TRUNCATION OF SPHERICAL CONVOLUTION INTEGRATION WITH AN ISOTROPIC KERNEL

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Abstract

There are many applications where the spherical convolution integration is employed, especially those dealing with the Earth and it's gravity field, i.e. in geophysics, geodesy, oceanography, meteorology, etc. The truncation of the integration is often necessary as detailed input data are usually not available around the world. In this contribution an elegant symmetrical apparatus how to treat the truncation problem properly is described. Some important aspects are mentioned and one practical example is shown as well.

Mathematical background

Let's consider a general convolution integral

$$\int_{\Omega'} K(\psi) f(\Omega') d\Omega', \qquad (1)$$

where K is an isotropic kernel, a function of spherical distance ψ between the evaluation and integration points. f is a continuous function on a sphere, $\Omega = (\varphi, \lambda)$ is a solid angle, and φ, λ are spherical latitude and longitude. Let us express the kernel K as a series of Legendre polynomials. We get

$$K(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} a_n P_n(\cos\psi), \qquad (2)$$

where

$$\forall n = 0,1,2,\ldots : a_n = \int_0^{\pi} K(\psi) P_n(\cos\psi) \sin\psi d\psi. \qquad (3)$$

This integral can also be written as

$$\int_{\Omega'} K(\psi) f(\Omega') d\Omega' = \int_C K(\psi) f(\Omega') d\Omega' + \int_{\Omega' - C} K(\psi) f(\Omega') d\Omega', \qquad (4)$$

where C denotes a spherical cap of an arbitrary radius $\psi_0 \leq \pi$. In practice the area C corresponds to such a domain where detailed input data are known.

Let us define a new kernel

$$K^*(\psi) = \begin{cases} K(\psi), & \text{for } \psi \le \psi_0, \\ 0, & \text{for } \psi_0 < \psi \le \pi. \end{cases} \tag{5}$$

K^* can be expressed also as a series of Legendre polynomials as follows

$$K^*(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} s_n(\psi_0) P_n(\cos\psi), \tag{6}$$

where for $\forall n = 0,1,2,...;\ \psi_0 \le \pi$ we get

$$s_n(\psi_0) = \int_0^{\pi} K^*(\psi) P_n(\cos\psi)\sin\psi d\psi = \int_0^{\psi_0} K(\psi) P_n(\cos\psi)\sin\psi d\psi. \tag{7}$$

Then the first integral on the right hand side in eqn. (4) can be re-written as

$$\int_C K(\psi) f(\Omega')d\Omega' = \int_{\Omega'} K^*(\psi) f(\Omega')d\Omega' = \sum_{n=0}^{\infty} s_n(\psi_0) Y_n(\Omega), \tag{8}$$

where Y_n are Laplace surface spherical harmonics of f defined as follows

$$\forall n = 0,1,2,\ldots : Y_n(\Omega) = \sum_{m=0}^{n} \left[a_{nm} R_{nm}(\Omega) + b_{nm} S_{nm}(\Omega) \right]. \tag{9}$$

In eqn. (9), R_{nm} and S_{nm} are the fully normalised spherical harmonics and a_{nm} and b_{nm} are the fully normalised spherical harmonic coefficients of f defined by following expressions

$$\begin{pmatrix} a_{nm} \\ b_{nm} \end{pmatrix} = \int_{\Omega'} f(\Omega') \begin{pmatrix} R_{nm}(\Omega') \\ S_{nm}(\Omega') \end{pmatrix} d\Omega', \qquad (10)$$

$$\begin{pmatrix} R_{nm}(\Omega) \\ S_{nm}(\Omega) \end{pmatrix} = \sqrt{\left(2-\delta_{0m}\right)\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}} P_{nm}\left(\sin\varphi\right) \begin{pmatrix} \cos m\lambda \\ \sin m\lambda \end{pmatrix}. \qquad (11)$$

Symbol δ_{0m} in eqn. (11) stands for Kronecker delta and P_{nm} are associated Legendre functions.

Similarly we can define a complementary kernel K^{**} as

$$K^{**}\left(\psi\right) = \begin{cases} 0, & \text{for}\, \psi \le \psi_0, \\ K\left(\psi\right), & \text{for}\, \psi_0 < \psi \le \pi, \end{cases} \qquad (12)$$

and express it again in a series form

$$K^{**}\left(\psi\right) = \sum_{n=0}^{\infty} \frac{2n+1}{2} q_n\left(\psi_0\right) P_n\left(\cos\psi\right), \qquad (13)$$

where for $\forall n = 0,1,2,\ldots;\ \psi_0 \le \pi$ we get

$$q_n\left(\psi_0\right) = \int_0^{\pi} K^{**}\left(\psi\right) P_n\left(\cos\psi\right) \sin\psi d\psi = \int_{\psi_0}^{\pi} K\left(\psi\right) P_n\left(\cos\psi\right) \sin\psi d\psi. \qquad (14)$$

The second integral on the right hand side of the eqn. (4) may be than expressed as

$$\int_{\Omega'-C} K(\psi) f(\Omega') d\Omega' = \int_{\Omega'} K^{**}(\psi) f(\Omega') d\Omega' = \sum_{n=0}^{\infty} q_n(\psi_0) Y_n(\Omega). \quad (15)$$

Equations (8) and (15) provide a nice symmetrical apparatus for dealing with any truncation problem encountered in the studies of the Earth. Among other things, we note that the division of the integration area Ω' into a spherical cap and the rest of the sphere does not correspond to a separation of the two partial contributions: each partial integral must be expressed as a series containing all frequencies. Since the sum $K^{*}(\psi) + K^{**}(\psi)$ must be equal to $K(\psi)$, we have

$$\forall n = 0,1,2,\ldots; \psi_0 \leq \pi : s_n(\psi_0) + q_n(\psi_0) = a_n. \quad (16)$$

Indeed, putting together equations (4), (8) and (15) we get

$$\int_{\Omega'} K(\psi) f(\Omega') d\Omega' = \sum_{n=0}^{\infty} a_n Y_n, \quad (17)$$

as expected.

Example

A very nice example where the above mentioned theory is employed in geodesy, is the Stokes's integration during the geoid determination process. Here we want to describe how to treat this integration properly with an obviously available data and to show some numerical results from Canadian national project "Precise Geoid Determination for Geo-referencing and Oceanography". A deeper analysis of Stokes's integration can be found in (Martinec, 1993).

The famous Stokes's integral

$$N(\Omega) = \frac{R}{4\pi\gamma_0} \int_{\Omega'} \Delta g(\Omega') S(\psi) d\Omega', \qquad (18)$$

transforms the gravity anomalies Δg to disturbing potential T or geoidal height N if we assume the Bruns's formula. In this integral, R is the radius of the reference sphere, γ is the normal gravity on the reference ellipsoid and $S(\psi)$ is the Stokes's function. The Stokes's function may be represented in a spatial form (Heiskanen and Moritz, 1967, eqn.(2-164)) or (Vaníček and Krakiwsky, 1986, eqn.(22.16))

$$S(\psi) = 1 + \frac{1}{\sin\psi/2} - 6\sin\psi/2 - 5\cos\psi - 3\cos\psi \ln\left(\sin\psi/2 + \sin^2\psi/2\right), \qquad (19)$$

or in a spectral form

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n \cos(\psi), \qquad (20)$$

where $P_n(\cos\psi)$ is the Legendre polynomial of degree n. From the point of view of the boundary value problem solution, Stokes's function is simply a Green's function. It may also be regarded as an homogeneous and isotropic integration kernel (Vaníček and Krakiwsky, 1986). It is clear that the Stokes's formula (18) requires knowledge of the gravity anomalies over the whole Earth. How to solve this problem if there is a lack of gravity data in some regions? Vaníček and Kleusberg (1987) suggested to separate the summation over n in the Stokes's function (20) into low and high degree parts

$$S(\psi) = \sum_{n=2}^{\ell} \frac{2n+1}{n-1} P_n(\cos\psi) + \sum_{n=\ell+1}^{\infty} \frac{2n+1}{n-1} P_n(\cos\psi). \qquad (21)$$

Let us denote the low degree part as $S_\ell(\psi)$ and the high degree part as $S^\ell(\psi)$. A high degree part of the Stokes's kernel is called the *spheroidal Stokes's kernel*. If we substitute Stokes's function from (21) into Stokes's integral (18), we can split the geoidal height as well into a low degree part $N_\ell(\Omega)$ usually called *reference spheroid* and a high degree contribution $N^\ell(\Omega)$

$$N(\Omega) = N_\ell(\Omega) + N^\ell(\Omega), \qquad (22)$$

where

$$N_\ell(\Omega) = \frac{R}{4\pi\gamma} \int_{\Omega'} \Delta g(\Omega') S_\ell(\psi) d\Omega', \tag{23}$$

and

$$N^\ell(\Omega) = \frac{R}{4\pi\gamma} \int_{\Omega'} \Delta g(\Omega') S^\ell(\psi) d\Omega'. \tag{24}$$

It is possible to determine the low frequency part of the geoid $N_\ell(\Omega)$ from satellite measurements (global geopotential models) with a sufficient accuracy. In our example ℓ=20. The Stokes's integration is employed for high degree part of the geoidal height only according eqn. (24). We can see that the problem with lack of the gravity data in some regions in eqn. (24) still remains. In order to solve this problem let us to split the integration domain into a spherical cap ψ_0 (in our example ψ_0 = 6 degree) and the rest of the reference sphere. Integration over a spherical cap somehow estimates the high frequency part of the geoidal height. Integration over the rest of the reference sphere is usually much smaller and thus it is possible to determine it from the global geopotential model with sufficient accuracy. This part we call a *truncation error*. Moreover it is possible to minimise the truncation error by modifying the spheroidal Stokes's function. Such a modification is introduced e.g. in (Vaníček and Kleusberg, 1987) in a least-square sense according idea of (Molodensky et al., 1960). The Stokes's kernel after modification we call the *modified spheroidal Stokes's kernel*.

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REFERENCE SPHEROID
COMPUTED FROM GEOPOTENTIAL MODEL

FINAL GEOID MODEL

HIGH DGREE PART OF THE GEOID
COMPUTED USING MODIFIED SPHEROIDAL
STOKES‘S KERNEL AND INTEGRATED IN
6 DEGREE SPHERICAL CAP

TRUNCATION ERROR