THE UNB TECHNIQUE FOR
PRECISE GEOID DETERMINATION
HOW TO COMPUTE GEOID IN SEVEN (NOT SO) EASY STEPS

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1. Formulation of the appropriate Boundary Value Problem (BVP) of the third (Robin's?) kind.

*Region of Interest*: Space outside the geoid;
*Boundary*: The geoid (unknown);
*Unknown Function*: Disturbing potential \( T(\cong W-U) \); has the following properties:
\[
\nabla^2 T(r) = -4\pi G \rho(r) \quad \text{(non-homogeneity is a problem),}
\]
\[
\lim_{r \to \infty} T(r) = O(r^{-3}) \quad \text{(Hörmander condition)};
\]
*Boundary Values:*
\[
\frac{\partial T(r)}{\partial r}
\]
\[
-2 r g \cdot T(r) 
\]
\[
\Omega = F(T(r_g)) \quad \text{(cannot be obtained from surface observations).}
\]

2. Transformation of BVP into a harmonic space.
Potential \( W \) has to be transformed to another potential \( W^h \), harmonic everywhere above the geoid for which the boundary value equation can be linked to observations in the harmonic space:
\[
T(\vec{r}) \rightarrow T^h(\vec{r});
\]
\[
\nabla^2 T(\vec{r}) = 0 \quad \text{(homogeneous equation)};
\]
geoid \( \rightarrow \) co-geoid.
Many choices of harmonic spaces exist.
3. Selection of an appropriate harmonic space

Helmert's space:
\[ T^h(\vec{r}) = T(\vec{r}) - V'(\vec{r}) + V''(\vec{r}) - V^a(\vec{r}) + V^c(\vec{r}), \]

where \( t \) stands for topography, \( a \) for atmosphere and \( c \) for condensation.

Choice of condensation scheme: Helmert's second condensation. Why Helmert's scheme?
We use the scheme that preserves earth mass in the transformation to Helmert's space (=> spherical model must be used).

Hörmander's condition is violated.

4. Formulation of boundary values on the geoid (really Helmert's co-geoid)

4.1. Transformation of observations into Helmert's space

Observed gravity \( g(\Omega) \) transformed to:
\[ g^h(\vec{r},\Omega) = g(\vec{r},\Omega) - DTE(\vec{r},\Omega) - DAE(\vec{r},\Omega), \]

where
\[ DTE(\vec{r}) = \frac{\partial}{\partial H} \left(V'(\vec{r}) - V''(\vec{r})\right) \]
\[ DAE(\vec{r}) = \frac{\partial}{\partial H} \left(V^a(\vec{r}) - V^c(\vec{r})\right) \]

(these are complicated integrals for spherical models). See Figure 1.

4.2. Determination of Helmert's anomalies

(Helmert's) gravity \( g^h(\vec{r},\Omega) \) on the earth surface is converted to

(Helmert's) anomalies:
\[ \Delta g^h(\vec{r},\Omega) = g^h(\vec{r},\Omega) - \gamma \left(r_o + H''(\Omega),\Omega\right) + \]
\[ -\frac{2}{R} H''(\Omega) \Delta g^h(\Omega) - SITE(\vec{r},\Omega) - SIAE(\vec{r},\Omega) \]
\[ = -\frac{\partial T^h(\vec{r},\Omega)}{\partial H} \bigg|_{\Omega} + \frac{1}{\gamma(\vec{r},\Omega)} \frac{\partial \gamma(\vec{r},\Omega)}{\partial n} \bigg|_{\Omega} T^h(\vec{r} - Z(\Omega),\Omega). \]

4.3. Use of simple Bouguer's anomaly

Instead of using observed gravity \( g(\vec{r},\Omega) \) we use simple Bouguer anomaly
\[ \Delta g^B(\Omega) : \]
\[ g(\vec{r},\Omega) - \gamma \left(r_o + H''(\Omega),\Omega\right) = \Delta g^FA(\Omega) = \Delta g^B(\Omega) - 2\pi \rho_o H''(\Omega). \]
4.4 Averaging gravity anomalies

For easier numerical integration (later on) we convert point values \( \Delta g^h(r, \Omega) \) to mean values:

\[
\bar{\Delta g}^h(r, \Omega) = \frac{1}{A} \int_A \Delta g^h(r, \Omega) dA.
\]

This is done for 5' by 5' cells. The size of cells is dictated by available gravity data density.

4.4 Evaluation of 'fundamental gravimetric equation'

\[
\Delta g^h(r, \Omega) = -\frac{\partial T^h(r, \Omega)}{\partial r} \bigg|_{r=r_g} - T^h(r - Z(\Omega), \Omega) + \varepsilon_n(r, \Omega)
\]

\[
-\varepsilon_{\delta g}(r, \Omega) = F\left[T^v(r, \Omega)\right] + EC(r, \Omega).
\]

This equation links the observed values \( g(r, \Omega) \) to boundary values \( F\left[T^v(r, \Omega)\right] \), it is valid for all \( r \geq r_g \).
4.5. **Downward continuation of \( F(T^h) \) to the geoid**

Since \( T^h \) is harmonic above geoid, so is \( r \frac{\partial T^h}{\partial r} \) and \( \nabla^2 \{ rF[T^h(r, \Omega)] \} = 0 \), \( \forall r \geq r_g \).  Direchlet's BVP can be formulated and solved \( \Rightarrow \) Poisson's solution can be used.  (Actually, \( EC(r, \Omega) \) is also harmonic for \( \forall r \geq r_g \), but we have elected to work with \( rF[T^h] \) only.  We get:

\[
F[T^h(r_g, \Omega)] = \Delta g^h(r_g, \Omega) - EC(r_g, \Omega).
\]

We have elected to continue \( T' \) by \( S' \) mean values rather than point values.  (Poisson's solution appears to be a linear operation.)  Mean values \( F[T^h(r_g, \Omega)] \) are the discrete boundary values for the BVP of the third kind.

5. **Solution of the BVP of the third kind.**

5.1. **Reformulation (generalization) of the BVP**

To take advantage of the availability of global field \( (T)_L \) from satellite orbit analysis we construct \( (T^h)_L \) through transformation to Helmert's space (Hemertization).  We take \( L=20 \); for \( L>20 \) global field consists of mostly noise.

\[
T^h = (T^h)_{20}^2 + (T^h)_{20}^2
\]

Note: \( W^h = U + T^h = U + (T^h)_{20}^2 + (T^h)_{20}^2 \) and \( U + (T^h)_{20}^2 \) is a new reference potential, \( [U + (T^h)_{20}^2] \) is a new reference spheroid.

5.2. **Stokes's solution to the BVP**

This is the classical solution of Green's kind:

\[
T^h(r_g, \Omega) = 4\pi R \int_{\text{geoid}} F[T^h(r_g, \Omega)] S(r_g, \Omega, \Omega') d\Omega'
\]

\[
S(r_g, \Omega, \Omega') = \sum_{j=2}^{\infty} \left( \frac{R}{r_g} \right)^{j+1} \sum_{m=-j}^{j+1} \frac{2j+1}{j-1} P_j(\cos \Psi).
\]

For the generalized BVP:

\[
T^h(r_g, \Omega) = (T^h(r_g, \Omega))_{20}^2 + 4\pi R \int_{\text{spheroid}} F[T^h(r_g, \Omega)] S_{20}^2(r_g, \Omega, \Omega') d\Omega'
\]

\[
S_{20}^2(r_g, \Omega, \Omega') = \sum_{j=21}^{\infty} \left( \frac{R}{r_g} \right)^{j+1} \sum_{m=-j}^{j+1} \frac{2j+1}{j-1} P_j(\cos \Psi).
\]

So far, we have taken only \( r_g' = R \).
5.3. Splitting the integration domain

We evaluate the Stokes integral in its above form only over a spherical cap $C_0$ of radius $\Psi_0$. For the rest of the domain (spheroid - $C_0$) we use the spectral form. We choose $\Psi_0 = 6^\circ$ (somewhat arbitrarily).

Using a simplified notation we write:

$$
\int_{\text{sph}} F S^{20} = \int_{\text{sph}} F^{20} S^{20} = \int_{\text{sph}} F^{20} \left( S^{20} + M_{20} \right) =
$$

$$
= \int_{C_0} F^{20} S^{20} + \int_{\text{sph}-C_0} F^{20} S^{20} + M_{20}.
$$

$S^{20}$ is a modified spheroid Stokes kernel. We have chosen $M_{20}$ such as to minimize the upper bound of the far-zone contribution (second term).

We evaluate this FZC in a spectral form using a global field model. For computational reasons we construct the $F^{20}$ already on the topography and only $F^{20} [T^h(r, \Omega)]$ is continued down to the geoid.

6. Solution of the geoid

Still in the Helmert space, we construct the (Helmert co-) geoid:

$$
N^h(\Omega) = \frac{1}{\gamma(\Omega)} \left[ \left( T^h\left( r^h, \Omega \right) \right)_{20} + \left( T^h\left( r^h, \Omega \right) \right)_{20} \right]
$$

$$
= \left( N^h(\Omega) \right)_{20} + \left( N^h(\Omega) \right)_{20}.
$$

The first term is the reference spheroid, the second term is the residual geoid form the solution to the BVP of the third kind.