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Geoid, topography, and the Bouguer plate or shell

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Abstract. Topography plays an important role in solving many geodetic and geophysical problems. In the evaluation of a topographical effect, a planar model, a spherical model or an even more sophisticated model can be used. In most applications, the planar model is considered appropriate: recall the evaluation of gravity reductions of the free-air, Poincaré–Prey or Bouguer kind. For some applications, such as the evaluation of topographical effects in gravimetric geoid computations, it is preferable or even necessary to use at least the spherical model of topography. In modelling the topographical effect, the bulk of the effect comes from the Bouguer plate, in the case of the planar model, or from the Bouguer shell, in the case of the spherical model. The difference between the effects of the Bouguer plate and the Bouguer shell is studied, while the effect of the rest of topography, the terrain, is discussed elsewhere. It is argued that the classical Bouguer plate gravity reduction should be considered as a mathematical construction with unclear physical meaning. It is shown that if the reduction is understood to be reducing observed gravity onto the geoid through the Bouguer plate/shell then both models give practically identical answers, as associated with Poincaré's and Prey's work. It is shown why only the spherical model should be used in the evaluation of topographical effects in the Stokes–Helmert solution of Stokes' boundary-value problem. The reason for this is that the Bouguer plate model does not allow for a physically acceptable condensation scheme for the topography.

Key words: Geoid – Bouguer Reduction of Gravity – Stokes–Helmert's Problem

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1 Introduction

Periodically, people discover that planar and spherical models of topography give very different results for Bouguer anomalies. Similarly, the results for the direct and indirect topographical effects in the Stokes–Helmert technique for geoid computations obtained by means of the planar and spherical models are found to be quite different. Some people claim that the planar model can safely be used for "local work" while the spherical model has to be used for global work. Others still maintain that all these questions have already been sorted out and that they do not require any more of our attention. So what is going on?

When looking into this problem (Vaníček and Novák 1999) we discovered an interesting story, which we will try to recount here. To do so, we focus only on the "infinite plate" and the "spherical shell" models, leaving out the terrain effects. The corresponding difference of the planar and spherical models of terrain presents another fascinating story which, in our opinion, requires a separate and rather more extensive paper to deal with it adequately. The main point of this "terrain story" is the discovery that, contrary to popular belief, the spherical model terrain effect has to be considered globally. This point has been already discussed by Novák et al. (1998) and by Novák and Vaníček (1999). A more formal and complete paper on the subject of spherical terrain model is under review (Novák et al. submitted). Once published, it will complement the present paper.

In the present paper, we will focus our attention first on the difference between the Bouguer plate and Bouguer shell effects on Bouguer anomalies. Then, we will tackle another, more or less independent issue, that of the difference between the two models in the gravimetric geoid determination by means of the Stokes–Helmert technique, where the use of the two models has profound consequences. In fact, the planar model, the Bouguer plate, cannot be used at all if we wish to use a physically meaningful condensation scheme.

2 The story of Bouguer plate reduction

In order to see the pattern, let us show the gravitational potential, the gravitational attraction (negative first vertical derivative of the potential), and the vertical gradient of gravitational attraction (negative second derivative of the potential) of the topographical (Bouguer) plate and the topographical (Bouguer) shell side by side. To keep things simple, let us assume a constant density ϱ (say, the usual $2670\,\mathrm{kg\,m^{-3}}$) and the same thickness, H, for both the infinite plate and the shell of the inner radius R. This is all shown in Fig. 1. The three quantities of interest are computed at two points: one on the top and one at the bottom of the plate/shell. In addition, the second derivative, which is discontinuous on the top (and also at the bottom) of the plate/shell, is at this point computed in both directions: from above and from below. The expressions for the plate are derived from Eqs. (3.5) and (3.7) in Heiskanen and Moritz (1967) by simply extending the finite plate to infinity. The expressions for the shell are derived directly from the equations for the potential (of a spherical shell) in Wichiencharoen [1982, Eqs. (19), (24) and (25)].

Now, examining Fig. 1, how different really are the results for the planar and spherical models? Starting from the bottom, with the vertical gradient of attraction, and neglecting the higher-order terms (of the order of H/R and smaller) in the spherical model, the results are identical. The attraction of the plate at its top is only one half of that of the shell (at its top and neglecting the higher-order terms), while the attraction at the bottom of the plate is exactly opposite to that at the top. The attraction at the downside of the shell is zero, as it should be (Kellogg 1929). Note that the change in the attraction when vertically traversing the plate or shell is the same, except for higher-order terms. The situation for the potential is naturally different: as the potential of the infinite plate is infinite, we cannot make any direct comparison between the two models. We can only observe that in the spherical model, the potentials at the upside and on the downside of the shell differ only by higher-order terms.

What does it all mean? We wish to address here only the most interesting question of what this means in the context of the (incomplete, i.e. without the terrain correction) Bouguer gravity anomaly. The incomplete or simple Bouguer anomaly is computed from the following formula:

$$\Delta g = g + A + A^{\mathrm{B}} - \gamma \tag{1}$$

where g is the observed gravity on the Earth's surface (at altitude H), A is the "free-air reduction" (to the geoid) due to the Earth's masses enclosed within the inner radius of the spherical shell (including the latitude and altitude terms), A^{B} is the "Bouguer reduction" (to the geoid) due to the mass of the Bouguer plate, and γ is the normal gravity at the reference ellipsoid [Heiskanen and Moritz 1967, Eq. (3.19)]. In this case, the so-called Bouguer reduction is given by

$$A^{\mathrm{B}} = 2\pi G \varrho H \tag{2}$$

For the standard value of mean topographical density of $2670\ \mathrm{kg\,m^{-3}}$, the numerical value of the Bouguer reduction is $0.1119 \times H$ mGal.

Inspecting again Fig. 1, we can see rather easily that A^{B} is *not* the difference between the gravity values at the top and the bottom of the infinite plate! It is *not* the difference between gravity values on the upside and

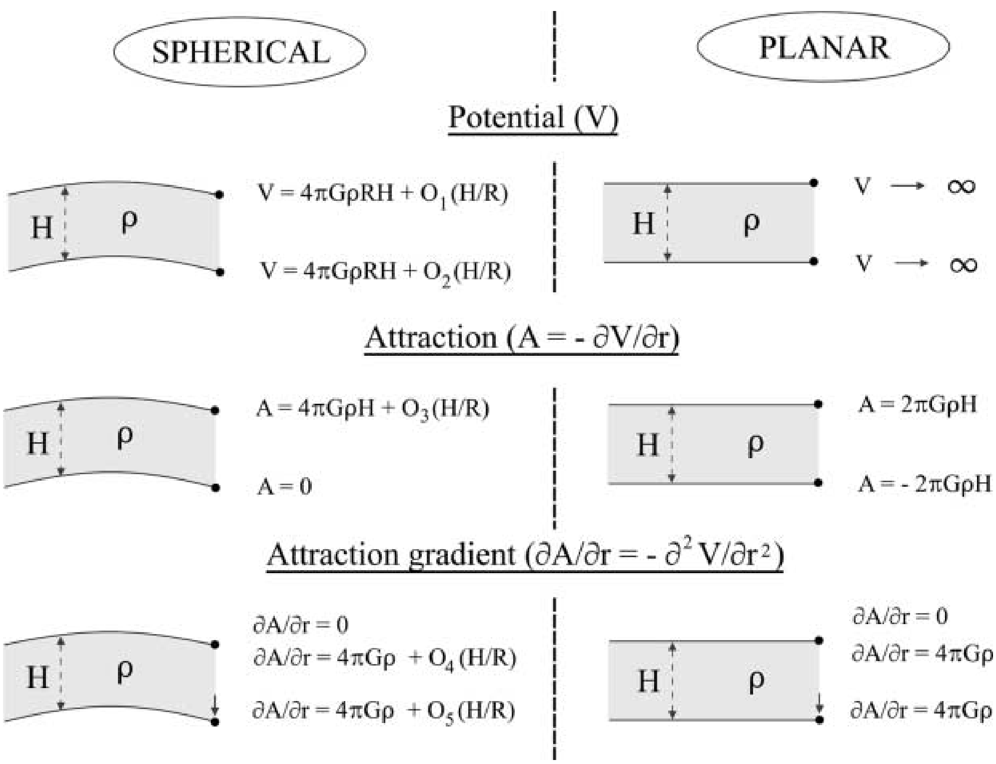

Fig. 1. The behaviour of gravity and gravity potential induced by Bouguer shell and Bouguer plate. The symbols $O_i(H/R)$ denote higher-order terms

downside of the spherical shell either! This difference, in the planar case, is equal to

$$A^{\mathrm{PP}} = 4\pi G\varrho H \tag{3}$$

known in geodesy as the "Poincaré–Prey gravity reduction" [Heiskanen and Moritz 1967, Eq. (3.64)]. In the spherical case the difference is the same except for the higher-order terms, which we will neglect for simplicity. (We note that we arrive at the stated conclusions, using either the attraction or the vertical gradient of attraction formulae, as we should.) For the standard topographical density, the numerical value of the Poincaré–Prey reduction is $0.2238 \times H$ mGal.

The discrepancy between the planar and spherical models was pointed out by Karl (1971). It should be mentioned that planar and spherical terrain models display the same discrepancy. Taking the observation point to be located on top of a thin tower of height H, its planar terrain correction will amount to $-2\pi G\varrho H$, while its spherical terrain correction will be $-4\pi G\varrho H$ (Novák et al., submitted, 2000).

Now, the standard explanation of the Bouguer reduction is that it represents the effect of the removal of the Bouguer plate, i.e. of a plate of infinite mass. This, in physical terms, is rather an unsatisfactory manipulation. On the other hand, inspecting again Fig. 1, we can see that the operation of the removal of the Bouguer shell (more satisfactory because of the finiteness of its mass) results again in the Poincaré–Prey rather than Bouguer gravity reduction. We note that in either case, the "removal of the plate or shell" results in the fact that the Bouguer anomaly refers to the point where the gravity g has been observed, i.e. to the surface of the Earth.

There is an alternative possibility. If we wish to view the Bouguer anomaly as being referred to the geoid rather than the Earth's surface, we have to understand the Bouguer reduction A^{B} as being the vertical gradient of gravity within the plate/shell between the geoid and the Earth's surface multiplied by the thickness of the plate/shell. We obtain, up to the higher-order terms

$$A^{\mathrm{B}} = -\frac{\partial^2 V}{\partial r^2} H \tag{4}$$

where the second derivative (negative attraction gradient) is evaluated at the top of the plate/shell. However, according to Fig. 1, the gradient has at this point two values, one in the limit taken from the outside and the other in the limit taken from the inside towards the top surface of the plate/shell. Thus, strictly speaking, the gradient is at this point undefined! The "Bouguer gradient" in Eq. (2) is the average of the outside and inside values, i.e. a mathematically and physically meaningless quantity, as already pointed out by Vaníček and Krakiwsky (1986). On the other hand, taking the appropriate value of the gradient within the mass (i.e. between the top and the bottom surfaces) valid for the inside of the plate/shell, both models give the same result, namely the Poincaré–Prey value discussed above.

The inevitable conclusion is that the apparently incorrect Bouguer gravity reduction is *not* coming from the use of the (physically less satisfactory) planar model and that we should therefore obtain the "correct" result by using the physically more satisfying spherical model. Rather, the Bouguer gravity reduction is the consequence of the physically unsatisfactory "removal of the Bouguer plate" of an infinite mass. Thus the Bouguer gravity anomaly, useful as it is in many geophysical applications, is really only an artificial construction. The difference between using the planar and the spherical model is of second order; it is the known spherical correction to the Bouguer plate reduction [Vaníček and Krakiwsky 1986; Eq. (21.43)]

$$\delta A_S^{\mathrm{B}} = -8\pi G\varrho \frac{H^2}{R} \tag{5}$$

For the standard topographical density of 2670 kg m^{-3}, the numerical value of the spherical correction is $0.4476 \times H^2/R$ mGal. An interesting interpretation may be offered for the standard Bouguer reduction A^{B}, if we wish to use the alternative interpretation of gravity anomaly: it numerically reduces the observed gravity from the surface of the Earth to the mid-point of the infinite Bouguer plate, or the mid-point of the Bouguer spherical shell. This is a result of a particular selection of the value of the vertical gradient of gravity rather than the selection of a particular (planar) model.

3 The direct and primary indirect topographical effects

Let us now turn our attention to the other issue which is only loosely connected to the problem of Bouguer reduction. Probably the most widely used technique in North America for solving the boundary-value problem of geodesy (leading to geoid determination from observed gravity anomalies) is the Stokes–Helmert technique (Vaníček and Martinec 1994). The essence of this technique is that topographical masses are replaced by a condensed mass layer on the geoid surface, resulting in the introduction of an abstract space, called Helmert space (Vaníček and Martinec 1994), in which the solution is sought. The main idea behind the technique is that the disturbing potential T^h sought in Helmert space is harmonic everywhere above the geoid. The Helmert disturbing potential T^h is related to the real disturbing potential T by the following equation:

$$T^h(r,\Omega) = T(r,\Omega) - V(r,\Omega) \tag{6}$$

where the residual topographical potential V is defined as

$$V(r,\Omega) = V^t(r,\Omega) - V^c(r,\Omega) \tag{7}$$

where V^t denotes the potential of topographical masses and V^c stands for the potential of the (condensed) mass layer. The symbols r and Ω stand for the geocentric distance and a spatial angle with components (θ, λ) of spherical co-latitude and longitude.

The transformation of observed gravity (at the surface of the Earth) in the real space to its counterpart (Helmert's gravity) in the abstract Helmert space is achieved by subtracting from it the "Direct Topographical Effect" (DTE) given by the following formula (Vaníček and Martinec 1994):

$$\mathrm{DTE}(\Omega) = -\frac{\partial V(r,\Omega)}{\partial r}\bigg|_{r=r_t} \tag{8}$$

where the partial derivative (in units of acceleration) is evaluated at the surface of the Earth, i.e. on the topography, for $r(\Omega) = r_t(\Omega)$. The transformation of the resulting geoidal height in Helmert space (Helmert's co-geoidal height) to the real geoidal height (geoidal height in the real space) is realized by adding to it the "Primary Indirect Topographical Effect" (PITE) given by the following formula (Vaníček and Martinec 1994)

$$\mathrm{PITE}(\Omega) = \frac{V(r_g,\Omega)}{\gamma} \tag{9}$$

where γ is the normal gravity at the reference ellipsoid. We note that PITE is evaluated at the geoid (in Helmert space), i.e. for $r(\Omega) = r_g(\Omega)$, and is in units of length. It is based on Bruns' formula [Vaníček and Krakiwsky 1986, Eq. (21.4)], which links disturbing potential T to geoidal height N. There is also another, much smaller effect, called "Secondary Indirect Topographical Effect" (SITE), which we will not discuss here as it can be neglected under most circumstances (Vaníček et al. 1999).

Let us now concentrate on the two terms DTE and PITE. They can be evaluated by numerical integration over topography, considering the real topographical density $\varrho(r,\Omega)$, and using one of many possible mass condensation schemes. We will deal with only the average topographical density of

$$\varrho(r,\Omega) = \varrho_0 = 2670 \text{ kg m}^{-3} \tag{10}$$

although a more realistic topographical density model has to be used in accurate geoid determination (Martinec 1993; see also Fraser et al. 1998; Tsiavos and Featherstone 2000; Huang et al. submitted). Finally, we will show the models for three different mass condensation schemes:

(1) the *mean density* condensation, which gives the condensation layer density σ as

$$\sigma(\Omega) = \varrho_0\, H(\Omega) \tag{11}$$

where H is the orthometric height of the terrain (Vaníček and Kleusberg 1987);

(2) the *mass conservation* condensation, which preserves the total mass of the Earth when transforming from the real to the Helmert space:

$$\sigma(\Omega) = \varrho_0 H(\Omega)\left[1 + \frac{H(\Omega)}{R} + \frac{H^2(\Omega)}{3\,R^2}\right] \tag{12}$$

where R is the mean radius of the Earth (Wichiencharoen 1982);

(3) the *mass-centre conservation* condensation, which preserves the position of the centre of mass of the Earth in the transformation into the Helmert space (Wichiencharoen 1982):

$$\sigma(\Omega) = \varrho_0 H(\Omega)\left[1 + 3\frac{H(\Omega)}{2R} + \frac{H^2(\Omega)}{R^2} + \frac{H^3(\Omega)}{4\,R^3}\right] \tag{13}$$

We will consider both planar and spherical models here, as the comparison of the two is our main objective. Unfortunately, however, only the first condensation scheme can be used in conjunction with the planar model; the other two schemes do not make sense in their planar form – Eqs. (12) and (13)-have been derived for a spherical model and their planar counterparts do not exist.

For any of the condensation schemes, both the DTE and PITE can be expressed as a sum of the contribution of the Bouguer spherical shell (or an infinite Bouguer plate, in the case of the planar model) of thickness H, plus the contribution of the real terrain on top of the shell/plate. It turns out that the terrain contribution (called the topographical roughness term by Martinec and Vaníček 1994a, b) is not too sensitive to the selection of the mass condensation scheme. Once again, the terrain contributions are dealt with by Novák et al. (submitted) and here we will concentrate only on the Bouguer shell/plate contributions and denote them by $\mathrm{DTE}^{\mathrm{B}}(\Omega)$ and $\mathrm{PITE}^{\mathrm{B}}(\Omega)$. (We note that the Bouguer shell/plate contribution is, mathematically speaking, nothing other than the contribution of the singularity of Newton's integral at the computation point.) Table 1 gives an overview of the results for the three different condensation schemes and for the two models.

What can we say about the individual contributions? Is there, for instance, any indication that one condensation scheme is better than the others? To answer this question, we should evaluate the total topographical effect for each of the condensation schemes and compare them to establish whether the results are identical or not. In order to evaluate the total topographical effect, the DTE has first to be transferred from the Earth's surface to the geoid (downward continued), then convoluted with Stokes, kernel, and finally added to the PITE. Symbolically, we can write the following algorithm for the total topographical plate or shell model effect $\delta N^{\mathrm{B,total}}(\Omega)$ on the geoid:

$$\mathrm{DTE}^{\mathrm{B}}(\Omega) = \mathrm{DTE}^{\mathrm{B}}(r_t,\Omega) \rightarrow \mathrm{DTE}^{\mathrm{B}}(r_g,\Omega) \rightarrow \ \delta N^{\mathrm{B}}(r_g,\Omega) = \delta N^{\mathrm{B}}(\Omega) \tag{14}$$

$$\delta N^{\mathrm{B,total}}(\Omega) = \delta N^{\mathrm{B}}(\Omega) + \mathrm{PITE}^{\mathrm{B}}(\Omega) \tag{15}$$

The problem here is with the downward continuation $\mathrm{DTE}^{\mathrm{B}}(r_t,\Omega) \rightarrow \mathrm{DTE}^{\mathrm{B}}(r_g,\Omega)$ of the DTE. A harmonic function does have a uniquely defined downward continuation, which can be obtained by means of solving a boundary-value problem of Dirichlet type, leading to the solution in the form of the Poisson integral. However, the downward continuation of a non-harmonic function cannot be evaluated through Poisson's integral! It is easy to prove that the residual

Table 1. Expressions (and references) for the direct (DTE) and primary indirect (PITE) topographical effects for the three most often used condensation schemes in spherical and planar models

Spherical model	Planar model
Mean-density condensation	
$DTE^B(\Omega) = -4\pi G\varrho H^2(\Omega)/R$	$DTE^B(\Omega) = 0$
(Martinec and Vaníček 1994b)	(Vaníček and Kleusberg 1987)
$PITE^B(\Omega) = 2\pi G\varrho H^2(\Omega)/\gamma$	$PITE^B(\Omega) = -\pi G\varrho H^2(\Omega)/\gamma$
(Vaníček and Martinec 1994)	(Vaníček and Kleusberg 1987)
Mass-conservation condensation	
$DTE^B(\Omega) = 0$	Not defined
(Martinec 1993)	
$PITE^B(\Omega) = -2\pi G\varrho H^2(\Omega)/\gamma$	Not defined
(Martinec and Vaníček 1994a)	
Mass-centre-conservation condensation	
$DTE^B(\Omega) = -2\pi G\varrho H^2(\Omega)/R$	Not defined
(Martinec 1993)	
$PITE^B(\Omega) = -4\pi G\varrho H^2(\Omega)/\gamma$	Not defined
(Martinec 1993)	

topographical potential V is not a harmonic function within the topography:

(1) the disturbing potential T satisfies the following Poisson equation within the topography:

$$\Delta T(r,\Omega) = -4\pi G\varrho(r,\Omega), \ \forall r_g \leq r < r_t \tag{16}$$

where Δ stands for the Laplacian operator;

(2) the Helmert disturbing potential T^h, on the other hand, satisfies the Laplace equation everywhere above the geoid

$$\Delta T^h(r,\Omega) = 0, \ \forall r > r_g \tag{17}$$

(3) substituting for T^h in Eq. (17) from Eq. (6) and considering Eq. (16), we obtain

$$\Delta V(r,\Omega) = 4\pi G\varrho(r,\Omega), \ \forall r_g \leq r < r_t \tag{18}$$

which concludes the proof. As V is not harmonic, there is no reason to believe that V^B is harmonic either, and the downward continuation of V^B, and therefore even of the DTE^B, cannot be evaluated without bringing the density $\varrho(r,\Omega)$ into the discussion, which makes the proposition much more involved.

We thus have to conclude that there is no simple way of theoretically comparing the performance of the three condensation schemes. All that can be ascertained is that the first scheme changes both the mass and the centre of mass, the second changes the centre of mass, while the third changes the mass of the Earth in Helmert space. Thus the resulting geoid in Helmert space has to be corrected for scale, by subtracting −4.9 cm from all the geoidal heights (Martinec 1998), or for the shift of the geoid with respect to the centre of mass (Hörmander's corrections), amounting to $(-0.6, -1.5, 0.2\ \text{cm})$ (Hörmander 1976), or for both (in the case of mean density condensation). From the numerical point of view, the scheme that preserves the mass of the Earth should be recommended because the effects are the smallest and thus the easiest to evaluate accurately.

Is there any indication that the spherical model involving Bouguer's shell gives significantly better results than the planar model involving Bouguer's plate? Not from the discussion above! To begin with, when investigating the relative performance of the planar and spherical models in the evaluation of DTE and PITE, we can no longer disregard the terrain effect. However, our numerical experiments, where we took both the plate/shell and the terrain into account, have shown (Novák et al., submitted, 2000) that significant differences are encountered when spherical and planar models are used. Since the spherical model is indubitably closer to reality than the planar model, we conjecture, based on the above discussion, that the spherical model is the theoretically more correct one and should be used whenever a higher accuracy of results is desired. It may be argued that the numerical evaluation of the spherical terrain correction can be affected by systematic errors embedded in global elevation data. This is obviously the computational advantage of the planar model, which also produces a smoother field more suitable for prediction of gravity data. Most of these systematic errors will, however, be cancelled out using Helmert's condensation. For more details on this, see Novák et al. (submitted, 2000).

4 Conclusions

The investigation of the difference between the Bouguer plate and shell model effects has led to several rather interesting discoveries. First, it became clear that the choice of either the plate or the shell does not affect the Bouguer plate reduction appreciably: the difference is of second order compared to the reduction itself and is described by the "curvature correction" as we might have expected. Second, we have confirmed that the physical interpretation of the Bouguer plate reduction is not the removal of the effect of the plate/shell; rather it is the removal of the effect of the upper half of the plate/shell. Alternatively, the reduction can be viewed as purely formal because it uses a value for the vertical gradient of gravity that does not make physical sense.

In geoidal applications the differences between the plate and the shell effects are more subtle, yet more fundamental. First, we see that the planar model does not allow us to use any of the physically meaningful condensation models: the mass conservation condensation or the mass centre conservation condensation. This alone, in our opinion, should disqualify the planar model from use in geoidal studies. Even when the planar model can be used, i.e. for the mean density condensation, the effects appear to be quite different. For the reason of having to downward continue a non-harmonic function, it is impossible to compare the results from the two models theoretically. We thus cannot show theoretically the difference between the effects of the two models.

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