On the numerical evaluation of the truncated geoid

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A b s t r a c t : The truncated geoid, defined by the so called "Truncated Stokes Integral", which is an integral transform of the gravity anomalies in a limited domain of a spherical cap of specified radius with the Stokes function being the kernel, has been widely used by geodesists as an approximation to the geoid in a variety of approaches to the (terrestrial) geoid determination. The possible physical interpretation of the truncated geoid, i.e., its relation to the density distribution generating the surface gravity has been investigated by the authors. Preliminary results are being published as the study progresses. This paper focuses on numerical aspects of the evaluation of the truncated geoid. It can be computed either in spectral form, providing the geopotential is given in a spectral form, or directly by numerical integration over mean gravity anomalies. Alternatively, the truncated geoid may be evaluated analytically from geoidal undulations via another integral transform, namely the so called "Altimetry Integral". This becomes particularly useful when interpreting the truncated geoid on sea, where the geoidal undulations of global coverage and good quality can be obtained after procesing the readily available satellite altimetry data. However, the numerical evaluation of the "Altimetry Integral" poses a challenge. We devote most of the attention, in this paper, to overcoming the problems encountered in the numerical evaluation of the above integral transform. We review a couple of numerical techniques used previously and introduce a new, more flexible numerical technique for the truncated geoid from geoidal undulations evaluation, that was developed by us recently.

Key words: gravimetric inversion, truncated geoid, numerical methods

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1. Introduction

The truncated geoid, defined by proper mathematical aparatus in section 2, is evaluated by integrating over the gravity anomalies in a spherical cap of a radius, which is usually referred to as "the truncation radius". In the geodetic boundary value problem, the geoid is computed by the Stokes integral of gravity anomalies over the entire globe [Stokes, 1849]. Here, the truncated geoid may be encountered in several techniques for the terrestrial (local, regional) geoid determination, where it is treated purely as a mathematical object, serving as an approximation to the geoid, while the "truncation error", which is the contribution of the integral over the omitted domain - i.e., the counterpart of the spherical cap to the whole sphere, is either neglected or modelled (e.g., [Molodenskij, 1962; Vaníček and Kleusberg, 1987; Vaníček et al., 1995].

The authors, along with Prof. Sjoeberg, Prof. Kleusberg, and Dr. Martinec, were likely the first ones to consider the physical meaning of the truncated geoid (in the sequel abbreviated as the TG). They started investigating the relation of the TG to the density distribution generating the surface gravity more closely in 1992. The first study was carried out by means of computer simulation and was limited to density distributions represented by point mass anomalies. Gravity inversion for one point mass anomaly using the TG together with its theoretical verification was presented in [Vajda and Vaníček, 1996]. An inversion algorithm, named the "truncation technique", which was designed for sets of point mass anomalies as a result of multiple computer simulations was presented in [Vajda and Vaníček, 1997]. The investigation is in progress and now it will concentrate on continuous density distributions.

This paper focuses on numerical aspects of the evaluation of the TG. The surface of truncated geoidal undulations (the TG) can be computed either in spectral form, providing the geopotential is given in a spectral form, or by direct numerical integration over mean gravity anomalies on a mesh within the cap. This was the case in the context of our research, where we were generating synthetic gravity anomalies, in either spectral form or on a mesh, by sets of point masses. Both these evaluations are ralatively straight forward and will be presented in section 3.

Eventually we would like to evaluate the TG from geoidal undulations for the sake of the off-shore geophysical interpretation in terms of the TG. There is an integral transform for evaluating the TG analytically from geoidal undulations, which was derived by Vaníček et al., [1987]. However, the numerical evaluation of such integral transform poses a challenge. The difficulties one has to face include the fact, that the integration needs to be carried out over the entire globe, which is impractical if not impossible, and also that there is a singularity, which can be removed analytically, but numerically causes a problem. Here we mention a couple of numerical techniques used previously and demonstrate a new, more flexible numerical technique for its evaluation, that was developed by us recently.

2. Truncated geoid

The truncated geoid is defined by the *truncated Stokes integral* (e.g., [Vaníček et al., 1987])

$$N^{\psi_0}(P) = \frac{R}{4\pi\gamma} \iint\limits_{\Re(\psi_0)} \Delta g(Q) S_{PQ} \,\mathrm{d}\,\sigma. \tag{1}$$

The truncated geoidal height N^{ψ_0} is evaluated at the computation point P as a convolution of the gravity anomalies Δg on a spherical cap $\Re(\psi_0)$ of radius ψ_0 centred at P, with the Stokes function S [Stokes, 1849] being the kernel. Point Q is the dummy integration point. R is the radius of the boundary sphere (mean earth) and γ is normal gravity. The increament $\mathrm{d}\,\sigma$ is the surface element. Integral (1) can be expressed in the local polar coordinates of point P as (cf. Fig.1):

$$N^{\psi_0}(P) = \frac{R}{4\pi\gamma} \int\limits_0^{\psi_0} \int\limits_0^{2\pi} \Delta g(\psi, \alpha) S(\psi) \sin(\psi) \,\mathrm{d}\,\psi \,\mathrm{d}\,\alpha, \tag{2}$$

where ψ is the spherical distance between points P and Q, and α is azimuth of point Q.

Radius ψ_0, called the truncation radius, is a free parameter of the TG. This is why we usually refer to it as the "truncation parameter". We also refer to a TG as to a surface of the truncated geoidal heights evaluated for a particular value of the truncation parameter ψ_0. The TG (the shape of this surface) changes when changing the value of ψ_0. That is the reason why we often talk about different TGs for different values of ψ_0. To be complete, the surface of the TG is shaped by

1. the mass distribution generating the surface gravity and therefore the TG, as well
2. the value of ψ_0.

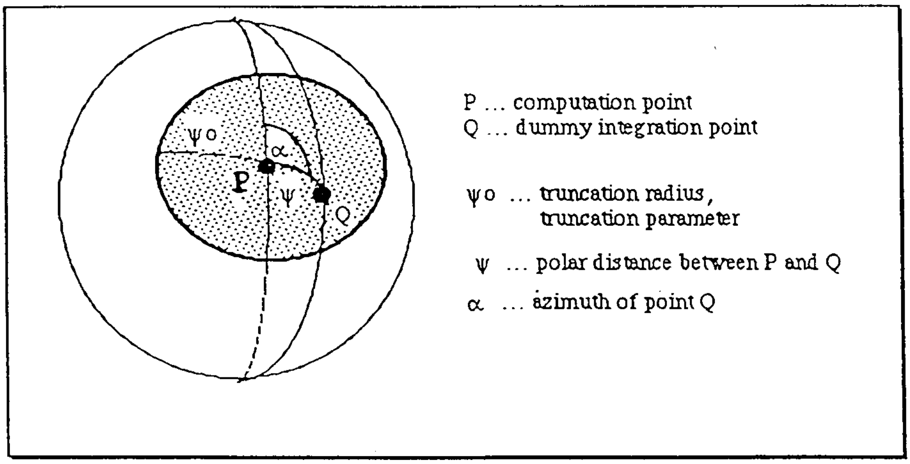

Fig. 1. The truncated Stokes integration.

The TG may alternatively be evaluated from the geoidal heights (undulations) via the *altimetry integral*, which expressed already in the polar coordinates of the computation point reads as follows [Vaníček et al., 1987]:

$$N^{\psi_0}(P) = N(P) - \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \frac{N(\psi,\alpha)}{\cos(\psi)-\cos(\psi_0)} T^{\psi_0}(\psi)\sin(\psi)\,\mathrm{d}\,\psi\,\mathrm{d}\,\alpha, \tag{3}$$

where N is geoidal undulation, the T kernel is given by the series

$$T^{\psi_0}(\psi) = Q_2(\psi_0)P_1(\cos\psi) + \sum_{n=2}^{\infty}\left(\frac{2n+1}{2}\right) t_n(\psi_0)P_n(\cos\psi) \tag{4}$$

with coefficients

$$t_n(\psi_0) = \frac{n(n-2)}{2n+1}Q_{n-1}(\psi_0) + \frac{n(n+1)}{2n+1}Q_{n+1}(\psi_0) - \\ - (n-1)\cos(\psi_0)Q_n(\psi_0), \tag{5}$$

where $Q_n(\psi_0)$ are the Molodenskij truncation coefficients [Molodenskij et al., 1962; Vaníček et al., 1987, eqn.(5.19)]

$$Q_n(\psi_0) = \int\limits_{\psi_0}^{\pi} S(\psi) P_n(\cos\psi) \sin(\psi) \, \mathrm{d}\,\psi, \tag{6}$$

with P_n being the Legendre polynomials. The integral in eqn. (3) has a weak singularity at $\psi = \psi_0$. This singularity can be removed as follows [Vaníček et al., 1987]:

$$N^{\psi_0}(P) = N(P) - \frac{1}{2} \int\limits_{0}^{\pi} \frac{\overline{N}(\psi) - \overline{N}(\psi_0)}{\cos(\psi) - \cos(\psi_0)} T^{\psi_0}(\psi) \sin(\psi) \, \mathrm{d}\,\psi, \tag{7}$$

since due to the orthogonality of the Legendre polynomials

$$\frac{1}{2}\overline{N}(\psi_0) \int\limits_{0}^{\pi} \frac{1}{\cos(\psi) - \cos(\psi_0)} T_{\ell}^{\psi_0}(\psi) \sin(\psi) \, \mathrm{d}\,\psi = 0, \tag{8}$$

and where the over-bar represents azimuthal average

$$(\overline{*}) = \frac{1}{2\pi} \int\limits_{0}^{2\pi} (*) \, \mathrm{d}\alpha. \tag{9}$$

Note, that integral (7) is to be evaluated over the entire globe to compute the truncated geoidal height in one point of computation.

3. Numerical evaluation of the TG from gravity anomalies

3.1 Computation of the truncated geoid in spectral form

Suppose that we are given the disturbing potential on the boundary sphere in spectral form, with the zero-th and the first degree spherical harmonics being absent:

$$T(\varphi, \lambda) = \sum_{n=2}^{\infty} T_n(\varphi, \lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[A_{nm}^{T} Y_{nm}^{c}(\varphi, \lambda) + B_{nm}^{T} Y_{nm}^{s}(\varphi, \lambda) \right], \tag{10}$$

where $Y^c_{nm}(\varphi,\lambda) = \cos(m\lambda)P_{nm}(\sin\varphi)$, $Y^s_{nm}(\varphi,\lambda) = \sin(m\lambda)P_{nm}(\sin\varphi)$ are spherical harmonic functions with $P_{nm}(\sin(\varphi))$ being the associated Legendre polynomials. The truncated geoid can be then evaluated from the disturbing potential as

$$N^{\psi_0}(P) = \frac{1}{\gamma}\sum_{n=2}^{\infty}\left(1 - \frac{n-1}{2}Q_n(\psi_0)\right)\sum_{m=0}^{n}\Big(A^T_{nm}Y^c_{nm}(\varphi_P,\lambda_P)+ + B^T_{nm}Y^s_{nm}(\varphi_P,\lambda_P)\Big), \qquad (11)$$

where again $Q_n(\psi_0)$ are the Molodenskij truncation coefficients (cf. eqn.(6)) and can be numerically evaluated by Paul's algorithm [Paul, 1973]. Having a spectral representation of the disturbing potential up to a certain preselected degree is equivalent to having such a spectral representation of the gravity anomalies or the geoidal undulations, as there is a simple linear relation between the spectral coefficients of these three quantities (e.g., [Vaníček and Krakiwsky, 1986, page 518]).

In the context of our computer simulation we were generating synthetic gravity anomalies on a mesh covering the entire boundary sphere and performing spherical harmonic analysis, using the "SHA routine" [Martinec, 1992], to obtain the spherical harmonic coefficients of the gravity anomalies up to the desired degree and order $n_{\max}$. The disturbing potential harmonic coefficients were then computed simply as (e.g., Vaníček and Krakiwsky, 1986, eqn.(22.8))

$$A^T_{nm} = \frac{R}{n-1}A^{\Delta g}_{nm},\; B^T_{nm} = \frac{R}{n-1}B^{\Delta g}_{nm},\; n = 2,\ldots,n_{\max},\; m = 0,\ldots,n. \qquad (12)$$

The Molodenskij truncation coefficients were computed using the "TCPAL subroutine" [Kleusberg, 1992] based on Paul's algorithm.

When computing a real global TG (based on actual gravity field data), the spherical harmonic coefficients come from a specific global spectral solution to the geopotential, such as GFZ93a [Gruber and Anzenhofer, 1993] or GEM-T3 [Lerch et al., 1992], etc. Naturally, the spectral content of such a real global TG is only as good as is the spectral content of the geopotential model used for its computation.

3.2 Computation of the truncated geoid by numerical integration over mean gravity anomalies

Suppose that mean gravity anomalies are given on a uniform geographical grid $\Delta\varphi = \Delta\lambda = \Delta^{IG}$ (IG stands for "integration grid"), where a value of $\Delta g(\varphi, \lambda)$ is assigned to the centre of each grid cell of size $R\Delta\varphi \times R\cos(\varphi)\Delta\lambda$. When expressing the truncated Stokes integral (eqn. (1) or (2)) in geographical coordinates $[\varphi, \lambda]$,

$$N^{\psi_0}(P) = \frac{R}{4\pi\gamma} \iint_{\Re(\psi_0)} \Delta g\,(\varphi, \lambda) S(\psi) \cos(\varphi)\,\mathrm{d}\,\varphi\,\mathrm{d}\,\lambda, \tag{13}$$

where the polar distance is evaluated from the geographic coordinates of points P and Q (e.g., [Vaníček and Krakiwsky, 1986, eqn.(20.50)])

$$\cos(\psi) = \sin(\varphi_P)\sin(\varphi) + \cos(\varphi_P)\cos(\varphi)\cos(\lambda - \lambda_P), \tag{14}$$

and where the elementary surface element reads as

$$\mathrm{d}\,\rho = \sin(\psi)\,\mathrm{d}\,\psi\,\mathrm{d}\,\alpha = \cos(\varphi)\,\mathrm{d}\,\varphi\,\mathrm{d}\,\lambda, \tag{15}$$

we have to face a weak singularity at $Q \equiv P$, ($\psi = 0$). After a proper treatment of this removable singularity (e.g., [Vajda, 1995, page 135]), the integration can be performed numerically as

$$\begin{aligned} N^{\psi_0}(P) = {} & \frac{R}{\gamma}\sqrt{\frac{\cos(\varphi_P)}{\pi}}\Delta g(P)\Delta^{IG} + \\ & + \frac{R}{4\pi\gamma}\sum_{Q_i \in D} \Delta g(Q_i) S(\psi_i)\cos(\varphi_i)\left(\Delta^{IG}\right)^2, \end{aligned} \tag{16}$$

where the first term is the contribution of the cell which is concentric with the computation point P, and the second term is the contribution of all the cells whose centres fall inside the spherical cap of radius ψ_0 excluding the cell concentric with P (denoted here as domain D). Note that the gravity anomalies have to be given on an area extending the computation area in any direction by $R\psi_0^{\max}$, where $\psi_0^{\max}$ is the largest desired value of the truncation parameter of the TG under investigation. The step of the geographical grid ($\Delta\varphi$) dictates the smallest possible value of ψ_0, ($\psi_0^{\min}$), that can be considered, as well as the smallest possible step in the truncation parameter $\Delta\psi_0^{\min}$ when computing the TGs with systematically decreasing the value of the truncation radius from $\psi_0^{\max}$ to $\psi_0^{\min}$, which is the case in our computer simulation.

In the context of real gravity anomalies given on a mesh in a geographic area (locally, regionally), the same limitations, imposed upon the values of the truncation parameter, as above, apply.

4. Numerical evaluation of the truncated geoid from geoidal undulations

The integration in eqn. (7) has to be carried out over the entire reference sphere, which is not possible. The problem of a too large integration domain can be by-passed, following the method of Vaníček et al., [1987], by splitting the computation into low-degree and high- degree parts, evaluating the low–degree part in spectral form, and introducing an approximation in the numerical integration of the high–degree part by truncating the integration domain to a manageable size.

First we split the truncated geoid into low- and high–degree parts at a spheroidal degree ℓ

$$N^{\psi_0} = N_\ell^{\psi_0} + \delta N_\ell^{\psi_0}. \tag{17}$$

The low-degree, or long-wavelength, truncated geoid is evaluated in spectral form

$$\begin{aligned} N_\ell^{\psi_0}(P) = \frac{1}{\gamma} \sum_{n=2}^{\ell} \left(1 - \frac{n-1}{2} Q_n(\psi_0)\right) \sum_{m=0}^{n} \Big(& A_{nm}^T Y_{nm}^c(\varphi_P, \lambda_P) + \\ & + B_{nm}^T Y_{nm}^s(\varphi_P, \lambda_P)\Big) . \end{aligned} \tag{18}$$

Note that disturbing potential, or equivalently the geoidal undulations, in spectral form up to degree ℓ has/have to be known. In the context of our computer simulations the spherical harmonic coefficients of the disturbing potential were obtained by spherical harmonic analysis (cf. section 3.1). When computing a real TG, these coefficients must come from a specific solution to the geopotential, such as global models or global satellite models (cf. section 3.1).

The high-degree or short-wavelength truncated geoid reads as (note here that again, due to the orthogonality of Legendre polynomials on $\langle 0; \pi\rangle$, we removed the low-degree part of the T kernel)

$$\delta N_\ell^{\psi_0}(P) = \delta N^\ell(P) - \frac{1}{2}\int_0^\pi \frac{\delta\overline{N}^\ell(\psi) - \delta\overline{N}^\ell(\psi_0)}{\cos(\psi) - \cos(\psi_0)} T_\ell^{\psi_0}(\psi)\sin(\psi)\,\mathrm{d}\,\psi, \tag{19}$$

where the spheroidal T kernel is given by

$$T_\ell^{\psi_0}(\psi) = \frac{\ell+1}{2}\Big\{\ell Q_{\ell+1}(\psi_0)P_\ell(\cos\psi) - (\ell-1)Q_\ell(\psi_0)P_{\ell+1}(\cos\psi)\Big\} + \\ + \sum_{n=\ell+1}^{\infty}\left(\frac{2n+1}{2}\right) t_n(\psi_0)P_n(\cos\psi) \tag{20}$$

and where

$$\delta N^\ell(\varphi,\lambda) = N(\varphi,\lambda) - \frac{1}{\gamma}\sum_{n=2}^{\ell}\sum_{m=0}^{n}\Big[A_{nm}^T Y_{nm}^c(\varphi,\lambda) + B_{nm}^T Y_{nm}^s(\varphi,\lambda)\Big], \tag{21}$$

Vaníček et al., [1987] have shown, that when the integration domain in eqn. (19) is truncated from the entire sphere to a cap of radius that of the first positive maximum of the spheroidal T kernel (ϕ_0), the result approximates that of eqn. (19) very well. Only spectral band $\langle \ell, \ell+25\rangle$ of the (high-degree) truncated geoid gets slightly distorted. This phenomenon has not been properly explained, yet, thus there is a lot of room for further investigation. We get

$$\delta N_\ell^{\psi_0}(P) \doteq \delta N^\ell(P) - \frac{1}{2}\int_0^{\phi_0} \frac{\delta\overline{N}^\ell(\psi) - \delta\overline{N}^\ell(\psi_0)}{\cos(\psi) - \cos(\psi_0)} T_\ell^{\psi_0}(\psi)\sin(\psi)\,\mathrm{d}\,\psi. \tag{22}$$

Compared to the fairly straightforward analytical removal of the weak singularity at $\psi = \psi_0$, its removal in the numerical integration remains still a challenge, due to the discretization. Contrary to Vaníček et al. [1987], who perform the double numerical integration in the local (ψ, α) coordinate system of the computation point P (the "ring integration technique", which in fact does not remove the singularity completely), and contrary to Zhang [1988], who furthermore approximates geoidal undulations by polynomial functions to treat the singularity numerically in a better way (the "improved ring integration technique"), we have chosen to perform the integration in geographical coordinates (φ, λ) to avoid further approximations.

Suppose the mean (high-degree) geoidal undulations are given on a uniform geographical grid. We propose that it be performed by making use of four sub-integration domains (see Fig. 2) as follows:

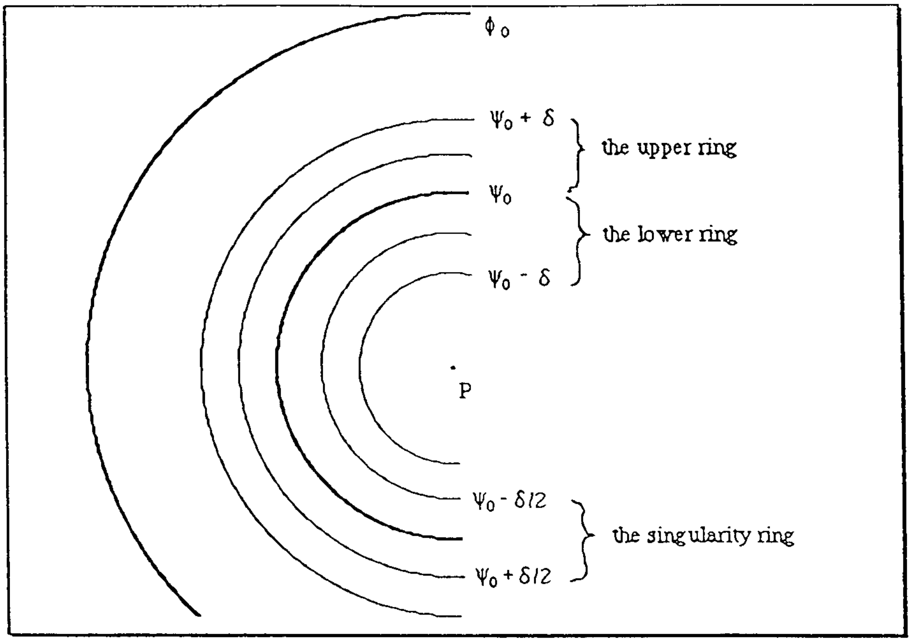

Fig. 2. The integration domain and the sub-domains for the numerical integration over the geoidal undulations.

Thus

$$\delta N_\ell^{\psi_0}(P) = \delta N^\ell(P) - \frac{1}{4\pi} \sum_{i;Q_i \in D_1} \frac{\delta N^\ell(Q_i) - C_1}{\cos(\psi_i) - \cos(\psi_0)} T_\ell^{\psi_0}(\psi_i) \cos(\varphi_i) \Delta^2 - \\ - T_\ell^{\psi_0}(\psi_0) C_P(\psi_0), \qquad (23)$$

where the D_1 sub-domain is the spherical cap of radius ϕ_0 with a removed ring of radius ψ_0 and thickness δ, the singularity ring (domain D_2), and where

$$C_1 = \frac{1}{n_2} \sum_{i;Q_i \in D_2}^{n_2} \delta N^\ell(Q_i) \qquad (24)$$

is the contribution of the singularity ring. The third term in eqn. (23) takes care of the removable weak singularity and is computed as an azimuthal average of the slope of the geoid in the ring at $\psi = \psi_0$, and the C_P constant is evaluated by means of the upper and lower rings (cf. Fig.2) as

$$C_P(\psi_0) = \frac{1}{\sin(\psi_0)} \left[\frac{1}{n_u} \sum_{i;Q_i \in UR}^{n_u} \delta N^\ell(Q_i) - \frac{1}{n_\ell} \sum_{i;Q_i \in LR}^{n_\ell} \delta N^\ell(Q_i) \right] \frac{1}{\delta} \cdot \qquad (25)$$

Numerical testing has lead to choosing $\delta = \Delta^{IG}$ for the integration.

The above described numerical technique was tested against the TGs computed from gravity anomalies. It proved reasonably accurate in the context of our computer simulations in terms of the interpretation of the TG using the truncation technique. It is difficult to say more about the discrepancy caused by the approximation due to the auxiliary truncation (replacing the entire globe by the spherical cap of radius ϕ_0). More testing needs to be performed in order to asses the accuracy in a more proper way. The fact that the integration up to the first positive maximum of the spheroidal T kernel (ϕ_0) gives such a good approximation has not been theoretically explained, yet. In comparison to the existing techniques, namely *the ring integration* [Vaníček et al., 1987] and *the improved ring integration* [Zhang, 1988], our numerical technique seems to be no less accurate, but more flexible. In fact, those two techniques were specifically tuned-up for a particular value of the truncation parameter ψ_0, which matched the objective of the investigation, for which they were designed [Vaníček et al., 1987]. Our technique works equally well for any value of ψ_0 under the limitations discussed in section 3.2. The mesh, on which the geoidal undulations are given, must extend the computation area in any direction by $R\phi_0$, which depends on the values of ψ_0 and ℓ. The relation between ℓ, ψ_0, and ϕ_0 is tabulated in Vajda [1995].

5. Conclusions

Procedures for evaluating the truncated geoid for any value of the truncation parameter from gravity anomalies, geoidal undulations, and in spectral form are presented along with the respective numerical techniques for their performance. Although these techniques were designed, tested and used on synthetic gravity data, they are equally useful and accurate on the real gravity data. Above mentioned techniques served as the core numerical tools in our computer simulations, based on which we demonstrated the behaviour of the truncated geoid and its potential within the frame of geophysical exploration by gravity inversion. Our aim is to use them in the interpretation of the real surface gravity in the future.

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