) - V^{ca}(r,\Omega) \tag{1.6}$$

By subtracting the normal gravity potential $U(r,\Omega)$ from eqn. (1.4), the disturbing potential $T^{H}(r,\Omega)$ in Helmert's space becomes

$$T^{H}(r,\Omega) = W^{H}(r,\Omega) - U(r,\Omega) \tag{1.7}$$

Helmert's disturbing potential $T^{H}(r,\Omega)$ is harmonic above the geoid, so that it satisfies the Laplace equation

$$\nabla^{2} T^{H}(r,\Omega) = 0 \tag{1.8}$$

To determine $T^{H}(r,\Omega)$, the boundary value problem of the third kind outside the geoid has to be solved. Therefore, the boundary values on the a-priori unknown geoid are needed. In this problem, the Helmert gravity anomalies on the geoid serve as the boundary values. To find a relation between the disturbing potential and the Helmert gravity anomalies, let us introduce the radial derivative of the Helmert disturbing potential

$$\frac{\partial\, T^{H}(r,\Omega)}{\partial\, r} = \frac{\partial\, W^{H}(r,\Omega)}{\partial\, r} - \frac{\partial\, U(r,\Omega)}{\partial\, r} \tag{1.9}$$

The negative radial derivative of the Helmert disturbing potential $T^{H}(r,\Omega)$ defines the Helmert gravity disturbance $\delta g^{H}(r,\Omega)$, i.e.,

$$-\frac{\partial\, T^{H}(r,\Omega)}{\partial\, r} = \delta g^{H}(r,\Omega) - \varepsilon_{\delta g}(r,\Omega) \tag{1.10}$$

The second term on the right hand side of eqn. (1.10) is the ellipsoidal correction to the gravity disturbance (Vaníček *et al.*, 1999).

Since the geoidal height above the ellipsoid are not usually available, the gravity disturbance is not considered to be a measurable quantity on the surface of the Earth. Therefore Helmert's gravity disturbance $\delta g^{H}(r,\Omega)$ has to be transformed to more commonly available quantity, which is the Helmert gravity anomaly $\Delta g^{H}(r,\Omega)$. This transformation is achieved by adding a term $\Gamma(r,\Omega)$ to the gravity disturbance. This term accounts for the change in normal gravity due to the difference between the geodetic height $h(\Omega)$ and the commonly available orthometric height $H^{O}(\Omega)$. This expression can be written with a sufficient accuracy as

$$\Gamma(r,\Omega)=\frac{T^H(r,\Omega)}{\gamma(r,\Omega)}\frac{\partial\,\gamma(r,\Omega)}{\partial\,n} \tag{1.11}$$

In solving the boundary value problem it is convenient to introduce the following spherical approximation

$$\frac{1}{\gamma(r,\Omega)}\frac{\partial\,\gamma(r,\Omega)}{\partial\,n}\cong-\frac{2}{R} \tag{1.12}$$

The error caused by this approximation is called the ellipsoidal correction $\varepsilon_n(r,\Omega)$ for the spherical approximation.

Substituting eqns. (1.11) and (1.12) into eqn. (1.10), we can formulate the boundary value problem in the Helmert space by the following equation

$$-\frac{\partial\,T^H(r,\Omega)}{\partial\,r}-\frac{2}{R}T^H(r,\Omega)=\Delta g^H(r,\Omega)+\varepsilon_n(r,\Omega)-\varepsilon_{\delta g}(r,\Omega) \tag{1.13}$$

This equation represents the fundamental equation of the physical geodesy valid in the Helmert space. It relates the known boundary values $\Delta g^H(r,\Omega)$ to the unknown disturbing potential $T^H(r,\Omega)$ outside and on the geoid. As the boundary values are the Helmert gravity anomalies on the geoid, the next two sections deal with the derivation of this quantity from the measurements.

Evaluation of the Helmert gravity anomalies on the earth surface

Helmert gravity anomaly $\Delta g^H(r,\Omega)$ referred on the earth surface is given by

$$\Delta g^H(r_t(\Omega))=g^H(r_t(\Omega))-\gamma(H^N(\Omega)) \tag{1.14}$$

where $\gamma(H^N(\Omega))$ is the normal gravity on the telluriod in the Helmert space (see Figure 1).

The Helmert gravity $g^H(r_t(\Omega))$ on the earth surface is obtained from the observed gravity $g(r_t(\Omega))$, by adding the direct topographical effect and the direct atmospheric effect:

$$g^H(r_t(\Omega))=g^H(r_t(\Omega))-\left.\frac{\partial\,\delta V^t(r,\Omega)}{\partial\,r}\right|_{r=r_t(\Omega)}-\left.\frac{\partial\,\delta V^a(r,\Omega)}{\partial\,r}\right|_{r=r_t(\Omega)} \tag{1.15}$$

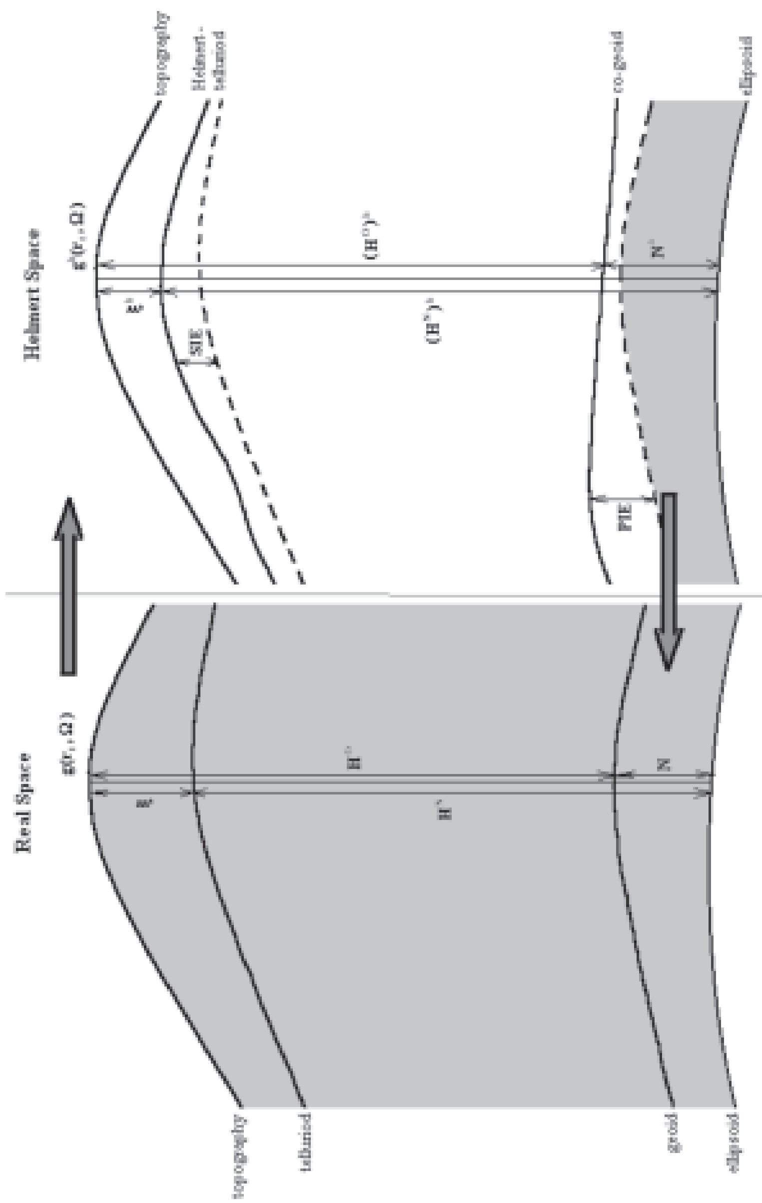

Figure 1. The quantities involved in real and Helmert's space.

The direct topographical effect (DTE) on gravity, given by a radial derivative of the residual gravitational potential $\delta V^{t}(r,\Omega)$ of topographical masses, is the gravitational attraction of topographical masses minus the gravitational attraction of condensed topographical masses. It should be evaluated on the earth surface. Analogous, the direct atmospheric effect (DAE) on gravity is gravitational attraction of atmospheric masses minus the gravitational attraction of condensed atmospherical masses.

Normal gravity $\gamma\left(H^{N}(\Omega)\right)$ on the telluriod in Helmert's space is obtained by upward continuation of normal gravity at the corresponding point on the mean geocentric ellipsoid. The height used for this computation should be the normal height $H^{N}(\Omega)$ in Helmert's space, which is the height of Helmert's telluroid above the ellipsoid. In practice, however, the heights on gravity observations on the earth surface are orthometric heights $H^{O}(\Omega)$ and the upward continuation of normal gravity is computed using $H^{O}(\Omega)$ instead of $H^{N}(\Omega)$. Therefore, a correction has to be added to the normal gravity. This correction consists of two parts. The first one, which is caused by the different position of the telluroid in the real and Helmert's space, is called the secondary indirect effect. The second part, which is due to the difference between the normal and orthometric height, is called the correction for the orthometric height. The expression for the normal gravity on Helmert's telluroid is then given by

$$\gamma^{H}\left(H^{N}(\Omega)\right)=\gamma\left(H^{O}(\Omega)\right)-\frac{2}{R}H^{O}(\Omega)\Delta g^{SB}\left(r_{t}(\Omega)\right)-\frac{2}{r_{t}(\Omega)}\delta V^{t}\left(r_{t}(\Omega)\right)-\frac{2}{r_{t}(\Omega)}\delta V^{a}\left(r_{t}(\Omega)\right) \qquad (1.16)$$

where $\Delta g^{SB}\left(r_{t}(\Omega)\right)$ is the simple Bouguer gravity anomaly.

The second term on the right-hand side of eqn. (1.16) is the geoid-quasigeoid correction to the boundary value problem. The third and fourth terms stands for the secondary indirect topographical effect on gravity (SITE) and the secondary indirect atmospheric effect on gravity (SIAE).

Inserting eqns. (1.14) and (1.15) back into eqn. (1.14) and considering also the free-air gravity anomaly given by

$$\Delta g^{FA}\left(r_{t}(\Omega)\right)=g\left(r_{t}(\Omega)\right)-\gamma\left(H^{O}(\Omega)\right) \qquad (1.17)$$

the boundary value problem in the Helmert space (eqn. (1.14)) takes the following form

$$\Delta g^{H}\left(r_{t}(\Omega)\right)=g^{FA}\left(r_{t}(\Omega)\right)+\left.\frac{\partial\,\delta V^{t}(r,\Omega)}{\partial r}\right|_{r=r_{t}(\Omega)}+\left.\frac{\partial\,\delta V^{a}(r,\Omega)}{\partial r}\right|_{r=r_{t}(\Omega)}+\frac{2}{r_{t}(\Omega)}\delta V^{t}\left(r_{t}(\Omega)\right)+$$
$$+\frac{2}{r_{t}(\Omega)}\delta V^{t}\left(r_{t}(\Omega)\right)+\frac{2}{R}H^{O}(\Omega)\Delta g^{SB}\left(r_{t}(\Omega)\right)+\varepsilon_{n}(r,\Omega)-\varepsilon_{\delta g}(r,\Omega) \qquad (1.18)$$

Free-air gravity anomalies are not very smooth, so that are not suitable for interpolation and averaging. Therefore, in practice the smoother complete Bouguer gravity anomalies $\Delta g^{CB}(r_t(\Omega))$ are used to produce the mean gravity data. The complete Bouguer anomaly is defined as

$$\Delta g^{CB}(r,\Omega) = \Delta g^{SB}(r,\Omega) + \delta g^{tc}(r,\Omega) = \Delta g^{FA}(r,\Omega) - 2\pi\rho_o G H^O(\Omega) + \delta g^{tc}(r,\Omega) \qquad (1.19)$$

where $\delta g^{tc}(r,\Omega)$ is the gravimetric terrain correction, i.e. a correction for the attraction of the earth's topography relative to the height at the evaluation point.

Downward continuation of Helmert gravity anomaly

To solve the Stokes boundary value problem in the Helmert space the Helmert's gravity anomalies $\Delta g^H(r_t(\Omega))$ have to be downward continued onto the geoid.

The Helmert disturbing potential is a harmonic function above the Helmert co-geoid. Poisson's solution to Dirichlet's problem of upward continuation of a harmonic function can be applied in finding a solution to the inverse problem, i.e. downward continuation.

The relation between Helmert's gravity anomalies $\Delta g^H(r_t(\Omega))$ on the geoid and Helmert's gravity anomalies $\Delta g^H(r_t(\Omega))$ on the earth surface is given by Poisson's integral

$$\Delta g^H(r_t(\Omega)) = \frac{R}{4\pi\, r_t(\Omega)} \iint_{\Omega' \in \Omega_O} \Delta g^H(r_g(\Omega))\, K[r_t(\Omega), \psi(\Omega,\Omega'), R]\, d\Omega' \qquad (1.20)$$

where $K[r_t(\Omega), \psi(\Omega,\Omega'), R]$ is the Poisson integral kernel. The solution to the problem of downward continuation is then given as a inverse solution of an integral equation of the first kind, where $\Delta g^H(r_t(\Omega))$ is known and $\Delta g^H(r_g(\Omega))$ is being determined. This integral equation can be solved iteratively according to Jacobi's iteration approach.

Stokes's boundary value problem

According to eqn. (1.3), the geoidal height $N(\Omega)$ can be obtained from the Bruns formula if the disturbing gravity potential $T(r_g(\Omega))$ on the geoid is known. The relation between the gravity anomaly on the geoid and the disturbing gravity potential referred on the geoid is given by Stokes theorem.

Solving Stokes's boundary value problem in the Helmet space, the co-geoid heights $N^H(\Omega)$ is given by the Stokes integral

$$N^H(\Omega) = \frac{R}{4\pi\, \gamma_o(\Omega)} \iint_{\Omega' \in \Omega_o} \Delta g^H(r_t(\Omega))\, S(\psi(\Omega,\Omega'))\, d\Omega' \qquad (1.21)$$

where $S(\psi(\Omega,\Omega'))$ is the Stokes integral kernel.

Transformation of geoidal heights from the Helmert space to the real space

To obtain the geoid, the co-geoidal heights $N^H(\Omega)$ have to be transformed from the Helmert space back to the real space. This transformation is done by evaluating the primary indirect topographical effect (PITE) on the geoidal height according to the following equation

$$N^t(\Omega) = \frac{\delta V^t(r_g(\Omega))}{\gamma_o(\Omega)} \qquad (1.22)$$

where $\delta V^t(r_g(\Omega))$ is the residual gravitational potential of topographical masses, which is reckoned at the geoid.

The geoid is finally computed by

$$N(\Omega) = N^H(\Omega) + N^t(\Omega) \qquad (1.23)$$

The primary indirect atmospheric effect as well as the secondary indirect atmospheric effect can be neglected, as it is shown in (Novák, 2000). Equation (1.23) is then final equation in the precise geoid determination process.

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