THE UNB TECHNIQUE FOR PRECISE GEOID DETERMINATION

HOW TO COMPUTE GEOID IN SEVEN (NOT SO) EASY STEPS

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1. Formulation of the appropriate Boundary Value Problem (BVP) of the third (Robin's?) kind.

*Region of Interest: Space outside the geoid;
*Boundary: The geoid (unknown);
*Unknown Function: Disturbing potential T(W-U); has the following properties:

$$^{2}T(\vec{r}) = -4\pi G\rho(\vec{r})$$ (non-homogeneity is a problem),

$$\lim_{r \to \infty} T(\vec{r}) = O(r^{-3})$$ (Hörmander condition);

*Boundary Values:

$$\left.\frac{\partial T(\vec{r})}{\partial r}\right|_{r=rg} - \frac{2}{r_g} T\left(r_g - Z(\Omega), \Omega\right) = F\left(T(\vec{r}_g)\right)$$ (cannot be obtained form surface observations).

2. Transformation of BVP into a harmonic space.

Potential W has to be transformed to another potential W^h, harmonic everywhere above the geoid for which the boundary value equation can be linked to observations in the harmonic space:

$T(\vec{r}) \to T^h(\vec{r})$;

$^{2}T(\vec{r}) = 0$ (homogeneous equation);

geoid $\to$ co-geoid.

Many choices of harmonic spaces exist.

3. Selection of an appropriate harmonic space

Helmert's space:

$$T^h(\vec{r}) = T(\vec{r}) - V^t(\vec{r}) + V^{ct}(\vec{r}) - V^a(\vec{r}) + V^{ca}(\vec{r}),$$

where t stands for topography, a for atmosphere and c for condensation.

Choice of condensation scheme: Helmert's second condensation. Why Helmert's scheme?

We use the scheme that <u>preserves earth mass</u> in the transformation to Helmert's space (=> spherical model <u>must</u> be used).

Hörmander's condition is violated.

4. Formulation of boundary values on the geoid (really Helmert's co-geoid)

4.1. Transformation of observations into Helmert's space

Observed gravity $g(H^{\circ},\Omega) = g(r_t,\Omega)$ transformed to:

$$g^h(r_t,\Omega) = g(r_t,\Omega) - DTE(r_t\Omega) - DAE(r_t,\Omega),$$

where

$$DTE(\vec{r}) = \frac{\partial}{\partial H}\left(V^t(\vec{r}) - V^{ct}(\vec{r})\right)$$

$$DAE(\vec{r}) = \frac{\partial}{\partial H}\left(V^a(\vec{r}) - V^{ca}(\vec{r})\right)$$

(these are complicated integrals for spherical models). See Figure 1.

4.2. Determination of Helmert's anomalies

(Helmert's) gravity $g^h(r_t,\Omega)$ on the earth surface is converted to (Helmert's) anomalies:

$$\Delta g^h(r_t,\Omega) = g^h(r_t,\Omega) - \gamma\left(r_o + H^o(\Omega),\Omega\right) +$$

$$-\frac{2}{R}H^o(\Omega)\Delta g^B(\Omega) - SITE(r_t,\Omega) - SIAE(r_t,\Omega)$$

$$= -\left.\frac{\partial T^h(r,\Omega)}{\partial H}\right|_{r=r_t} + \frac{1}{\gamma(r_t,\Omega)}\left.\frac{\partial\gamma(r,\Omega)}{\partial n}\right|_{r=r_t} T^h(r_t - Z(\Omega),\Omega).$$

4.3. Use of simple Bouguer's anomaly

Instead of using observed gravity $g(r_t,\Omega)$ we use simple Bouguer anomaly $\Delta g^B(\Omega)$:

$$g(r_t,\Omega) - \gamma\left(r_o + H^o(\Omega),\Omega\right) = \Delta g^{FA}(\Omega) = \Delta g^B(\Omega) - 2\pi\rho_o H^o(\Omega).$$

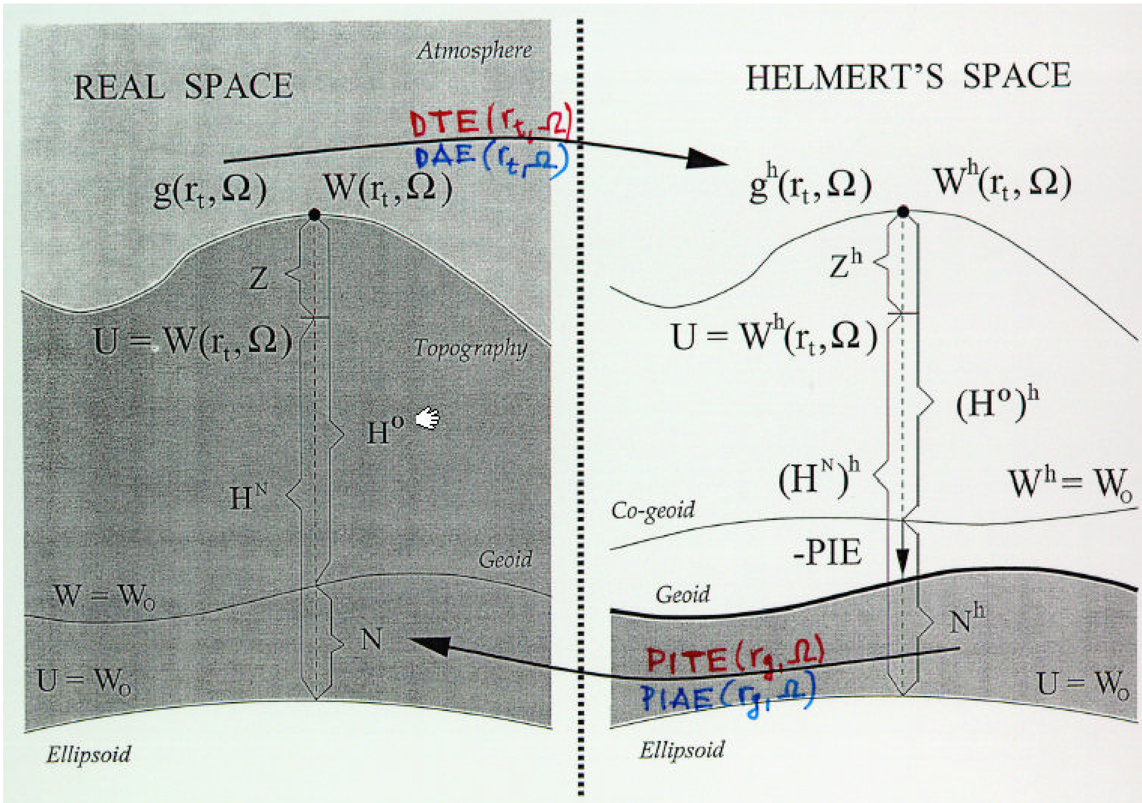

Figure 1 Real and Helmert spaces.

4.4 Averaging gravity anomalies

For easier numerical integration (later on) we convert point values $\Delta g^h(r_t,\Omega)$ to mean values:

$$\bar{\Delta} g^h(r_t,\Omega) = \frac{1}{A}\int_A \Delta g^h(r_t,\Omega)\,dA.$$

This is done for 5' by 5' cells. The size of cells is dictated by available gravity data density.

4.4. Evaluation of 'fundamental gravimetric equation'

$$\Delta g^h(r_t,\Omega) = -\frac{\partial T^h(r,\Omega)}{\partial r}\bigg|_{r=r_t} - T^h(r_t - Z(\Omega),\Omega) + \varepsilon_n(r_t,\Omega)$$

$$-\varepsilon_{\delta g}(r_t,\Omega) = F\left[T^h(r_t,\Omega)\right] + EC(r_t,\Omega).$$

This equation links the <u>observed values</u> $g(r_t,\Omega)$ to <u>boundary values</u> $F\left[T^h(r_g,\Omega)\right]$, it is valid for all $r \geq r_g$!

4.5. Downward continuation of $F\left(T^h\right)$ to the geoid

Since T^h is harmonic above geoid, so is $r\dfrac{\partial \mathrm{T}^h}{\partial r}$ *and* $^{2}\left(rF\left[\mathrm{T}^h(r,\Omega)\right]\right)=0$, $r \geq r_g$. Direchlet's BVP can be formulated and solved <u>Poisson's solution can be used.</u> (Actually, $EC(r,\Omega)$ is also harmonic for $r \geq r_g$, but we have elected to work with $rF^h[\mathrm{T}]$ only. We get:

$$F\left[\mathrm{T}^h\left(r_g,\Omega\right)\right]=\Delta g^h\left(r_g,\Omega\right)-EC\left(r_g,\Omega\right).$$

We have elected to continue 5' by 5' mean values rather than point values. (Poisson's solution appears to be a linear operation.) Mean values $\overline{F}\left[\ T^h\left(r_g,\Omega\right)\right]$ are the <u>discrete boundary values</u> for the BVP of the third kind.

5. Solution of the BVP of the third kind.

5.1. Reformulation (generalization) of the BVP

To take advantage of the availability of global field $(\mathrm{T})_L$ from satellite orbit analysis we construct $\left(T^h\right)_L$ through transformation to Helmert's space (Hemertization). We take L=20; for L>20 global field consists of mostly noise.

$$T^h=\left(T^h\right)_{20}+\left(T^h\right)^{20}$$

Note: $W^h=U+T^h=U+\left(T^h\right)_{20}+\left(T^h\right)^{20}$ *and* $U+\left(T^h\right)_{20}$ is a new <u>reference potential</u>, $[U+\left(T^h\right)_{20}]/\gamma$ is a new <u>reference spheroid</u>.

5.2. Stokes's solution to the BVP

This is the classical solution of Green's kind:

$$T^h\left(r_g,\Omega\right)=4\pi R\int_{geoid}F\left[T^h\left(r_g',\Omega\right)\right]S\left(r_g',\Omega,\Omega\right)d\Omega'$$

$$S\left(r_g',\Omega,\Omega'\right)=\sum_{j=2}^{\infty}\left(\frac{R}{r'_g}\right)^{j+1}\sum_{m=-j}^{i}.\frac{2j+1}{j-1}P_j(\cos\Psi).$$

For the generalized BVP:

$$\mathrm{T}^h\left(r_g,\Omega\right)=\left(\mathrm{T}^h\left(r_g,\Omega\right)\right)_{20}+4\pi R\int_{spheroid}F\left[T^h\left(r_g',\Omega\right)\right]S^{20}\left(r_g',\Omega,\Omega'\right)d\Omega'$$

$$S^{20}\left(r_g',\Omega,\Omega'\right)=\sum_{j=21}^{\infty}\left(\frac{R}{r_g'}\right)^{j+1}\sum_{m=-j}^{j}.\frac{2j+1}{j-1}P_j(\cos\Psi).$$

So far, we have taken only $r_g'=R$.

5.3. Splitting the integration domain

We evaluate the Stokes integral in its above form only over a spherical cap C_0 of radius Ψ_0. For the rest of the domain (spheroid - C_0) we use the spectral form. We choose $\Psi_0 = 6^\circ$ (somewhat arbitrarily).

Using a simplified notation we write:

$$\int_{sph} FS^{20} = \int_{sph} F^{20} S^{20} \left(= \int_{sph} F^{20} S \right) = \int_{sph} F^{20} \underbrace{\left(S^{20} + M_{20} \right)}_{S^{*20}} =$$

$$= \int_{C_0} F^{20} S^{*20} + \int_{sph - C_0} F^{20} S^{*20} \quad .$$

S^{*20} is a modified spheroid Stokes kernel. We have chosen M_{20} such as to minimize the upper bound of the <u>far-zone contribution</u> (second term). We evaluate this FZC in a spectral form using a global field model. For computational reasons we construct the F^{20} already on the topography and only F^{20} $[T^h(r, \Omega)]$ is continued down to the geoid.

6. Solution of the geoid

Still in the Helmert space, we construct the (Helmert co-) geoid:

$$N^h(\Omega) = \frac{1}{\gamma_0(\Omega)} \left[\left(T^h(r_g, \Omega) \right)_{20} + \left(T^h(r_g, \Omega) \right)^{20} \right]$$

$$= \left(N^h(\Omega) \right)_{20} + \left(N^h(\Omega) \right)^{20} \quad .$$

The first term is the reference spheroid, the second term is the <u>residual geoid</u> form the solution to the BVP of the third kind.