On the Accuracy of Modified Stokes's Integration in High-frequency Gravimetric Geoid Determination

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Abstract

Two numerical techniques are used in recent regional high-frequency geoid computations in Canada: discrete numerical integration and the fast Fourier transform. These two techniques have been tested for their numerical accuracy using a synthetic gravity field. The synthetic field was generated by artificially extending the EGM96 spherical harmonic coefficients to degree 2160, which is commensurate with the regular $5'$ geographical grid used in Canada. This field was used to generate self-consistent sets of synthetic gravity anomalies and synthetic geoid heights with different degree variance spectra, which were used as control on the numerical geoid computation techniques. Both the discrete integration and the fast Fourier transform were applied within a 6° spherical cap centered at each computation point. The effect of the gravity data outside the spherical cap was computed using the spheroidal Molodenskij approach. Comparisons of these geoid solutions with the synthetic geoid heights over western Canada indicate that the high-frequency geoid can be computed with an accuracy of $\sim$ 1cm using the modified Stokes technique, with discrete numerical integration giving a slightly, though not significantly, better result than the fast Fourier transform.

Keywords : geoid determination · Stokes's integration · fast Fourier transform

1 Introduction

The gravimetric determination of the geoid relies upon the solution of the spherical geodetic boundary-value problem, and requires the evaluation of Stokes's surface convolutive integral. In practice, the gravimetric geoid is computed using a combination of terrestrial and satellite-derived gravity data. The approach taken in this contribution is to spectrally decompose the geoid height into the reference spheroid (low-frequency geoid), which is computed from a satellite-derived spherical harmonic global model, and the high-frequency geoid, which is computed from terrestrial gravity data.

The high-frequency component of the geoid in this data combination requires the numerical evaluation of adapted Stokes's (1849) formula. Generally, its solution can be obtained by:

1. using discrete numerical integration (i.e. quadrature-based summation); or

2. converting Stokes's convolutive integral from the space domain into a product of the spectra of Stokes's function (or a modification thereof) with that of gravity data in the frequency domain, and back again.

The latter method is usually referred to as the fast Fourier transform (FFT) technique (eg. Schwarz, Sideris and Forsberg, 1990). Since only the one-dimensional fast Fourier transform or 1D-FFT (Haagmans *et al.*, 1993) allows for evaluation of Stokes's formula without any planar approximation or simplification of the kernel, it is considered here to be the only realistic alternative to discrete numerical integration.

The major purpose of this contribution is to examine the numerical accuracy of both approaches based on the use of the Molodenskij-modified spheroidal Stokes's formula (Vaníček and Kleusberg 1987; see also Vaníček and Sjöberg, 1991; Martinec and Vaníček, 1996). The integration domain for the modified Stokes integral is divided into a spherical cap of radius 6° about each computation point, and the remainder of the sphere. The contribution of the gravity data within the spherical cap is computed using both the discrete numerical integration and the 1D-FFT technique. The contribution of the distant gravity data in the region outside this spherical cap is computed by employing the spheroidal Molodenskij-type approach (*ibid.*).

A synthetic gravity field based on spherical harmonics will be used to assess and compare the accuracy of the gravimetrically computed geoid models (cf. Tziavos 1996). The coefficients to degree

360 of the EGM96 spherical harmonic model of the Earth's gravity field (Lemoine *et al.*, 1998) were supplemented with synthetically generated high-frequency coefficients out to degree 2160. The EGM96 and synthetically generated coefficients were used to create self-consistent gravity anomalies and geoid heights on a regular $5'$ geographical grid over a test area in the Canadian Rocky Mountains. These synthetic gravity data grids were used to compute the high-frequency geoid via the discrete numerical integration and the 1D-FFT technique. The difference between the computed and synthetic geoid heights was used to assess the relative numerical accuracy of these methods and the high-frequency geoid determination based on the Molodenskij-modified spheroidal Stokes formula.

2 Spectral decomposition of the geoid

Terrestrial gravity data provide detailed local information about the medium and high-frequency components of the Earth's gravity field. However, due to incomplete gravity data coverage (and availability) and long-wavelength biases in the observed gravity data (eg. due to the drift of gravimeters), the long-wavelength components of the Earth's gravity field are less accurately determined by terrestrial gravity observations. At present, the only reliable global information about the Earth's gravity field is based on artificial-satellite dynamics (i.e. the analysis of satellites' orbital perturbations). However, due to the attenuation of the strength of the Earth's gravity field with increasing distance from the geocentre, only the low-frequency component of the Earth's gravity field can reliably be detected in this way.

In the sequel, a spectral form of the Earth's gravity field is used, with a distinction made between the low and high-frequency components. The threshold value of $\ell = 20$ is used throughout the text. This reflects our belief that the frequencies up to this degree can correctly be derived from satellite dynamics, and do not have to be further improved by terrestrial gravity data in a combined global geopotential model. However, this choice of $\ell = 20$ is somewhat subjective and no proof is given to justify this value, apart from the arguments in the preceding paragraph.

Based on the above frequency decomposition of gravity data, the geoid height can similarly be decomposed into the low-frequency *reference spheroid* N_ℓ, and the so-called *residual geoid* N^ℓ, which is the high-frequency component of the geoid. The reference spheroid is computed from the low-frequency gravity disturbing potential T_ℓ at the geoid level r_g (estimated from the low-degree spherical harmonic coefficients of a global geopotential model) as follows (Bruns, 1878; Heiskanen and Moritz, 1967)

$$N_\ell(\Omega) \; = \; \frac{T_\ell(r_g,\Omega)}{\gamma} \; = \; \frac{GM}{r_g \; \gamma} \sum_{n=2}^{\ell} \left(\frac{R}{r_g} \right)^n T_n(\Omega) \; \doteq \; R \sum_{n=2}^{\ell} T_n(\Omega) \; , \tag{1}$$

where γ is the normal gravity acceleration on the surface of the geocentric reference ellipsoid, GM is the product of the Newtonian gravitational constant G and the mass of the Earth M, T_n are the zonal coefficients of the disturbing gravity potential T (derived from fully-normalized, unitless spherical harmonic coefficients obtained from a global geopotential model), and R is the radius of the geocentric reference sphere upon which the spherical harmonic expansion of the coefficients T_n reduces to the Laplace harmonics. Throughout the sequel, spherical coordinates $(r,\varphi,\lambda) = (r,\Omega)$, represented by the geocentric latitude φ, the geocentric longitude λ, and the length of the geocentric radius vector r, are used to describe a location of points of interest. The expression on the right-hand side of Eq. (1) is derived for the spherical approximation of the geoid by the reference sphere, i.e. $r_g \doteq R$, and for $GM/R^2 \doteq \gamma$.

3 Review of the spheroidal Molodenskij approach

This section provides a review of the spheroidal Molodenskij approach to high-frequency geoid computation as developed and used at the University of New Brunswick. The theoretical bases for these approaches are given elsewhere in the geodetic literature (eg. Vaníček and Kleusberg, 1987; Vaníček and Sjöberg, 1991; Martinec and Vaníček, 1996). This section simply presents them in a single, coherent framework, which describes how to practically implement them for high-frequency geoid computation.

The geodetic boundary-value problem is used for the solution of the high-frequency gravimetric geoid that is residual to the reference spheroid. It is assumed that there are no external topographical and atmospheric masses above the geoid (i.e. the regularization of the geoid is not described or considered in this contribution). Accordingly, the high-frequency disturbing gravity potential T^ℓ is a harmonic function everywhere outside the geoid and its behaviour is controlled by the *Laplace differential equation* (eg. Heiskanen and Moritz, 1967)

$$\forall \; r \; > \; r_g \; : \; \nabla^2 \, T^\ell(r,\Omega) \; = \; 0 \; . \tag{2}$$

A boundary condition to the homogeneous elliptical equation (Eq. 2) for the solution of the unknown function T^ℓ at the geoid level r_g is the *fundamental gravimetric equation*, which reads in a spherical approximation as (Martinec and Vaníček, 1996)

$$r \ = \ r_g : \ - \Delta g^{\ell}(\Omega) \ \doteq \ \frac{\partial}{\partial r} \, T^{\ell}(r,\Omega) \Big|_{r} + \ \frac{2}{R} \, T^{\ell}(r,\Omega) \ , \tag{3}$$

where the high-frequency gravity anomaly at the geoid is defined as (eg. Heiskanen and Moritz, 1967)

$$\Delta g^{\ell}(\Omega) = \ \Delta g(\Omega) \ - \ \frac{GM}{r_g^2} \sum_{n=2}^{\ell} \left(\frac{R}{r_g} \right)^n (n-1) \, T_n(\Omega) \ \doteq \ \Delta g(\Omega) \ - \ \gamma \sum_{n=2}^{\ell} (n-1) \, T_n(\Omega) \ . \tag{4}$$

The solution of Eqs. (2) and (3) for the unknown function T^{ℓ} exists and is unique when T^{ℓ} is regular at infinity and when Δg^{ℓ} do not contain any first-degree harmonics (eg. Heiskanen and Moritz, 1967), which can be satisfied by the proper selection of the geocentric reference ellipsoid. The solution is given by the *spheroidal Stokes integral* (Martinec and Vaníček, 1996; also see Vaníček and Sjöberg, 1991; Vaníček and Kleusberg, 1987), which is analogous to the spherical Stokes (1849) integral

$$T^{\ell}(r_g,\Omega) \ = \ \frac{R}{4\pi} \iint_{\Omega_{\oplus}} \Delta g^{\ell}(\Omega^{'}) \, \mathcal{S}^{\ell}(\psi) \, d\Omega^{'} \ , \tag{5}$$

where Ω defines the geocentric position of the computation point on the geoid, $\Omega^{'}$ defines the geocentric position of the integration point, $\Omega_{\oplus}$ denotes the surface of a sphere of unit radius, and ψ is the spherical distance between the integration point and computation point.

The function $\mathcal{S}^{\ell}$ for the computation of the high-frequency geoid in Eq. (5) is the *spheroidal Stokes kernel* (Vaníček and Kleusberg, 1987), and can conveniently be computed according to

$$\mathcal{S}^{\ell}(\psi) \ = \ \mathcal{S}(\psi) \ - \sum_{n=2}^{\ell} \frac{2n+1}{n-1} \ P_n(\cos\psi) \ , \tag{6}$$

where $\mathcal{S}$ is the spherical Stokes kernel (Stokes, 1849). The high-frequency geoid N^{ℓ} can be computed from the high-frequency disturbing potential T^{ℓ} using Bruns's theorem (Bruns, 1878)

$$N^{\ell}(\Omega) \ = \ \frac{T^{\ell}(r_g,\Omega)}{\gamma} \ , \tag{7}$$

which is then added to the reference spheroid N_{ℓ} in Eq. (1) to yield the total geoidal height N.

Due to the incomplete coverage or availability of terrestrial gravity data, the spheroidal Stokes integral (5) cannot be evaluated over the full spatial angle $\Omega_{\oplus}$. Therefore, a limited integration domain $\Omega_{\odot}$, represented by a spherical cap of radius ψ_o, must be used instead (Figure 1). This term is called the *contribution of the near zone* $N^{\ell}_{\odot}$ which can be computed from the terrestrial gravity data using the modified spheroidal Stokes integral

$$N^{\ell}_{\odot}(\Omega) \ = \ \frac{R}{4\pi\gamma} \iint_{\Omega_{\odot}} \Delta g^{\ell}(\Omega^{'}) \, \mathcal{S}^{\ell}(\psi,\psi_o) \, d\Omega^{'} \ . \tag{8}$$

The solution of the convolutive integral in Eq. (8) can be computed in either the spatial or the spectral form. While the former method uses discrete numerical integration of high-frequency gravity data, the latter method converts the integral from the space domain into the product of spectra of the Stokes function with that of the high-frequency gravity data in the frequency domain, and back again, see Section (3.2). The accuracy of these two approaches will be investigated later in this paper. Problems arising from the singularity of Stokes's integral for $\psi = 0$ are addressed in Section (3.1).

The influence of the gravity information from the remainder of the globe $(\Omega_\oplus - \Omega_\odot)$ is accounted for using a term called the *contribution of the far zone* $N^\ell_{\oplus-\odot}$ (the truncation bias)

$$N^\ell_{\oplus-\odot}(\Omega) \; = \; \frac{R}{4\pi\gamma} \iint_{\Omega_{\oplus-\odot}} \Delta g^\ell(\Omega') \; \mathcal{S}^\ell(\psi,\psi_o) \; d\Omega' \; . \tag{9}$$

The contribution of the far zone can be evaluated from the global geopotential model, see Section (3.3). However, due to significant differences between available models (eg. Najafi, 1996; also see Lambeck and Coleman, 1983), one should try to keep the far zone contribution as small as possible. This can be achieved by modifying the spheroidal Stokes function. It is acknowledged that there are many different modifications to Stokes's formula. However, these will not be considered here since the modification introduced by Vaníček and Kleusberg (1987) is used for practical geoid computations in Canada.

The *modified spheroidal Stokes function* can be written as (*ibid.*)

$$\forall \; t_n \in \mathcal{R}^n : \; \mathcal{S}^\ell(\psi,\psi_o) \; = \; \mathcal{S}(\psi) \; - \sum_{n=2}^{\ell} \frac{2n+1}{n-1} \; P_n(\cos\psi) \; - \sum_{n=0}^{\ell} \frac{2n+1}{2} \; t_n(\psi_o) \; P_n(\cos\psi) \; , \tag{10}$$

where t_n are the *modification coefficients*. It can be shown that in order to minimize the effect of gravity from the remote zones, see Eq. (9), the following integral must be minimized (*ibid.*)

$$\forall \; t_n \in \mathcal{R}^n \; : \; \min_{t_n} \; \left\{ \int_{\psi=\psi_o}^{\pi} \left[\; S^\ell(\psi,\psi_o) \; \right]^2 d\Omega' \right\} . \tag{11}$$

The solution leads to the system of linear equations for the unknown modification coefficients t_n

$$m \; \leq \; n \; : \; \sum_{n=0}^{\ell} \frac{2n+1}{2} \; R_{n,m}(\psi_o) \; t_n(\psi_o) \; = \; Q^\ell_m(\psi_o) \; , \tag{12}$$

where the coefficients $R_{n,m}(\psi_o)$, introduced by Paul (1973), are

$$m \; \leq \; n \; : \; R_{n,m}(\psi_o) \; = \; \int_{\psi=\psi_o}^{\pi} P_n(\cos\psi) \; P_m(\cos\psi) \; \sin\psi \; d\psi \; . \tag{13}$$

The *spheroidal truncation coefficients* in Eq. (12) can then be computed as (cf. Molodenskij *et al.*, 1960; see also Heiskanen and Moritz, 1967; Martinec, 1993)

$$Q^\ell_m(\psi_o) \; = \; \int_{\psi=\psi_o}^{\pi} \mathcal{S}^\ell(\psi) \; P_m(\cos\psi) \; \sin\psi \; d\psi \; = \; Q_m(\psi_o) \; - \sum_{n=2}^{\ell} \frac{2n+1}{n-1} \; R_{n,m}(\psi_o) \; . \tag{14}$$

3.1 Contribution of the computation point

The spherical, spheroidal, and modified spheroidal Stokes integrals are weakly singular for the spherical distance $\psi = 0$. Therefore, in seeking the solution in the spatial form, its appropriate treatment must first be chosen. The classical method consists of adding and subtracting the gravity anomaly at the computation point as follows

$$N^{\ell}_{\odot}(\Omega) = \frac{R}{4\pi\gamma} \iint_{\Omega_\odot} \Big\{ \Delta g^{\ell}(\Omega)\ \mathcal{S}^{\ell}(\psi,\psi_o) + \Big[\Delta g^{\ell}(\Omega^{'}) - \Delta g^{\ell}(\Omega) \Big]\ \mathcal{S}^{\ell}(\psi,\psi_o) \Big\}\ d\Omega^{'}\ , \tag{15}$$

which can be split into the *contribution of the computation point*

$$N^{\ell}_{\diamondsuit}(\Omega)\ =\ \frac{R\ \Delta g^{\ell}(\Omega)}{2\gamma} \int_{\psi=0}^{\psi_o} \mathcal{S}^{\ell}(\psi,\psi_o)\ \sin\psi\ d\psi\ , \tag{16}$$

and the contribution of the rest of the cap

$$N^{\ell}_{\odot-\diamondsuit}(\Omega)\ =\ \frac{R}{4\pi\gamma} \iint_{\Omega_\odot} \Big[\Delta g^{\ell}(\Omega^{'}) - \Delta g^{\ell}(\Omega) \Big]\ \mathcal{S}^{\ell}(\psi,\psi_o)\ d\Omega^{'}\ . \tag{17}$$

The singularity is removed from Eq. (17) because the value of the integrand equals zero for $\psi = 0$. The integral of the Stokes function in Eq. (16) can be evaluated analytically as follows

$$\int_{\psi=0}^{\psi_o} \mathcal{S}^{\ell}(\psi,\psi_o)\ \sin\psi\ d\psi\ =\ \int_{\psi=0}^{\psi_o} \mathcal{S}(\psi)\ \sin\psi\ d\psi\ -\ \sum_{n=2}^{\ell} \frac{2n+1}{n-1} \int_{\psi=0}^{\psi_o} P_n(\cos\psi)\ \sin\psi\ d\psi$$

$$-\ \sum_{n=0}^{\ell} \frac{2n+1}{2}\ t_n(\psi_o) \int_{\psi=0}^{\psi_o} P_n(\cos\psi)\ \sin\psi\ d\psi\ . \tag{18}$$

The integral of the Stokes function can be written (Heiskanen and Moritz, 1967)

$$\int_{\psi=0}^{\psi_o} \mathcal{S}(\psi)\ \sin\psi\ d\psi\ =\ -\int_{\psi=\psi_o}^{\pi} \mathcal{S}(\psi)\ \sin\psi\ d\psi\ =\ -\ Q_0(\psi_o)\ , \tag{19}$$

and the integral of Legendre's polynomial reads (Paul, 1973)

$$\int_{\psi=0}^{\psi_o} P_n(\cos\psi)\ \sin\psi\ d\psi\ =\ -\int_{\psi=\psi_o}^{\pi} P_n(\cos\psi)\ \sin\psi\ d\psi\ =\ -\ R_{n,0}(\psi_o)\ . \tag{20}$$

The integral on the left-hand side of Eq. (18) is then of the form (Martinec, 1993)

$$\int_{\psi=0}^{\psi_o} \mathcal{S}^{\ell}(\psi,\psi_o)\ \sin\psi\ d\psi\ =\ -\ Q^{\ell}_0(\psi_o)\ +\ \sum_{n=0}^{\ell} \frac{2n+1}{2}\ R_{n,0}(\psi_o)\ t_n(\psi_o)\ =\ -\ \tilde{Q}^{\ell}_0(\psi_o)\ , \tag{21}$$

and the contribution of the computation point to the geoid is given by (*ibid.*)

$$N^{\ell}_{\diamondsuit}(\Omega)\ =\ -\ \frac{R\ \Delta g^{\ell}(\Omega)}{2\gamma}\ \tilde{Q}^{\ell}_0(\psi_o)\ . \tag{22}$$

3.2 Contribution of the near zone from discrete numerical integration and the 1D-FFT

The contribution of the remainder of the spherical cap to the high-frequency geoid can be evaluated by discrete summation over mean values of gravity anomalies on a regular geographical grid within the cap, cf. Figure (1). After accounting for the contribution of the computation point, the *contribution of the near zone* to the high-frequency geoid is

$$N^{\ell}_{\odot-\diamond}(\Omega) \;=\; \frac{R}{4\pi\gamma} \iint_{\Omega_\odot} \left[\, \Delta g^{\ell}(\Omega') \;-\; \Delta g^{\ell}(\Omega) \,\right] \, \mathcal{S}^{\ell}(\psi,\psi_o)\; d\Omega' \; . \tag{23}$$

Using the mean value theorem, the integration in Eq. (23) can be replaced by the summation over $(j-1)$ cells within the spherical cap of the product of discrete mean values of high-frequency gravity anomalies (Vaníček and Krakiwsky, 1986)

$$\overline{\Delta g^{\ell}}(\Omega_k) \;=\; \frac{1}{\Delta\Omega_k} \iint_{\Omega_k} \Delta g^{\ell}(\Omega')\; d\Omega' \; , \tag{24}$$

with corresponding point values of modified spheroidal Stokes's function. Equation (23) then reads

$$N^{\ell}_{\odot-\diamond}(\Omega) \;=\; \frac{R}{4\pi\gamma} \sum_{k}^{j-1} \left\{ \iint_{\Omega_k} \left[\, \Delta g^{\ell}(\Omega') \;-\; \Delta g^{\ell}(\Omega) \,\right] \, \mathcal{S}^{\ell}(\psi,\psi_o)\; d\Omega'_k \right\}$$

$$\doteq\; \frac{R}{4\pi\gamma} \sum_{k}^{j-1} \left[\, \overline{\Delta g^{\ell}}(\Omega_k) \;-\; \overline{\Delta g^{\ell}}(\Omega) \,\right] \, \mathcal{S}^{\ell}(\psi_k,\psi_o)\; \Delta\Omega_k \; , \tag{25}$$

where the value of $\mathcal{S}^{\ell}(\psi_k,\psi_o)$ is evaluated for the spherical distance ψ_k between the integration point and the centre of the k-th cell, and $\Delta\Omega_k$ is the surface area of the k-th cell, see Figure (1).

An alternative approach to the discrete numerical integration of Eq. (25) is represented by the solution in the spectral domain, which is based on the convolution theorem. Here, the discretised integration of mean gravity anomalies (Eq. 25) may be reformulated, without further approximation, as the sum (in latitude) of a series of one-dimensional discretised convolution integrals (in longitude). The discrete one-dimensional fast Fourier transform (1D-FFT), when applied to mean the high-frequency gravity anomalies within the spherical cap $(\Omega_\odot)$, can be written for a fixed latitude φ_k of the computation point as (Haagmans *et al.*, 1993)

$$N^{\ell}_{FFT}(\Omega) \;\doteq\; \frac{R\;\Delta\varphi\;\Delta\lambda}{4\pi\gamma} \;\; \mathcal{F}^{-1} \left\{ \sum_{k=1}^{i-1} \mathcal{F}\left[\, \overline{\Delta g}(\Omega_k)\;\cos\varphi_k \,\right] \cdot \mathcal{F}\left[\, \mathcal{S}^{\ell}(\psi_k,\psi_o) \,\right] \right\} \; , \tag{26}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the one-dimensional discrete Fourier transform operator and its inverse, respectively, i is the number of gravity data along the meridian φ_k, and $\Delta\varphi$ and $\Delta\lambda$ are the sampling intervals of a regular data grid. When applied correctly to the same gravity data, both the spatial form (Eq. 25) and spectral form (Eq. 26) should produce the identical high-frequency geoid. Note that the spectral form also faces the problem arising from the singularity of the Stokes function and the contribution of the computation point must be evaluated separately (*ibid.*).

Concerning the relative numerical efficiency of discrete numerical integration and the 1D-FFT, it has been argued for a long time that the discrete Fourier transform is computationally superior to discrete numerical integration. However, due to recent developments in the discrete numerical integration (Huang *et al.*, 2000), both methods use a comparable amount of computational time.

3.3 Contribution of the distant zone from spherical harmonic coefficients

The *contribution of the distant zone* to the high-frequency geoid has been derived by several authors. The approach taken here, originally derived by Molodenskij *et al.* (1960), uses the so-called Molodenskij coefficients (weights) to account for the influence of the distant zones omitted from the truncated integration in Eq. (25) or Eq. (26). The Molodenskij coefficients for the modified spheroidal Stokes function (Eq. 10) are (eg. Martinec, 1993)

$$\forall\ m\ >\ \ell\ :\ \tilde{Q}^{\ell}_{m}(\psi_o)\ =\ \int_{\psi=\psi_o}^{\pi} \mathcal{S}^{\ell}(\psi,\psi_o)\ P_m(\cos\psi)\ \sin\psi\ d\psi$$

$$=\ Q_m(\psi_o)\ -\ \sum_{n=2}^{\ell} \frac{2n+1}{n-1}\ R_{n,m}(\psi_o)\ -\ \sum_{n=2}^{\ell} \frac{2n+1}{2}\ t_n(\psi_o)\ R_{n,m}(\psi_o)\ , \qquad (27)$$

where the spherical truncation coefficients $Q_m(\psi_o)$ are given by Eq. (14). The contribution of the distant zone to the high-frequency geoid height can then be evaluated using a conversion of the spatial form in Eq. (9) to a spectral form based on spherical harmonics (*ibid.*)

$$N^{\ell}_{\oplus-\odot}(\Omega)\ =\ \frac{GM}{2R\gamma} \sum_{n=\ell+1}^{max} (n-1) \left(\frac{R}{r_g}\right)^{n+2} \tilde{Q}^{\ell}_{n}(\psi_o)\ T_n(\Omega)$$

$$\doteq\ \frac{R}{2} \sum_{n=\ell+1}^{max} (n-1)\ \tilde{Q}^{\ell}_{n}(\psi_o)\ T_n(\Omega)\ , \qquad (28)$$

with the spherical harmonic coefficients of the disturbing gravity potential T_n taken from a combined global geopotential model. The maximum degree $max = 120$ can be used for numerical evaluations, thus rendering the effect of higher-degree terms smaller than one millimetre (Martinec, 1993).

To conclude this Section, the final expression for the determination of the high-frequency geoid can be written for the discrete numerical integration as

$$N^{\ell}(\Omega) \; = \; N^{\ell}_{\diamond}(\Omega) \; + \; N^{\ell}_{\odot-\diamond}(\Omega) \; + \; N^{\ell}_{\oplus-\odot}(\Omega) \; = \; - \, \frac{R}{2\gamma} \, \Delta g^{\ell}(\Omega) \, \tilde{Q}^{\ell}_{0}(\psi_o)$$

$$+ \, \frac{R}{4\pi\gamma} \sum_{k}^{j-1} \Big[\, \overline{\Delta g^{\ell}}(\Omega_k) - \overline{\Delta g^{\ell}}(\Omega) \, \Big] \; \mathcal{S}^{\ell}(\psi_k, \psi_o) \; \Delta\Omega_k \; + \; \frac{R}{2} \sum_{n=\ell+1}^{120} (n-1) \; \tilde{Q}^{\ell}_{n}(\psi_o) \; T_n(\Omega) \; . \tag{29}$$

The final expression for the determination of the high-frequency geoid by the 1D-FFT is then

$$N^{\ell}(\Omega) \; = \; N^{\ell}_{FFT}(\Omega) \; + \; N^{\ell}_{\oplus-\odot}(\Omega)$$

$$= \; \frac{R \; \Delta\varphi \; \Delta\lambda}{4\pi\gamma} \;\; \mathcal{F}^{-1} \left\{ \sum_{k=1}^{i-1} \; \mathcal{F} \left[\; \overline{\Delta g}(\Omega_k) \;\; \cos\varphi_k \; \right] \cdot \mathcal{F} \left[\; \mathcal{S}^{\ell}(\psi_k, \psi_o) \; \right] \right\}$$

$$+ \, \frac{R}{2} \sum_{n=\ell+1}^{120} (n-1) \;\; \tilde{Q}^{\ell}_{n}(\psi_o) \; T_n(\Omega) \; . \tag{30}$$

4 Synthetic gravity field based on spherical harmonics

The test chosen to evaluate and compare the accuracy of the high-frequency geoid computation from discrete quadrature-based numerical integration (used in Eq. 29) and the discrete 1D-FFT (used in Eq. 30) uses a self-consistent set of synthetically generated geoid heights and gravity anomalies. This approach is analogous to that taken by Tziavos (1996), who uses a degree 360 global geopotential model. However, the current study extends the upper bound of the frequencies over which the accuracy of the geoid computation can be assessed.

Over the Canadian territory, mean gravity anomalies on a regular $5'$ geographical grid (prepared by the Geodetic Survey Division of Natural Resources Canada) are usually used for the determination of the regional gravimetric geoid (eg. Sideris and She, 1995; Vaníček *et al.*, 1990). The Nyquist frequency of this gravity grid corresponds to a maximum spherical harmonic expansion of degree and order 2160. Since freely available global geopotential models only contain coefficients up to degree and order 360 (notwithstanding the GPM98 models (Wenzel, 1998), which extend to degree 1800), it is necessary to artificially generate a set of synthetic coefficients complete to degree and order 2160.

The synthetic spherical harmonic expansion of the geopotential consists of two distinct parts. First, all spherical harmonic coefficients for degrees $\ell + 1 \leq n \leq 360$ were taken from the EGM96 global geopotential model (Lemoine *et al.*, 1998) with respect to the GRS80 reference ellipsoid. Second, the remaining synthetic coefficients for degrees $361 \leq n \leq 2160$ were generated from an artificially constructed sequence of pairs of real numbers. These were taken from the EGM96 coefficient pairs of orders $0 \leq m \leq 360$ at degree $n = 360$. For example, to generate the synthetic coefficients for degree 540, the EGM96 coefficient pairs were used for $0 \leq m \leq 360$ and used again for $361 \leq m \leq 540$.

However, using only this approach yielded an unrealistic degree variance (cf. Heiskanen and Moritz, 1967, p. 259) of the synthetic field, which did not agree with current expectations of the actual geopotential spectrum of the Earth (cf. Tscherning and Rapp, 1974). Therefore, each pair of high-degree synthetic coefficients was multiplied by a simple scale factor to yield a more realistic degree variance (described later). The degree-360 sequence of EGM96 coefficients was thus scaled and used in a repeated cycle, until all of required synthetic coefficients were produced. The aim of this approach was to generate a synthetic field that was reasonably realistic so that inferences made about the accuracy of the high-frequency geoid computation using the synthetic gravity field will apply to the real Earth.

The scale factor used is given by $(a^*/a)^{n-360}$, where a is the radius of the geocentric sphere upon which the spherical harmonic expansion of the EGM96 coefficients reduces to Laplace harmonics, and a^* is the radius of an arbitrary reference sphere that is of a similar magnitude to a, but slightly smaller. This combination of scale factor and synthetic coefficients provided a convenient means by which the degree variance of the synthetic field would smoothly extend from EGM96 into the higher degrees, while also providing a decaying degree variance similar to current expectations. Formulation of the scale factor using the a and a^* terms was effected for reasons of computational efficiency and convenience only. Therefore, no physical interpretation should be made regarding this scale factor or the synthetic gravity field.

The synthetic coefficients were used to produce two data-sets with different degree variances in the high-degree terms, which was achieved by changing the a^* term. The first synthetic field, called Data A, was generated using a value of $a^* = 6.34 \times 10^6$ m; the second synthetic field, called Data B, used $a^* = 6.35 \times 10^6$ m. The degree variances of these two data-sets were compared against the GPM98 tailored global geopotential models (Wenzel, 1998) in the region $360 \leq n \leq 1800$ to help ensure that the synthetic degree variances remained reasonably realistic. The value of the a^* term was selected in such a way that the high-degree frequencies of Data A are less powerful than the corresponding frequencies of GPM98B, and *vice versa* for Data B. The degree variances of Data A and GPM98B are shown in Figure (2); the degree variances of Data B and GPM98B are shown in Figure (3). Data A has less power in the very-high frequencies than Data B, which is apparent from the larger amplitude of higher frequency geoid undulations in Figures (4) and (5).

5 Methodology, results and discussion

A conceptually simple procedure, which follows that of Tziavos (1996), was employed to assess and compare the discrete numerical integration and the 1D-FFT in the modified spheroidal Stokes approach to high-frequency geoid determination. Two synthetic gravity fields were generated using the procedures described above. The synthetic gravity anomalies in the region $21 \leq n \leq 2160$ were computed directly for each set and are designated 'Data A' and 'Data B'. The corresponding synthetic geoid heights ($21 \leq n \leq 2160$) were likewise produced from each set and are designated 'Spheroid A' and 'Spheroid B'. The result is two self-consistent (to a spherical approximation) grids of geoid heights and gravity anomalies, which can be used to compare different methods for gravimetric geoid computation.

All gravity anomaly and geoid values were produced on identical, regular $5'$ geographical grids for a test region bounded by $49° \leq \varphi \leq 54°$ (N) and $236° \leq \lambda \leq 246°$ (E). This grid spacing corresponds with the gravity anomaly grids produced by the Canadian Geodetic Survey Division. Although this area covers the Canadian Rocky Mountains, it should be remembered that the synthetic geopotential coefficients for degrees $361 \leq n \leq 2160$ were not generated empirically from observational data. Therefore, they do not necessarily reflect the actual gravity field generated by the topography and geology of this region. Accordingly, no physical interpretation should be made from them.

The methodology is best understood by first considering the use of a single integration technique in conjunction with a single set of synthetic geoid heights and gravity anomalies. The high-frequency gravity anomalies ($21 \leq n \leq 2160$) were produced by subtracting the low-degree reference spheroid of degree $\ell = 20$ from synthetic gravity anomalies (Eq. 7). This approach was taken so as to replicate as closely as possible the approach that is used in practical geoid computations with observational gravity data. The contribution of the near zones to the high-frequency geoid ($21 \leq n \leq 2160$) was then obtained from the high-frequency synthetic gravity grid using both the discrete numerical integration (in Eq. 25) and the 1D-FFT (in Eq. 26). A cap radius of $\psi_o = 6°$ was used in each technique. The contribution of the distant zones to the high-frequency geoid was, in both cases, evaluated using spherical harmonics and the coefficients from the EGM96 global geopotential model to degree $max = 120$ (Eq. 28). The computed contributions from the near and distant zones were combined to give the high-frequency residual geoid, which was added to the ($\ell = 20$) reference spheroid to yield a total gravimetric geoid solution ($2 \leq n \leq 2160$) for the test region. This process was repeated for Data A and Data B, which resulted in four gravimetric geoid solutions computed from the synthetic gravity anomalies.

Figure (6) shows a schematic of the methodology used. Theoretically, the difference between the computed and synthetic geoids should be equal to zero everywhere across each grid for both Data A and Data B. Therefore, any computed differences are used as indicators of the *accuracy* of each numerical technique (i.e. numerical integration and 1D-FFT) when used in conjunction with the high-frequency geoid determination based on the Molodenskij-modified spheroidal Stokes formula. The term *accuracy* is used here since the exact geoid heights expected from the gravity anomalies is known (to a spherical approximation) from the synthetic field.

Tables (1) and (2) show a statistical summary of the differences between each synthetic geoid and the computed geoids, as obtained using the numerical integration and 1D-FFT, respectively. These are given for Data A (the synthetic field with less power than GPM98B in the medium and high-frequencies) and Data B (the synthetic field with more power than GPM98B in the medium and high-frequencies). The standard deviations in Tables (1) and (2) show that both the discrete numerical integration and the 1D-FFT deliver a precision of approximately one-centimetre for the test-region. Likewise, the root mean square in Tables (1) and (2) show that both methods deliver an accuracy of approximately one-centimetre for the test-region. The range (i.e. maximum minus minimum) and standard deviation of the discrepancies obtained through the use of the numerical integration procedure are slightly smaller than those obtained from the 1D-FFT, although the improvement is not statistically significant and lies within the numerical accuracy of the computer algorithms used. Thus, both methods provide comparable accuracy as one would expect from the theory. As stated earlier, the recent advances in discrete numerical integration make it as computationally fast as the 1D-FFT (Huang *et al.*, 2000). Therefore, there are no compelling arguments based on numerical accuracy nor efficiency that suggest that either technique is preferable for high-frequency geoid determination. Therefore, the choice of using discrete numerical integration or the 1D-FFT to compute the high-frequency geoid is merely one of computational convenience.

The discrepancies observed for both numerical approaches become slightly larger for Data B than for Data A. A visual confirmation of this point may be obtained by inspection of Figures (7) and (8), which plot the discrepancies between the synthetic geoid values those computed using numerical integration for Data A and Data B, respectively. This, and the results summarized in Tables (1) and (2), implies that the accuracy of each technique will decrease when applied to regions in which there is more power in the medium and high-frequency bands of the local gravity field. From this, it can be inferred than the accuracy of the computed geoid may be worse than one-centimetre in regions of the Earth for which this is true.

In addition to comparing the numerical accuracy of discrete numerical integration and the 1D-FFT, the results in Tables (1) and (2) show that the numerical implementation of the modified Stokes theory (Eqs. 29 and 30) can yield the high-frequency ($21 \leq n \leq 2160$) geoid accurate to approximately one centimetre in this test region. However, this has assumed that there are no errors in the input gravity data, which is not necessarily true when using observational gravity data.

6 Summary and conclusions

This contribution has reviewed the Molodenskij-modified spheroidal Stokes integral and given a strategy for its practical solution, which is based on a degree $\ell = 20$ satellite-derived reference spheroid, a contribution of gravity anomalies inside a spherical cap of 6° radius, and a contribution from the remote zones outside this cap from a degree 120 combined global geopotential model. This has been tested against a high-frequency synthetic gravity field to quantify the accuracy of the Molodenskij-modified spheroidal Stokes approach developed at the University of New Brunswick. The numerical evaluation of the near zone contribution to the high-frequency geoid as part of this approach has been computed using both discrete numerical integration and the 1D-FFT technique. Therefore, it has simultaneously given an accuracy evaluation of discrete numerical integration and the 1D-FFT technique.

From the comparison of the computed high-frequency geoids with the synthetic high-frequency geoid, the expected accuracy for the solution of this component of a gravimetric geoid is at the one-centimetre level. It is important to state that this test only really validates the accuracy of the high-frequency component of the computed gravimetric geoid. This is because any low-frequency errors in the reference spheroid ($n \leq 20$) are essentially invisible to the approach used here. Nevertheless, the tests do validate the numerical accuracy of the high-frequency geoid computations based on the Molodenskij-modified spheroidal Stokes theory. Also, the gravity and other data (such as topographical heights and densities) used in practical geoid computations do not currently allow for such a strict requirement in numerical accuracy. However, the theoretical and numerical principles are being formulated with respect to the goal of a one-centimetre geoid.

In addition to verifying the appropriateness of the modified Stokes theory, this investigation has also permitted a verification of the numerical accuracy of the near zone contribution to the high-frequency geoid, when computed from discrete numerical integration and the 1D-FFT technique. The computer algorithms for these two approaches delivered results of comparable accuracy, with the discrete numerical integration being slightly (though not significantly) more accurate. Therefore, the expected accuracy of the high-frequency geoid computed using the Molodenskij-modified spheroidal Stokes theory (assuming error-free gravity data) is at the one-centimetre level. However, it should be stressed that this value may not be achieved in practical geoid computations due to the accuracy of observational data and the approximations currently used for the regularization of the geoid.

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Statistic	Data A	Data B
Maximum	+ 0.026	+ 0.039
Minimum	− 0.017	− 0.030
Arithmetic Mean	+ 0.003	+ 0.003
Standard Deviation	± 0.008	± 0.010

Table 1: *Statistics of the differences between the high-frequency geoid from the synthetic field and discrete numerical integration (in metres).*

Statistic	Data A	Data B
Maximum	+ 0.033	+ 0.045
Minimum	− 0.026	− 0.036
Arithmetic Mean	+ 0.003	+ 0.003
Standard Deviation	± 0.009	± 0.011

Table 2: *Statistics of the differences between the high-frequency geoid from the synthetic field and the 1D-FFT (in metres).*

Figures :