Toward an improved height system for Canada

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Abstract

Height is a quality central to our understanding of the world. Heights in Canada are defined in the system of Orthometric heights, according to a method proposed by Helmert in 1890. However, much development in the theory of heights has been done since then, leading to a more rigorous definition of Orthometric heights, as summarized in Santos [2004b]. The new definition takes into account the effects of terrain roughness, laterally varying anomalous topographical density, and the NT geoid-generated gravity disturbance, which are not considered in the Helmert method.

This paper presents a calculation of corrections to Helmert orthometric heights, to update them to the more rigorous definition. The corrections for each effect, as well as a total correction comprised of all three effects, are evaluated for a Canadian test area comprised of several types of terrain. The correction is found to reach decimeters in some mountainous areas.

Introduction

Height is a quantity we deal with intimately every day. From climbing a staircase to piloting aircraft or designing roads, we use it to define the world and our interaction with it. An understanding of heights is essential to the study of any field of geomatics, and the Canadian height system – called Orthometric Heights – is directly related to earth's gravity field as well. This paper deals with the improvement of the Canadian height system, as a result of variations in its gravity field.

Orthometric height, $H^O(\Omega)$ of a point is defined as its geopotential number divided by the mean gravity along the plumbline between that point and the geoid. The geopotential number, defined as the difference between the potential on the goid and the potential at a surface point, may be easily calculated. The mean value of gravity along the plumbline, here shortened to *mean gravity*, is not so easily determined. Traditionally, mean gravity has been calculated according to Helmert's method, which accounts for gravity generated by the ellipsoid, and approximates topography with a plate extending to infinity, known as a Bouguer plate. Since his method was introduced, however, the attempt to calculate mean gravity has undergone significant evolution. As pointed out in Santos et al. [2004b], the relevant are the inclusion of local terrain effect in the calculation [Niethammer, 1932 and Mader, 1954], the introduction of effect of lateral anomalous density variations [Vaníček et al., 1995] and an examination of the effect of the geoid-generated gravity disturbance [Martin et al., 2003]. This paper, presents an evaluation of corrections to Helmert orthometric heights required to account for these effects. Corrections calculated in a Canadian context are presented here, and an assessment of their behavior is carried out. The theoretical background is presented in Santos et al. [2004b].

Corrections to Helmert orthometric heights

The Helmert method gives mean gravity along a plumbline as [Heiskanen and Moritz, p. 167]:

$$\bar{g}^{H}(\Omega) = g(r_t, \Omega) + 0.0424 H(\Omega)\,, \qquad \text{(Eq. 1)}$$

where $g(r_t, \Omega)$ is the value of gravity at the point at the surface of the earth with spherical coordinates Ω and geocentric radius r_t, and $H(\Omega)$ is the orthometric height of the surface at the same point. This in based upon application of the Poincaré-Prey reduction to surface gravity, assuming a linear gravity gradient along the plumbline. It takes into account both normal gravity and the effect of a Bouguer plate. However, as Santos et al. [2004b] explain, these effects alone do not provide a sufficiently accurate calculation of mean gravity in all parts of Canada.

Mean gravity may be defined more rigorously as a sum of calculable effects. Those related to topography are shown in Figure 1, where the black hills and valleys represent the contribution of terrain roughness, and the columns of varying shades of grey represent lateral density variations, and their contribution.

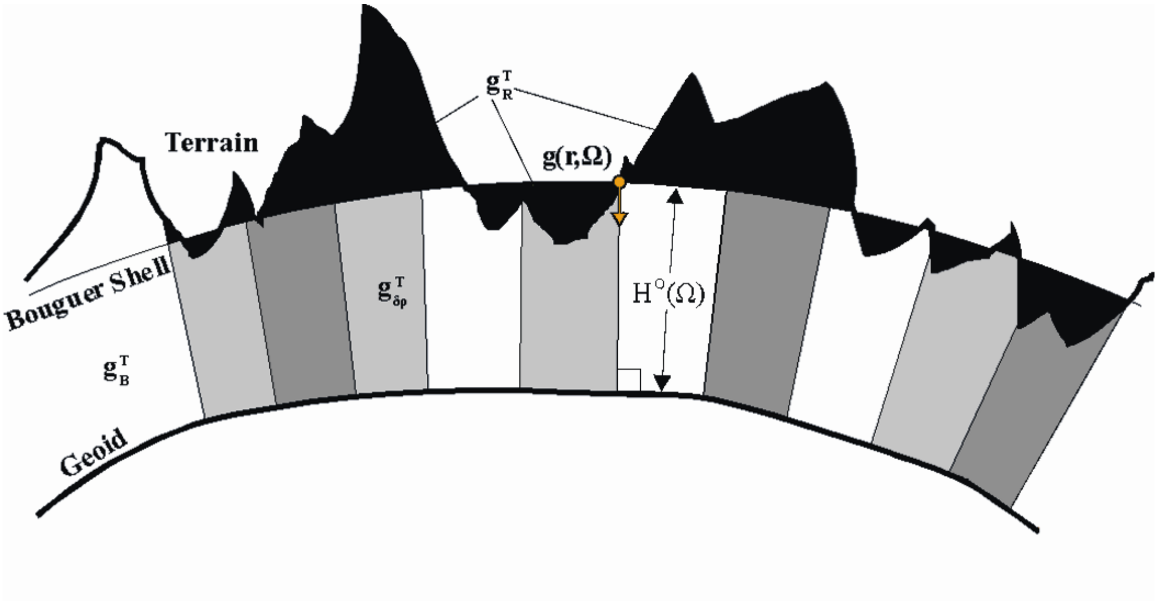

Figure 1: Components of rigorous gravity.

In addition to the contributions shown in Figure 1, gravity includes contributions not generated by topography. These are the gravity generated by the reference ellipsoid being used, and that generated by masses within the geoid. The NT space in which the latter exist was introduced in Vaníček et al. [2004], which may be referred to for further explanation.
Mathematically, the total gravity may be written:

$$g(r,\Omega) \cong \gamma(r,\Omega) + \delta g^{NT}(r,\Omega) + g_B^T(r,\Omega) + g_R^T(r,\Omega) + g^{\delta\rho}(r,\Omega)\,, \qquad \text{(Eq. 2)}$$

where the interpretation of the components is given in Table 1 below.

Table 1: Terms comprising the rigorous gravity formula.

Term	Meaning
$\gamma(r,\Omega)$	Normal gravity (generated by mass within the geodetic reference ellipsoid, e.g. GRS-80 or WGS-84)
$\delta g^{NT}(r,\Omega)$	Geoid-generated gravity disturbance (generated the mass within the geoid)
$g_B^T(r,\Omega)$	Effect on gravity of mass within a Bouguer shell of thickness $H^O(\Omega)$, assuming all mass has average crustal density
$g_R^T(r,\Omega)$	Effect on gravity of terrain variations, or roughness, above and below the Bouguer shell (i.e. hills and valleys) of average density
$g^{\delta\rho}(r,\Omega)$	Effect on gravity of lateral density variations from average crustal density, within the topography

Note that effects of atmospheric masses and of radial density variations are neglected throughout this paper. This is because the effect of atmospheric masses is very small, and the effect of radial density variations has been difficult to quantify due to insufficient data on the radial distribution of density within the crust.

It follows from Eq. 2 that gravity at a point on the surface, $g(r_t,\Omega)$, is rigorously defined as

$$g(r_t,\Omega) \cong \gamma(r_t,\Omega) + \delta g^{NT}(r_t,\Omega) + g_B^T(r_t,\Omega) + g_R^T(r_t,\Omega) + g^{\delta\rho}(r_t,\Omega)\,, \qquad \text{(Eq. 3)}$$

and mean gravity, $\bar{g}(r,\Omega)$, given by the integral mean of $g(r,\Omega)$, as

$$\bar{g}(r,\Omega) \cong \bar{\gamma}(\Omega) + \overline{\delta g}^{NT}(\Omega) + \bar{g}_B^T(\Omega) + \bar{g}_R^T(\Omega) + \bar{g}^{\delta\rho}(\Omega)\,; \qquad \text{(Eq. 4)}$$

where the bars over the terms represent mean quantities.

By substituting Eq. 3 into the expression for mean gravity according to Helmert's method, Eq. 1, we obtain the following expression for Helmert mean gravity:

$$\bar{g}^{H}(\Omega) \cong \gamma(r_t,\Omega) + \delta g^{NT}(r_t,\Omega) + g_B^T(r_t,\Omega) + g_R^T(r_t,\Omega) + g^{\delta\rho}(r_t,\Omega) + \\ + 0.0424 H(\Omega). \qquad \text{(Eq. 5)}$$

The correction to Helmert's mean gravity is then given by finding the difference between Eq. 5 and the rigorous mean gravity given by Eq. 4. In this operation, the contributions of normal gravity and gravity generated by mass within the Bouguer shell, along with the last term on the right hand side of Eq. 5, effectively cancel each other out [Santos et al., 2004b]. Thus,

$$\begin{aligned} \varepsilon_{\bar{g}}(\Omega) &\cong \bar{g}(\Omega) - \bar{g}^{H}(\Omega) \\ &\cong \overline{\delta g}^{NT}(\Omega) - \delta g^{NT}(r_t,\Omega) + \\ &\quad + \bar{g}_R^T(\Omega) - g_R^T(r_t,\Omega) + \\ &\quad + \bar{g}^{\delta\rho}(\Omega) - g^{\delta\rho}(r_t,\Omega), \end{aligned} \qquad \text{(Eq. 6)}$$

and the only terms of the correction which must be calculated are those resulting from the mean and surface effects on gravity of terrain roughness, the laterally varying density distribution, and the geoid-generated gravity disturbance. Once these terms have been calculated, the corresponding correction to Helmert orthometric heights may be determined using [c.f. Heiskanen and Moritz, 1981, p. 169]

$$\varepsilon_{H^O}(\Omega) = -\frac{H^O(\Omega)}{\bar{g}(\Omega)}\varepsilon_{\bar{g}}. \qquad \text{(Eq. 7)}$$

Corrections thus calculated may then be applied to leveling benchmarks, or any orthometric heights defined in the Canadian height system.

The test area

Software has been written to calculate the corrections to Helmert orthometric heights for the three effects shown in Eq. 6. These calculations were performed using a regular grid of points 5' apart, within a test area stretching from 49° to 54° in latitude, and 235° to 243° east in longitude. This area was chosen because its characteristics produce extreme results for each of the three effects calculated. Although it only represents a small segment of Canada's land mass, it contains both rugged and flat terrain, and seashore; so that the nature of the effects in many other areas may be judged from the results. Heights in the test area range from 0 m to 3227 m, and laterally varying densities range from 2490 kg/m^3 to 2980 kg/m^3. Since the average crustal density is 2670 kg/m^3, this corresponds to laterally varying anomalous densities between -180 kg/m^3 and 310 kg/m^3.

Geoid-generated gravity disturbances ranged from -266 mGal to -16 mGal. The distributions of height, anomalous density, and gravity anomalies are shown in Figures 2, 3 and 4, below. Figure 2 shows orthometric heights in the test area, with contour lines spaced at 500 m intervals. Figure 3 shows density polygons, with the colour scheme given in the bar to the right of the plot. Figure 4 shows gravity disturbances with contours spaced at 50 mGal intervals. In all plots in this paper, the colour gradient is given in the bar to the right of the plot.

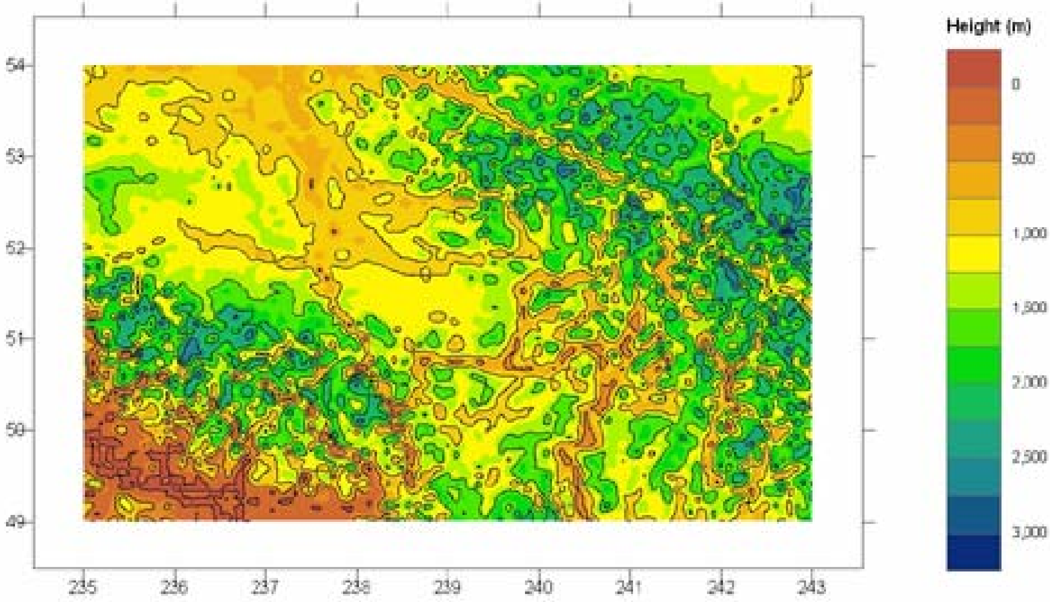

Figure 2: Heights within the test area.

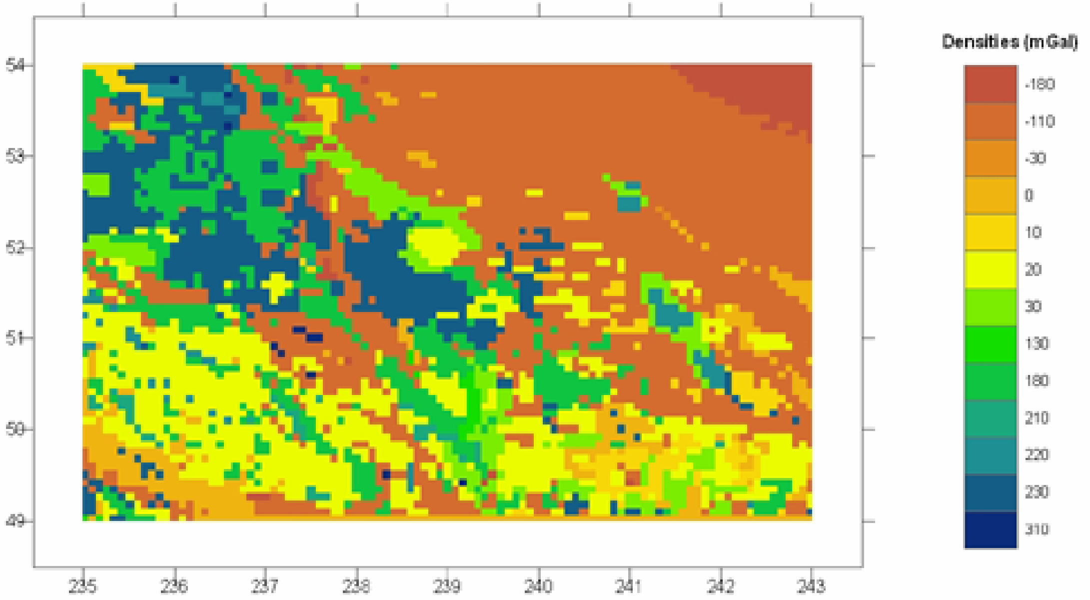

Figure 3: Laterally-varying anomalous densities within the test area.

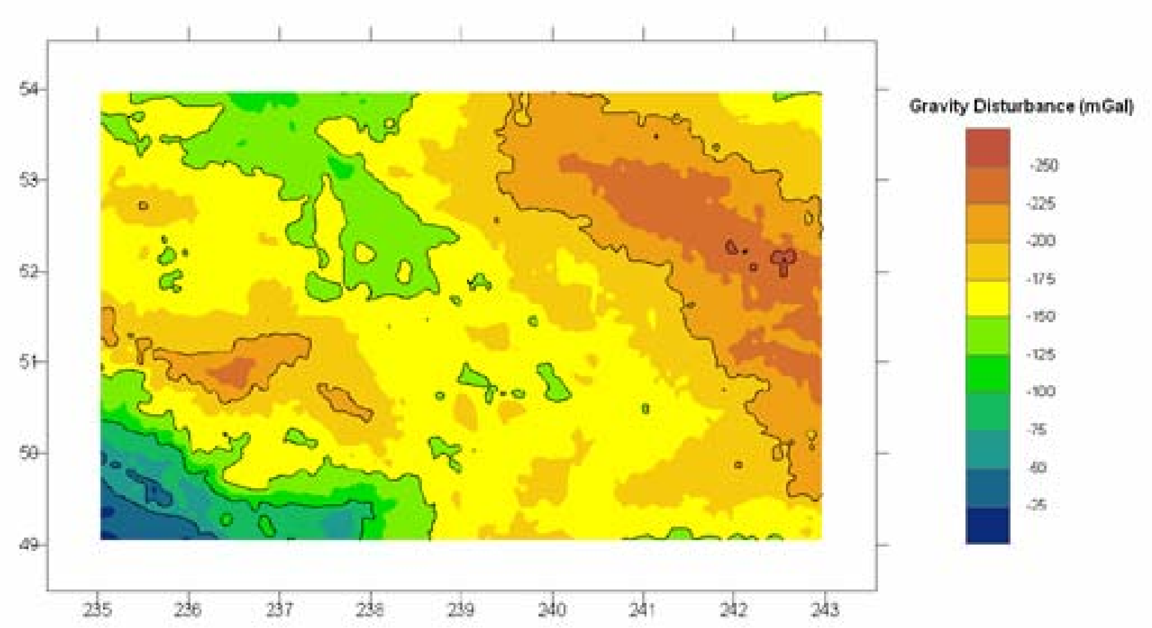

Figure 4: NT Geoid-generated gravity disturbances within the test area.

The effect of terrain roughness

The effect of terrain roughness, sometimes called just "the terrain effect" on gravity may be evaluated by integration of effects of surplus mass above the Bouguer shell, and mass deficit below the Bouguer shell, indicated by the black shaded area in Figure 1.

The terrain effect on mean gravity may be evaluated either as a simple average of the effect on gravity at the earth's surface and on the geoid, according to the method of Mader (1954), or using an integral mean. If the integral mean approach is applied, it may be done using the Niethammer approach (1932), in which the integral mean is approximated by calculating the effects on gravity at a series of points along the plumbline and averaging these results. Alternatively, according to a recent method developed at the University of New Brunswick (U.N.B.), the contribution of the potential of terrain roughness to gravitational potential at the surface and on the geoid may be calculated, and divided by orthometric height, to determine an exact value of the effect on mean gravity [Santos et al. 2003; Santos et al. 2004a; Santos et al 2004b]. Although results from all three methods are similar, the present study confirms that the U.N.B. method is the most effective due to its speed and accuracy.

For the calculation of both effect, integration of mass surpluses and deficits is performed within a spherical cap of 3°, with the assumption that any masses outside of this area may be neglected in this calculation. DTM data in three integration zones provides a basis for numerical integration. In the innermost zone, a 15' by 15' square centered on the computation point, 3" data is used. In the inner zone, a 200' square, 30" data is used.

Within the rest of the spherical cap, 5’ data is used. Furthermore, interpolation is performed to divide the central 3” cell into four 1.5” cells to make the integration accurate enough. Integration is performed to determine both the effect on mean gravity, and the effect on surface gravity. Once these are found, they are subtracted according to Eq. 6 to determine a correction to Helmert’s mean gravity. Note that the total effect on Helmert’s mean gravity would have the opposite sign of this correction.

Values of this correction within the test area ranged from 106 mGal to -25 mGal, while the corresponding corrections to Helmert orthometric height, calculated according to Eq. 9, ranged from 4.6 cm to -31.0 cm, and may be seen in Figures 5 and 6. Contour spacing in these figures is 40 mGal and 10 cm respectively.

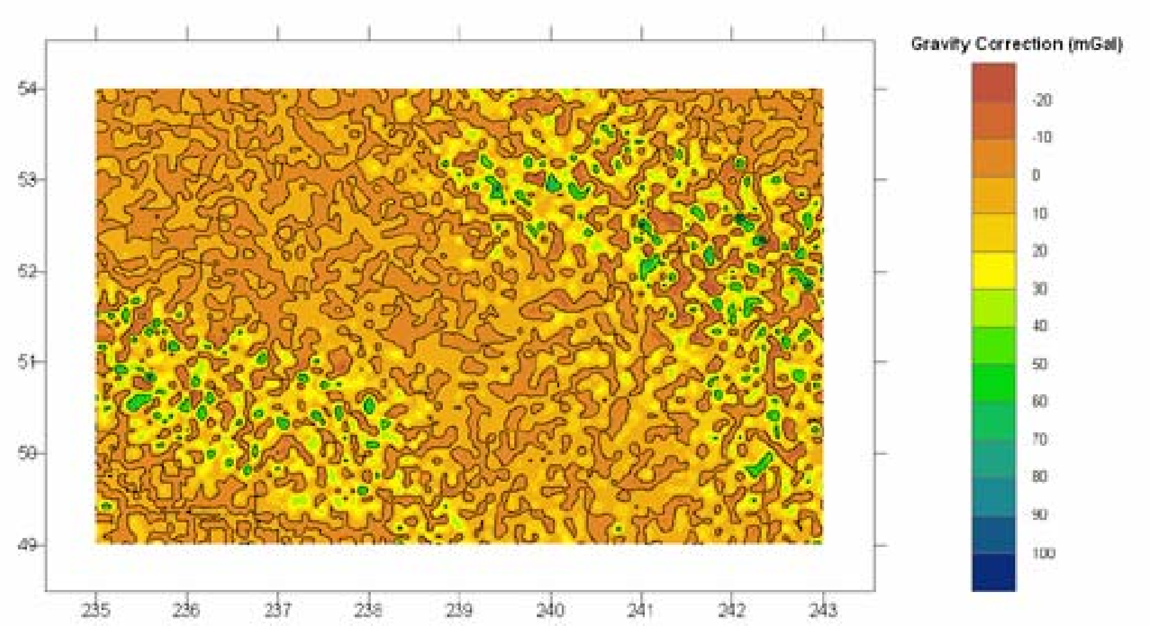

Figure 5: Correction to Helmert mean gravity for terrain roughness.

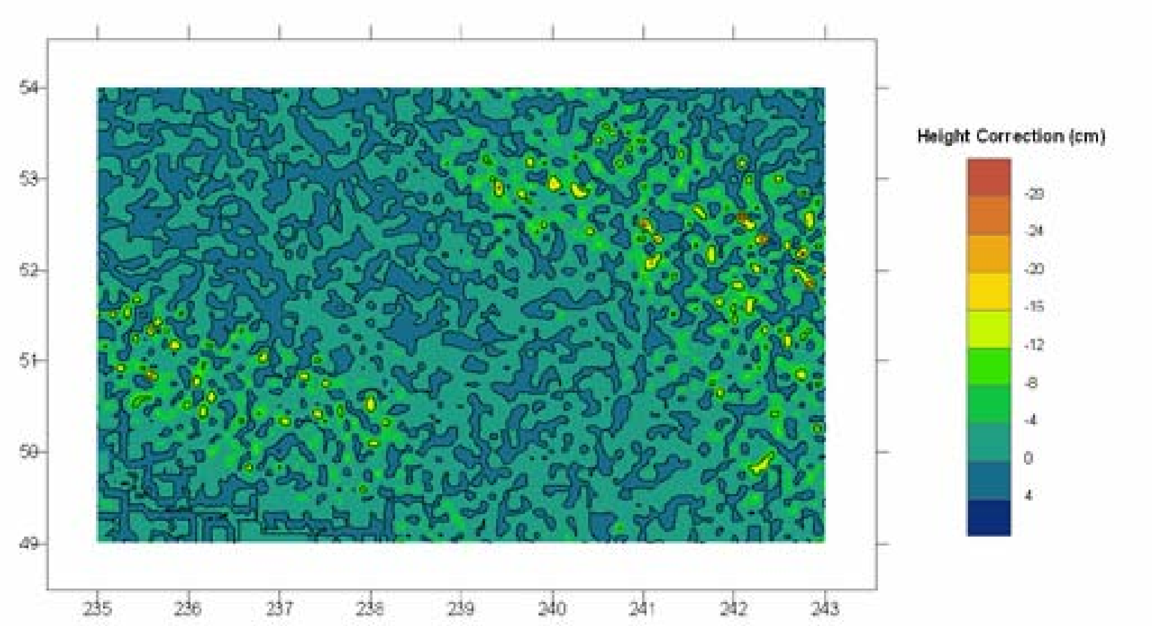

Figure 6: Correction to Helmert orthometric heights for terrain roughness.

Effects related to roughness should be expected to peak with the terrain, and this behaviour is seen in Figures 5 and 6. While it is may appear at first glance as though the correction follows the height of terrain almost exactly, the results show that this is not the case. Notice that the valleys which are distinctly defined in the topographic map of Figure 1 are not easily distinguished in the corrections given in Figures 5 and 6. Also, notice that the terrain in the middle of the map, having a more gradual slope, produces lower correction values; while the values peak sharply in rough terrain. While greater heights may amplify the correction, rough terrain is also necessary for it to be significant.

The effect of laterally-varying anomalous density

The effects of the Bouguer shell and of terrain are both evaluated assuming constant topographical density, and thus do not include the effect of density variations. While sufficient data is not available to calculate the effect of radial density variations, the effect of lateral variations may be calculated, thus accounting to some extent for the influence of density variations.

The effect of lateral density variations on gravity is calculated by first obtaining a series of density values for polygons within the area, and then subtracting from these the average crustal density of 2670 kg/m^3. This results in a series of density differences for the polygons, which may be thought of as columns of positive or negative density anomalies, extending from the surface of the geoid to the earth's surface, as indicated in Figure 1. For a more detailed description, see Huang et al. [2001].

To calculate exactly the effect of these anomalous density columns on gravity requires integration of the mass surpluses and deficits within these columns, using the same Newton kernel as for the terrain effect. Practically, however, the variations in height of these columns above and below the Bouguer shell – i.e., their influence on the terrain effect – may be neglected.

Integration was performed over two integration zones, the inner zone and the middle zone. Within the inner zone, which was 5’ square, integration points were spaced 3” apart, and density anomaly values were interpolated to these points. Within the middle zone, which was 10’ square, integration points were spaced 30” apart and were located in the center of cells of the input density data.

Calculation of the effect on mean gravity may be performed using the same three methods as for calculation of the terrain roughness effect. However, the difference in results using these three methods is small. Still, the U.N.B. method is preferred and was used for calculations, for the same reasons as with the terrain effect: speed, and accuracy.

The correction to Helmert mean gravity for laterally-varying anomalous density over the test area varied from 14 mGal to -24 mGal. Corresponding corrections to height varied from 6.5 cm to -4.5 cm, with both corrections correlated to both density and height. These results are also similar to those presented in Tenzer et al. [2003a], whose corrections, calculated for Canadian leveling points, ranged from 3.4 cm to -1.9 cm. Because the actual density in the test area was generally greater than average topographic density, the effect of the laterally-varying density distributions on height was normally positive. It is also noteworthy that the effect on mean gravity is very small, and might normally be negligible, the magnitude of the correction coming overwhelmingly from the effect on surface gravity. Results for the test area are shown in Figures 7 and 8, where Figure 7 uses a contour interval of 10 mGal, and Figure 8 uses 2 cm.

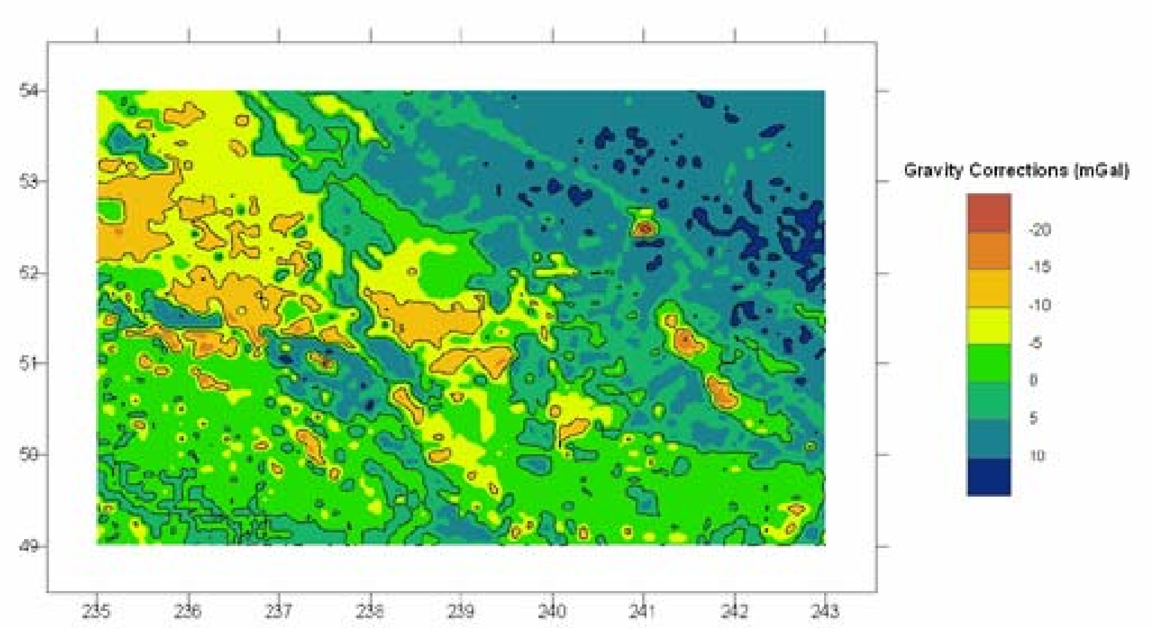

Figure 7: Correction to Helmert mean gravity for laterally-varying anomalous density.

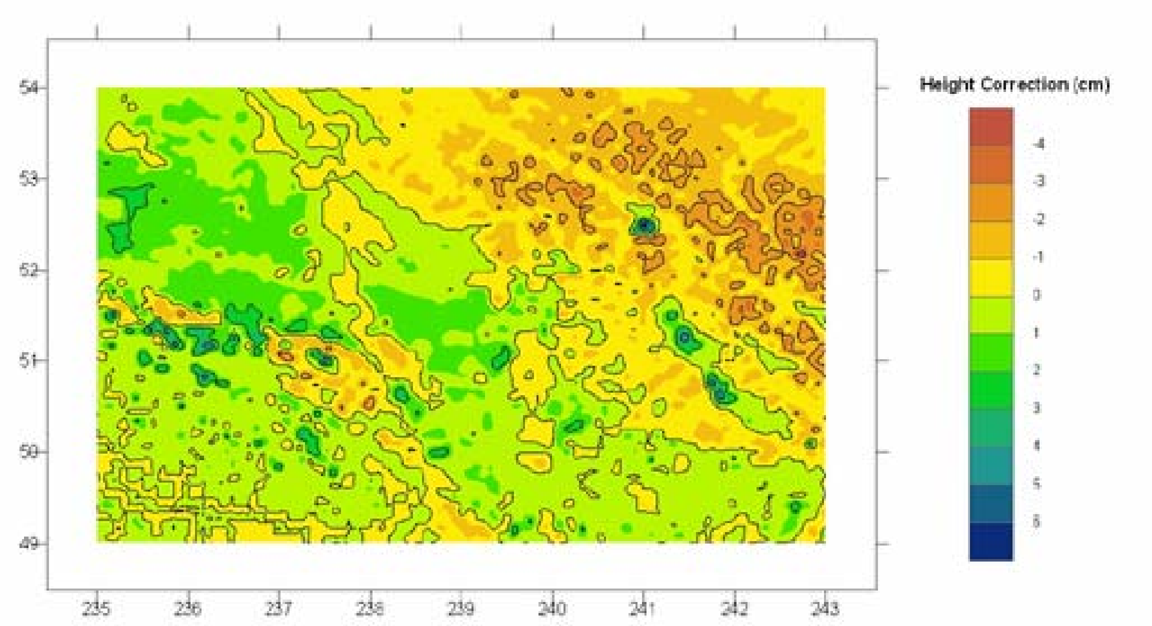

Figure 8: Correction to Helmert orthometric heights for laterally-varying anomalous density.

Figures 7 and 8 show a tendency for both the mean gravity and height corrections to peak where the terrain does, as with the terrain roughness effect; though the tendency is not so pronounced here. Also, the magnitude of this effect is generally small, in part because anomalous densities are smaller than average topographical density.

The geoid-generated gravity disturbance

The geoid-generated gravity disturbance represents the effect of the geoid's mass on gravity. The effect on gravity at the surface may be found by upward continuation of the disturbance, referred to the geoid's surface, to the earth's surface. This is done using Poisson's integral and radial integration over a spherical cap with radius of 7° from the computation point. Disturbances referred to the geoid may be found by removing the secondary indirect topographical effect from the NT gravity anomaly referred to the geoid [Santos et al., 2004b]. Note, however, that this effect may also be calculated directly using downward continuation of gravity disturbances referred to the earth's surface.

The upward continuation calculation used gravity disturbance data given in a regular grid with 5' spacing. Integration was done in two zones, with integration in a 10' square inner zone performed separately from that over the rest of the 7° spherical cap. In the inner zone, a planar approximation of distance was applied in evaluation of the Poisson kernel, and the 5' input data was interpolated to create a regular 30" grid.

The mean effect on gravity may either be approximated by upward continuation of the disturbance to a point halfway along the plumbline, or directly evaluated using an integral mean. In the latter case, a formula for the indefinite integral of the Poisson kernel has been provided by Tenzer et al. [2004, Eq. 15], which is employed with some adaptations in calculation of the mean geoid-generated gravity disturbance. Both techniques produced similar results, but the results from the integral mean approach were accepted as more accurate [Martin et al., 2003].

Results for the correction to Helmert mean gravity over the test area vary from -31 mGal to 16 mGal, and for the correction to Helmert orthometric heights from -2.8 cm to 8.6 cm. This is consistent with the results produced by Martin et al. [2003] for a similar test area, who found corrections ranging from -3.4 cm to 7.9 cm. As expected, the differences between gravity disturbances on the geoid, and mean or surface gravity disturbances, vary according to height of the computation point. It also follows that the magnitude of corrections is correlated with height of the computation point, though it is also correlated with the magnitude of the geoid-generated gravity disturbances. Distribution of the results is shown in Figures 9 and 10, below. Contour intervals are 10 mGal for Figure 9, and 2 cm for Figure 10.

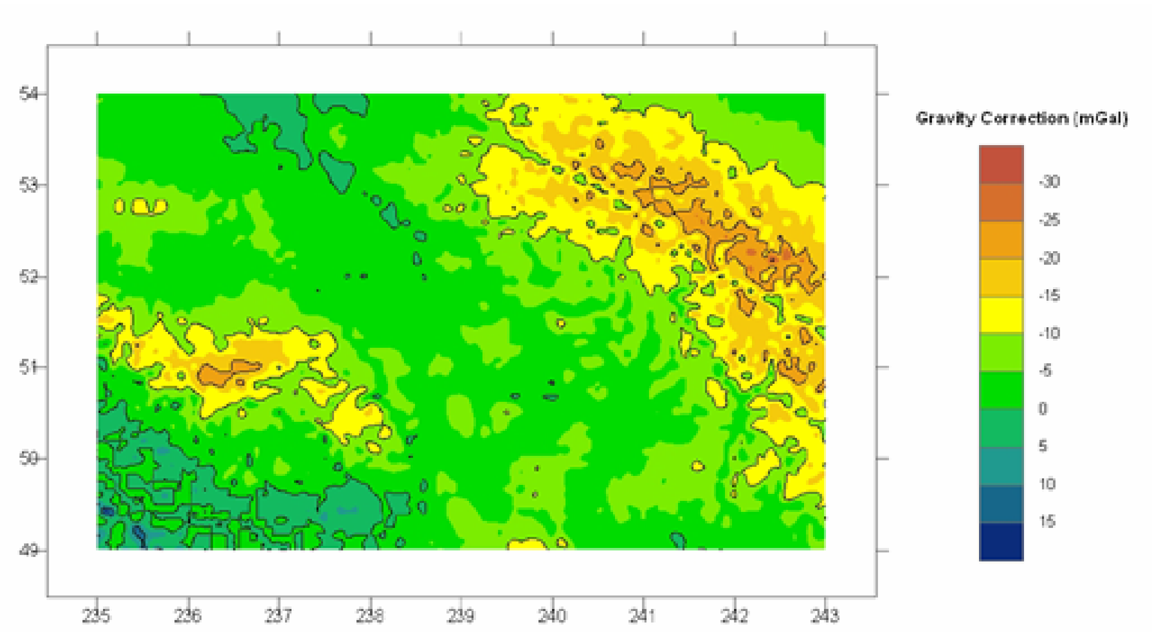

Figure 9: Correction to Helmert mean gravity for the geoid-generated gravity disturbance.

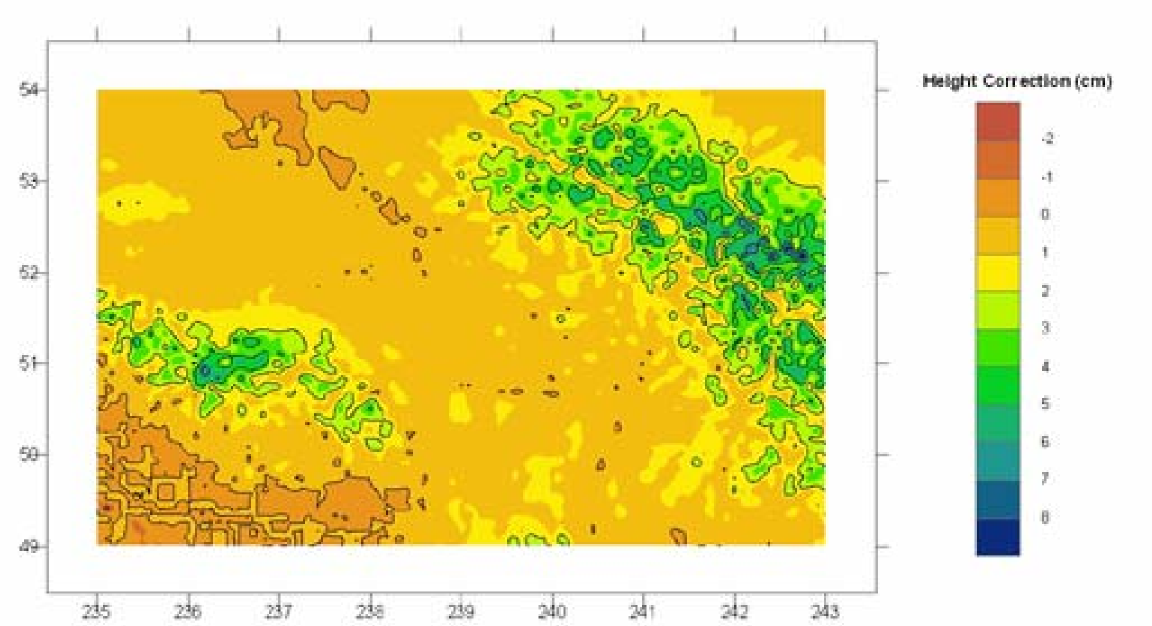

Figure 10: Correction to Helmert orthometric heights for the geoid-generated gravity disturbance.

It is difficult with the correction to mean gravity to distinguish the influence of terrain from that of the geoid-generated gravity disturbance, since the disturbance itself follows the terrain. Their contributions may be separated, however, because the geoid-generated gravity disturbance is very smooth while the terrain is very rough. Thus, the overall trend of the correction to gravity results from the geoid-generated disturbance, while the bumpy areas along the mountain ranges correspond to the effect of terrain height. The correction to height, as with the other two corrections, shows a more pronounced correlation with the terrain.

Total corrections to Helmert orthometric height

These three corrections must be added together to determine a final correction to Helmert orthometric heights. The results for the final correction to mean gravity ranges from -45 mGal to 95 mGal. These correspond to corrections to Helmert's orthometric height from -27.8 cm to 8.6 cm. In the final summation, the geoid-generated gravity disturbances and the terrain roughness often cancel each other within the test area, where the geoid-generated gravity disturbances are often negative. While the correction resulting from anomalous density occasionally worked in the opposite direction to the terrain effect, in the highest parts of the test area – where the influence of the density correction is amplified – it worked in the same direction as the terrain effect, making the overall correction larger. The corrections to gravity over the test area are given in Figure 11, below; and the corrections to height in Figure 12. Contours occur every 40 mGal in Figure 11, and every 10 cm in Figure 12.

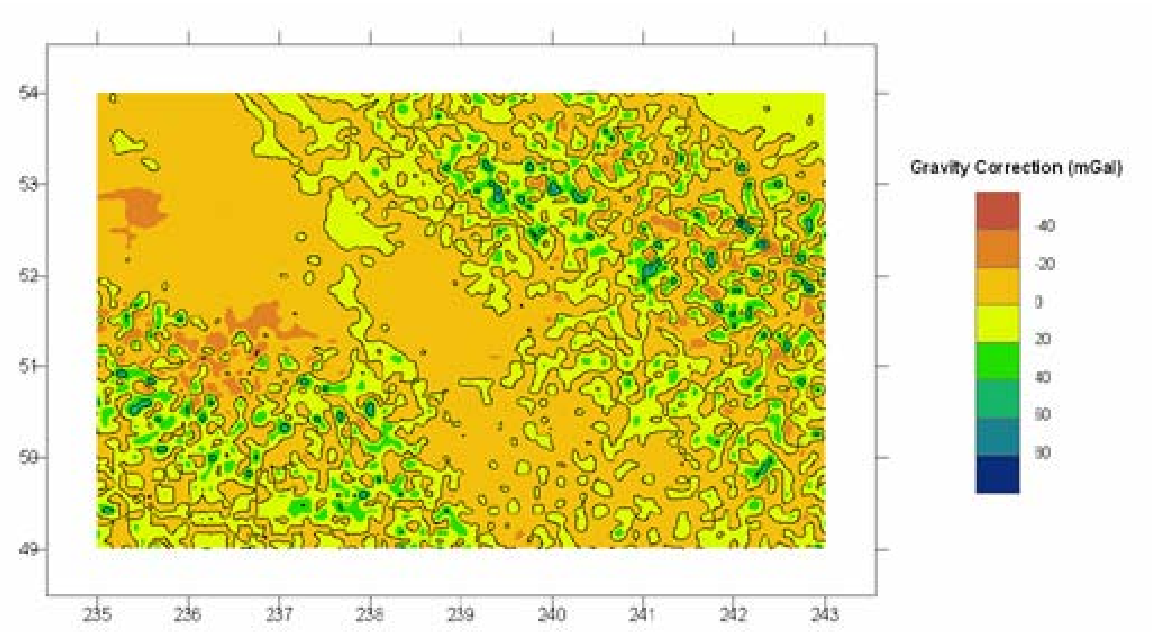

Figure 11: Total corrections to Helmert mean gravity.

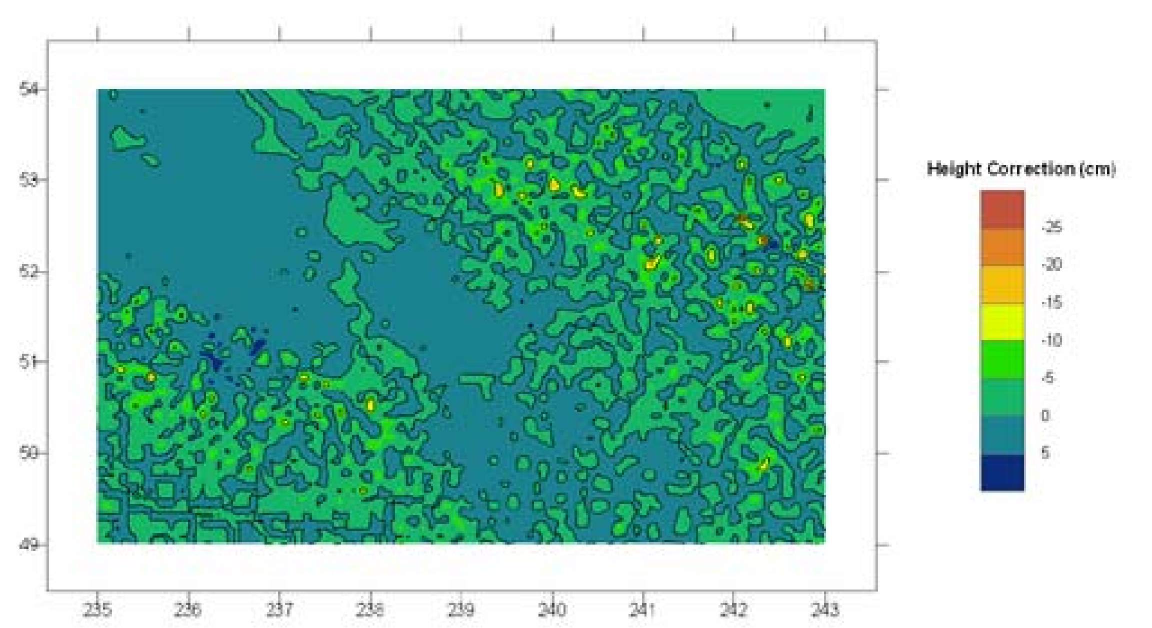

Figure 12: Total corrections to Helmert orthometric height.

Figure 11 shows that the correction to gravity has a tendency to peak in areas of rough terrain, where the main factor in these corrections – the terrain roughness effect – is at its greatest, and where the influences of height on the other two corrections is also at a maximum. While these gravity corrections may be the most effective means of seeing which effects have the greatest contribution to the final height corrections, it is the height corrections themselves which this paper sets out to describe. They are generally small, with a tendency to peak in rough terrain at high elevations. In most low-lying areas, the total correction is under 5 cm; while in more rugged areas they reach as high as 30 cm in magnitude.

A summary of all contributions and total effects is provided in Table 2, below.

Table 2: Corrections to Helmert orthometric height.

Correction	Minimum	Maximum	Mean
Terrain roughness	-31.0 cm	4.6 cm	-1.3 cm
Laterally-varying anomalous density	-4.5 cm	6.5 cm	-0.1 cm
Geoid-generated gravity disturbance	-2.4 cm	8.6 cm	1.1 cm
Total correction	-27.8 cm	8.6 cm	-0.3 cm

Summary and conclusions

Helmert's method for determining orthometric height is not adequate to determine heights having accuracy less than 1 cm in all areas of Canada. In a test area in the Rocky Mountains, corrections for three effects were calculated: the effect of terrain roughness, the effect of laterally-varying anomalous density, and the geoid-generated gravity disturbance. The total correction reached a maximum decimeters in magnitude. If Canadian heights are to be considered rigorous, these effects must be taken into consideration – especially in mountainous areas, like the Rockies.

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