THE THEORY OF MOTION OF THE HORIZONTAL PENDULUM WITH A ZÖLLNER SUSPENSION AND SOME INDICATIONS FOR THE INSTRUMENTAL DESIGN

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Summary: *The aim of this paper is to examine the physical laws which govern the motion of a horizontal pendulum, mounted with the familiar Zöllner suspension geometry and to establish the equations of its motion. It is possible in this process to determine calibration parameters with greater precision and confidence and to demonstrate that procedures based upon period measurement can inherently claim a precision superior to some modern alternatives. Moreover in approaching the problem from a theoretical viewpoint it is possible to pinpoint those features of design which give rise to the troublesome aberrations. This facility makes it possible to comment upon design improvements so that a closer approach may be made to an ideal instrument.*

1. INTRODUCTION

As in all experimental geophysical work the problem of calibration of horizontal pendulums, when used for the measurement of earth tidal tilts, is severe. The conventional instruments rely upon a simple relationship which exists between $\mathscr{T}_v$, the free period of the pendulum mounted vertically, and $\mathscr{T}_1$, the operational period of the pendulum mounted horizontally but also in such a way that the pivot used for the determination of $\mathscr{T}_v$ lies in a line joining the contact points of the suspension filaments with the supporting framework. More recent instruments, notably the Verbaandert/Melchior quartz pendulums, are not designed in a manner which allows $\mathscr{T}_v$ to be measured, and are subject to an initial calibration in the laboratory when their response to known tilts is related to operating period. In this case the known tilts are applied by an independently calibrated footstand, a crapaudine, which dilates and contracts in response to a changing head of mercury communicating with an internal cavity in the stand. Although it is possible to have an automatic calibration system in such cases, whereby the footstand is used operationally and made to effect a displacement to the pendulum position at regular intervals, most instruments of this type rely for their sensitivity in toto or in part upon the precision with which the operating period can be determined. Because of dissatisfaction in this procedure some current instruments have been designed in a more sophisticated way so that the pendulum may be mounted in a less critical and sensitive fashion and its microscopic movements in response to tilt are then amplified electronically. The responsibility for calibration and for stability is therefore partly shifted from the mechanics of the instrument to the electronic components, but this change in emphasis may not be altogether wise, merely introducing a new range of problems.

Dissatisfaction arises from the fact that it is not easy to measure the operating period of a horizontal pendulum since experience shows that period is dependent in a complex fashion

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upon the amplitude of swing. The phenomenon has received attention from many workers in the earth tide discipline notably Skalský, Pícha, Schneider, Mittelstrasse, Goloubitskij et al., and appropriate references are given in an appendix to this paper. Procedures have been devised to counter the problem, in particular Schneider derived a formula to represent the phenomenon, and this formula is vindicated by the present investigation. Nevertheless, such hypotheses, without exception, are empirically based in that they were suggested by the characteristic form of long term period measurement. Although physical and mechanical justification for the observed phenomena has been suggested by certain authors this has been intuitive, without the confirmation of theory which has always proved intractable. Inevitably a proportion of these deductions are now shown to be false.
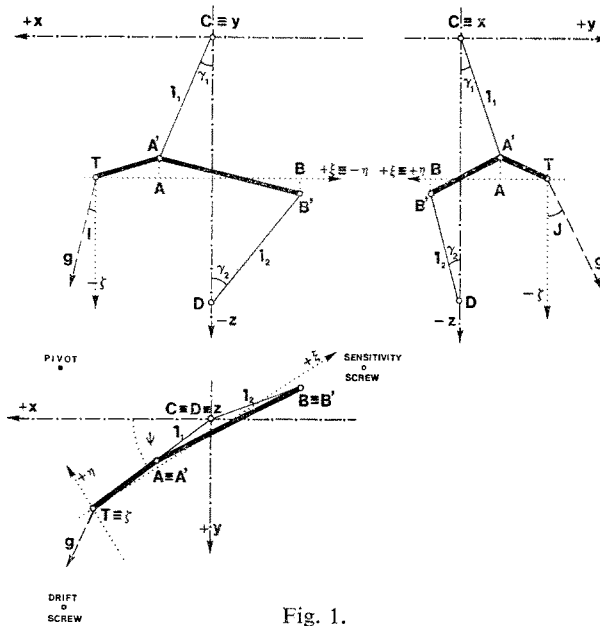

Fig. 1.

Let the pendulum, with its supporting frame, be represented by a system of five points A', B', T, C, D where A', B' are the points where the suspensions are attached to the pendulum beam, T is the centre of gravity of the pendulum beam, and C, D are the points where suspensions are attached to the supporting frame (Fig. 1).

Now consider the first co-ordinate system $\mathscr{B} \equiv (C; x, y, z)$, rectangular and positive, linked with the supporting frame so that the $-z$ axis passes from C to D (Fig. 1) and $+x$ is parallel with the vertical plane going through the "pivot" and "sensitivity screw" (Fig. 1).

Then consider a second co-ordinate system $\mathscr{B}' \equiv (T; \xi, \eta, \zeta)$, rectangular and positive, linked with the pendulum beam. Its orientation will be specified later. For the point T two equations of motion can be written:

$$(1); (2) \qquad \ddot{r} = \sum_{i=1}^{n} f_i / \mathscr{M} \; ; \quad \mathfrak{I}\dot{\omega} + \omega \times \mathfrak{I}\omega = \sum_{i=1}^{n} N_{i,T} \; .$$

Here: r refers to the radius-vector of T in $\mathscr{B}$; $\mathscr{M}$ the mass of T (i.e., the mass of the pendulum beam); f_i ($i = 1, 2, ..., n$) forces acting on the system $TA'B'$ expressed in $\mathscr{B}$; $\mathfrak{I}$ the tensor of inertia of the pendulum beam in T; ω the rotation of $\mathscr{B}'$; $N_{i,T}$ ($i = 1, 2, ..., n$) moments of forces f_i towards T.

Let us denote vectors in $\mathscr{B}$ by small latin letters and in $\mathscr{B}'$ by small greek letters. If we denote time variations (derivatives with respect to time, i.e., speeds) in $\mathscr{B}$ by $\cdot$ and $\mathscr{B}'$ by $*$ we can write for a free vector α:

$$(3) \qquad \dot{\alpha} = {}^*\alpha + \omega \times \alpha .$$

Denoting the radius vector of a point P in $\mathscr{B}'$ by ϱ_P we obtain similarly:

$$(4) \qquad \dot{r} = {}^*\varrho_P + \omega \times \varrho_P + \dot{r}_T .$$

Since $\varrho_P = r_P - r_T$, using the same symbolism, we obtain for accelerations:

$$(5) \qquad \ddot{\alpha} = {}^{**}\alpha + 2\omega \times {}^*\alpha + \dot{\omega} \times \alpha + \omega \times (\omega \times \alpha) ,$$

$$(6) \qquad \ddot{r}_P = \ddot{r}_T + {}^{**}\varrho_P + 2\omega \times {}^*\varrho_P + \dot{\omega} \times \varrho_P + \omega \times (\omega \times \varrho_P) .$$

Let us introduce now the vector called "moment of motion" of the point i with respect to P:

$$(7) \qquad a_{i,P} = (r_i - r_P) \times \dot{r}_i \mathscr{M}_i .$$

$\mathscr{M}_i$ is the "mass of point i". Taking especially $P \equiv C$ we get $a_{i,C} = r_i \times \dot{r}_i \mathscr{M}_i$. Since the "moment of f_i" (f_i being a force acting in i) is defined by the following equation:

$$(8) \qquad n_{i,P} = (r_i - r_P) \times f_i$$

and particularly the moment towards C:

$$(9) \qquad n_{i,C} = r_i \times f_i$$

we can write:

$$(10) \qquad \dot{a}_{i,C} = n_{i,C} .$$

The moment of motion of the body $\mathscr{E}$, represented in our case by $TA'B'$, with respect to C will be given by:

$$(11) \qquad a_{\mathscr{E},C} \equiv a = \int_{i \in \mathscr{E}} (r_i \times \dot{r}_i) \, \mathscr{M}_i .$$

Substituting for r_i and $\dot{r}_i$ from (4) and considering ${}^*\varrho_i = 0$ (i.e. considering $\mathscr{E}$ rigid) we obtain:

$$(11') \qquad a = \int_{i \in \mathscr{E}} \{(r_T \times \varrho_i) \times [(\omega \times \varrho_i) + \dot{r}_T]\} \, \mathscr{M}_i .$$

Eq. (11') can be rewritten as follows:

$$(11'') \qquad a = (r_T \times \dot{r}_T) \mathscr{M} - \dot{r}_T \times \int_{i \in \mathscr{E}} \varrho_i \mathscr{M}_i + r_T \times \left(\omega \times \int_{i \in \mathscr{E}} \varrho_i \mathscr{M}_i \right) +$$
$$+ \int_{i \in \mathscr{E}} [\varrho_i \times (\omega \times \varrho_i)] \, \mathscr{M}_i ,$$

where $\mathcal{M} = \int_{i\in\mathcal{E}} \mathcal{M}_i$. In (11″) $\int_{i\in\mathcal{E}} \varrho_i \mathcal{M}_i = \varrho_T \mathcal{M} = \mathbf{0}$ and

$$\int_{i\in\mathcal{E}} \left[\varrho_i \times (\omega \times \varrho_i) \right] \mathcal{M}_i = \int_{i\in\mathcal{E}} \left(\begin{Vmatrix} \zeta_i^2 + \eta_i^2, & -\xi_i\eta_i, & -\xi_i\zeta_i \\ -\eta_i\xi_i, & \xi_i^2 + \zeta_i^2, & -\eta_i\zeta_i \\ -\zeta_i\xi_i, & -\zeta_i\eta_i, & \eta_i^2 + \xi_i^2 \end{Vmatrix} \omega \right) \mathcal{M}_i = \mathfrak{J}\omega .$$

Thus (11) acquires the final form:

$$(11''') \qquad\qquad \mathbf{a} = \left(\mathbf{r}_T \times \dot{\mathbf{r}}_T \right) \mathcal{M} + \mathfrak{J}\omega .$$

According to (10) the moment of all the forces acting on $\mathcal{E}$ is given by the time derivatives of the moment of motion of $\mathcal{E}$ so that we can write:

$$(12) \qquad\qquad \sum_{i=1}^{m} \mathbf{n}_{i,C} = \dot{\mathbf{a}} = \left(\mathbf{r}_T \times \ddot{\mathbf{r}}_T \right) \mathcal{M} + \mathfrak{J}\dot{\omega} + \omega \times \mathfrak{J}\omega .$$

Considering (9) and taking into account that $\mathbf{n}_{i,T} = (\mathbf{r}_i - \mathbf{r}_T) \times \mathbf{f}_i$ we obtain:

$$(13) \qquad\qquad \sum_{i=1}^{m} \mathbf{n}_{i,C} = \sum_{i=1}^{m} \mathbf{n}_{i,T} + \left(\mathbf{r}_T \times \ddot{\mathbf{r}}_T \right) \mathcal{M}$$

which together with (12) gives us (2) with $\mathbf{N}$ replacing $\mathbf{n}$. The eq. (1) is obvious.

Let us, for simplicity, orient $\mathcal{B}'$ in such a way that the axes ξ, η, ζ will coincide with the main axes of the central ellipsoid of inertia of the pendulum beam. The matrix of the tensor of inertia thus degenerates into a diagonal form containing moments of inertia towards individual axes only, and (2) becomes:

$$(2') \qquad\qquad \mathfrak{J}_0 \dot{\omega} = \sum_{i=1}^{m} \mathbf{N}_{i,T} .$$

2. FORCES AND THEIR MOMENTS ACTING ON PENDULUM

In the following paragraph, we shall attempt to formulate the forces and moments acting on the pendulum as they arise from the physics of the instrument. This formulation will enable us then to write the differential equations of motion (1), $(2')$ in the final form valid for our particular case.

(a) Gravity force can be taken as acting in T and expressed by the following formula:

$$(14) \qquad\qquad \mathbf{g} \doteq g\mathcal{M}(I\mathbf{e}_1 + J\mathbf{e}_2 + \mathbf{e}_3) ,$$

where g is the gravitational acceleration; I is the angle between $\mathbf{g}$ and axis z in the xz plane; J is the angle between $\mathbf{g}$ and axis z in the yz plane; $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are vector-units in x, y, z axes. The approximation (14) holds because the angles I and J are always extremely small. As $\mathbf{g}$ acts in T its moment $\mathbf{N}_g$ is equal to $\mathbf{0}$.

(b) Tension in suspensions as a result of gravity force can be determined from the condition of equilibrium in A'. When we neglect z components, knowing

from experiments that the overwhelming part of the motion takes place in the plane x, y, we can write for the absolute value f_B of the reaction force acting in suspension $B'D$ (Fig. 1):

$$(15) \quad f_B \cos \gamma_2 \sqrt{[(x_A - x_B)^2 + (y_A - y_B)^2]} = g\mathcal{M} \sqrt{[(x_T - x_A)^2 + (y_T - y_A)^2]} \,.$$

Similarly for the absolute value f_A of the reaction force in suspension $A'C$:

$$(16) \quad f_A = (g\mathcal{M} + f_B \cos \gamma_2)/\cos \gamma_1 \,.$$

All the other forces will be much smaller so that we can take T, A', B' lying approximately in a vertical plane. Hence we can write:

$$(17) \quad x_A = x_T - \text{AT} \cos \psi \,, \quad y_A = y_T - \text{AT} \sin \psi \,;$$

$$x_B = x_T - \text{BT} \cos \psi \,, \quad y_B = y_T - \text{BT} \sin \psi \,.$$

Substituting (17) to (15) we can write:

$$(18) \quad \mathbf{f}_B = (\mathbf{r}_D - \mathbf{r}_{B'})f_B/l_2 = g\mathcal{M} \, \text{AT}(l_2 \, \text{AB} \cos \gamma_2)^{-1} (\mathbf{r}_D - \mathbf{r}_{B'}) = f_B'(\mathbf{r}_D - \mathbf{r}_{B'}).$$

Similarly, for $\mathbf{f}_A$ we can get:

$$(19) \quad \mathbf{f}_A = -\mathbf{r}_A f_A/l_1 = g\mathcal{M} \, \text{BT}(l_1 \, \text{AB} \cos \gamma_1)^{-1} (-\mathbf{r}_{A'}) = -f_A' \mathbf{r}_{A'} \,.$$

Moments of these forces can be, according to (8), expressed as follows:

$$(20) \quad \mathbf{N}_{A,T} = (\mathbf{r}_{A'} - \mathbf{r}_T) \times \mathbf{f}_A = f_A'(\mathbf{r}_T \times \mathbf{r}_{A'}) \,,$$

$$(21) \quad \mathbf{N}_{B,T} = (\mathbf{r}_{B'} - \mathbf{r}_T) \times \mathbf{f}_B = f_B'(\mathbf{r}_{B'} \times \mathbf{r}_D + \mathbf{r}_D \times \mathbf{r}_T + \mathbf{r}_T \times \mathbf{r}_{B'}) \,.$$

(c) R e a c t i o n a g a i n s t t o r s i o n i n s u s p e n s i o n s i n a c r o s s - s e c t i o n a l s e n s e is obviously bound to provide moments in A' and B'. According to [11, p. 297], we can write for the torsion torque in a suspension:

$$(22) \quad M_T = G\mathcal{I}_c \Phi/l \,,$$

where G is the modulus of torsion for the material used for the suspension filaments; $\mathcal{I}_c$ is the moment of inertia of a cross-section of the suspension filament; Φ is the angle of torsion; l is the length of the suspension. Taking for each suspension $\Phi = \psi - \psi_0$, where ψ_0 is the initial twist of the suspension we can write the moments:

$$(23) \quad \mathbf{N}_{A,T} = M_T l_1^{-1} \mathbf{r}_{A'} \doteq G\mathcal{I}_c(\psi - \psi_{0A}) \, l_1^{-2} \mathbf{r}_{A'} \,,$$

$$(24) \quad \mathbf{N}_{B,T} = M_T l_2^{-1}(\mathbf{r}_{B'} - \mathbf{r}_D) \doteq G\mathcal{I}_c(\psi - \psi_{0B}) \, l_2^{-2}(\mathbf{r}_{B'} - \mathbf{r}_D) \,.$$

Note that from the geometrical point of view, this initial twist refers to the angle between the plane xz and the neutral plane of the filament in an untwisted state.

(d) Internal torsional friction in suspensions in a cross-sectional sense is held to be responsible for the damping of the motion.* In accordance with the frictional law it must be proportional to the speed of motion taken in the negative sense. It will add to the moments in A' and B' and can thus be expressed by the following formulae:

(25); (26) $$\mathbf{N}_{A,T}^{F} = K\dot{\psi}l_1^{-2}\mathbf{r}_{A'}\; ; \quad \mathbf{N}_{B,T}^{F} = K\dot{\psi}l_2^{-2}(\mathbf{r}_{B'} - \mathbf{r}_D)\,,$$

where K is a constant characteristic for the suspension filaments.

(e) Air friction produces the same influence as the torsional friction in suspensions and can therefore be described by equations identical with eqs. (25) and (26), where only the coefficient K would have different meaning. Thus, we can deal with (25) and (26) only and interpret K as characterising not only the internal friction but also the air friction.

(f) Reaction against tension in suspensions due to longitudinal bend is a very difficult physical phenomenon to deal with from the mathematical point of view. The authors were not able to arrive at any satisfactory formula and have therefore decided to neglect the phenomenon altogether. Yet, the indication is, as correctly pointed out by Skalský, that it may play a significant role in the quantitative interpretation of the present theory. All that can be said about the phenomenon is that it would have a form similar to the eqs. (23) and (24).

3. DIFFERENTIAL EQUATION OF MOTION

It can be shown that the pendular motion of the pendulum can be described by differential equation of fourth order. Denoting the main moment of inertia of the pendulum beam towards the axis ζ by $\mathscr{C}_\zeta$ and $\mathscr{M}/\mathscr{C}_\zeta$ by Q we have:

(27) $$\frac{d^4\psi}{dt^4} + k_3 \frac{d^3\psi}{dt^3} + (k_1 + Qk_2x_0 + k_4)\frac{d^2\psi}{dt^2} + k_1k_3 \frac{d\psi}{dt} +$$

$$+ (k_1k_2Qx_0 + k_1k_4 - Qk_2^2)\,\psi = Qk_2Jg + k_1k_5 + \mathcal{O} + \ddot{\mathcal{O}}$$

where

(28) $$k_1 = (f'_A + f'_B)/\mathscr{M}\,, \quad k_2 = (f'_A\,\text{AT} + f'_B\,\text{BT})/\mathscr{M}\,,$$

$$k_3 = K(l_1^{-1}\cos\gamma_1 - l_2^{-1}\cos\gamma_2)/\mathscr{C}_\zeta\,, \quad k_4 = G\mathscr{J}_c(l_1^{-1}\cos\gamma_1 - l_2^{-1}\cos\gamma_2)/\mathscr{C}_\zeta\,,$$

$$k_5 = G\mathscr{J}_c(\psi_{0A}l_1^{-1}\cos\gamma_1 - \psi_{0B}l_2^{-1}\cos\gamma_2)/\mathscr{C}_\zeta$$

and $\mathcal{O}$ represents the non-linear terms:

(29) $$\mathcal{O} = \tfrac{1}{2}Qk_2(\tfrac{1}{3}Ig\psi^3 - Jg\psi^2)\,.$$

Substituting for f_i in (1) from (14), (18) and (19) we obtain:

$$(1') \qquad \ddot{\boldsymbol{r}} = \left[\boldsymbol{g} - f'_A \boldsymbol{r}_{A'} + f'_B(\boldsymbol{r}_D - \boldsymbol{r}_{B'})\right]/\mathscr{M} \,.$$

Limiting ourselves to the plane xy, making use of (17), and omitting suffixes T we can write:

$$(30) \qquad \ddot{x} = -\left(f'_A + f'_B\right) \mathscr{M}^{-1} x + Ig + \left(f'_A \,\mathrm{AT} + f'_B \,\mathrm{BT}\right) \mathscr{M}^{-1} \cos \psi \,,$$

$$(31) \qquad \ddot{y} = -\left(f'_A + f'_B\right) \mathscr{M}^{-1} y + Jg + \left(f'_A \,\mathrm{AT} + f'_B \,\mathrm{BT}\right) \mathscr{M}^{-1} \sin \psi \,.$$

Eqs. (30), (31) describe the motion of the centre of gravity in the xy plane.

Considering the moments given by equations (20), (21), (23)—(26) we can rewrite (2') as follows:

$$(2'') \qquad \mathfrak{I}_0 \dot{\omega} = f'_A(\boldsymbol{r}_T \times \boldsymbol{r}_{A'}) + f'_B(\boldsymbol{r}_{B'} \times \boldsymbol{r}_D + \boldsymbol{r}_D \times \boldsymbol{r}_T + \boldsymbol{r}_T \times \boldsymbol{r}_{B'}) +$$
$$+ \left[G\mathscr{I}_c(\psi - \psi_{0A})/l_1^2 + K\dot{\psi}/l_1^2\right] \boldsymbol{r}_{A'} + \left[G\mathscr{I}_c(\psi - \psi_{0B})/l_2^2 + K\dot{\psi}/l_2^2\right] (\boldsymbol{r}_{B'} - \boldsymbol{r}_D) \,.$$

Because the distribution of mass in the pendulum beam is more or less symmetrical along the horizontal plane as well as along the vertical plane passing through T, A', B', the ζ axis of the $\mathscr{B}'$ system can be taken as practically parallel to z. Thus only the ζ component of $\mathfrak{I}_0 \dot{\omega}$ will correspond to the z component of the right-hand side of (2''). For the reason mentioned above (section 2(b)) we can deal with the ζ and z components only. Taking $|\dot{\omega}| = \dot{\omega}_\zeta = \ddot{\psi}$ we get:

$$(32) \qquad \ddot{\psi} = \mathscr{C}_\zeta^{-1} \{ f'_A(x_T y_A - y_T x_A) + f'_B(x_B y_D - y_B x_D + x_D y_T - x_T y_D +$$
$$+ x_T y_B - y_T x_B) + \left[G\mathscr{I}_c(\psi - \psi_{0A}) \, l_1^{-2} + K\dot{\psi} l_1^{-2}\right] z_{A'} +$$
$$+ \left[G\mathscr{I}_c(\psi - \psi_{0B}) \, l_2^{-2} + K\dot{\psi} l_2^{-2}\right] (z_{B'} - z_D) \} \,.$$

Taking $x_D = y_D = 0$, $z_{A'} = -l_1 \cos \gamma_1$, z_B, $-z_D = l_2 \cos \gamma_2$, substituting for x_A, y_A, x_B, y_B from (17) and omitting suffixes T we can write:

$$(32') \qquad \ddot{\psi} = \mathscr{C}_\zeta^- \{ (l_2^{-1} \cos \gamma_2 - l_1^{-1} \cos \gamma_1) K\dot{\psi} + (l_2^{-1} \cos \gamma_2 -$$
$$- l_1^{-1} \cos \gamma_1) G\mathscr{I}_c \psi - (f'_A \,\mathrm{AT} + f'_B \,\mathrm{BT}) x \sin \psi +$$
$$+ (f'_A \,\mathrm{AT} + f'_B \,\mathrm{BT}) y \cos \psi + G\mathscr{I}_c(\psi_{0A} l_1^{-1} \cos \gamma_1 - \psi_{0B} l_2^{-1} \cos \gamma_2) \} \,.$$

This equation describes the angular motion of $\mathscr{B}'$ and therefore also the angular motion of the pendulum beam.

Making use of eq. (28) we can rewrite eqs. (30)—(32).

$$(33);\,(34) \qquad \ddot{x} + k_1 x = k_2 \cos \psi + Ig \,; \quad \ddot{y} + k_1 y = k_2 \sin \psi + Jg \,,$$

$$(35) \qquad \ddot{\psi} + k_3 \dot{\psi} + k_4 \psi = \mathscr{M}\mathscr{C}_\zeta^{-1} k_2(y \cos \psi - x \sin \psi) + k_5 \,.$$

Equations (33)—(35) represent the fundamental system which must be solved. Here all the coefficients (apart from J and k_5) are positive. J and k_5 could be either negative, zero or positive. Since a precise solution cannot be obtained, an approximate solution will now be attempted.

In order to transform (33)—(35) into a more soluble form let us approximate to $\sin \psi$ by $\psi - \frac{1}{6}\psi^3$ and to $\cos \psi$ by $1 - \frac{1}{2}\psi^2$. We can, thus, write:

$$(33');\,(34') \qquad \ddot{x} + k_1 x \doteq k_2 + Ig - \tfrac{1}{2}k_2\psi^2 \,; \quad \ddot{y} + k_1 y \doteq k_2\psi + Jg - \tfrac{1}{6}k_2\psi^3 \,.$$

Eq. (33′) can be regarded as practically independent from the other two and the relevant part of x:

$$(36) \qquad x \doteq \left(k_2 + Ig - \tfrac{1}{2}k_2\psi^2\right)/k_1 = x_0 - \tfrac{1}{2}k_2 k_1^{-1}\psi^2$$

substituted to (35). Hence we can write:

$$(35') \qquad \ddot{\psi} + k_3\dot{\psi} + \left(\mathcal{M}\mathcal{C}_\zeta^{-1}k_2 x_0 + k_4\right)\psi = \mathcal{M}\mathcal{C}_\zeta^{-1}k_2 y + k_5 - \\ - \mathcal{M}\mathcal{C}_\zeta^{-1}k_2\left(\tfrac{1}{2}y\psi^2 - \tfrac{1}{2}k_2 k_1^{-1}\psi^3 - \tfrac{1}{6}x_0\psi^3\right).$$

Eqs. (34′) and (35′) represent a system of two interrelated differential equations which must be solved together.

Let us rewrite the system (34′) and (35′) in the form of one differential equation of fourth order. Taking $+(\mathcal{M}/\mathcal{C}_\zeta)k_2 \cdot$ (34′) $+k_1 \cdot$ (35′) $+(\mathrm{d}^2/\mathrm{d}t^2)$ (35′) we obtain eq. (27). The terms of higher order in ψ in (35′) are, for a small amplitude of ψ, obviously much smaller than other terms in the equation. They can, therefore, be regarded as corrective terms influencing the result very little. Because their influence is small no significant error is made when substituting $(k_2\psi + Jg)/k_1$ from (34′) for y in the coefficient for ψ^2. Moreover, it can be shown that $\ddot{\mathcal{O}}$ is much smaller than the other corrective terms and may therefore be disregarded.

The coefficients k_1, k_2 can be rewritten as follows:

$$k_1 = g/\mathrm{AB}\left[\mathrm{BT}(l_1 \cos\gamma_1)^{-1} + \mathrm{AT}(l_2 \cos\gamma_2)^{-1}\right],$$
$$k_2 = g \, \mathrm{AT} \, \mathrm{BT}/\mathrm{AB}\left[(l_1 \cos\gamma_1)^{-1} + (l_2 \cos\gamma_2)^{-1}\right];$$

they obviously depend upon the geometry of the suspended beam only. So do, basically, the other coefficients, although k_3 is additionally proportional to the constant K, characteristic for internal friction in the suspensions and the air friction. k_4, k_5 are also proportional to the suspensions modulus of torsion and k_5 depends upon the initial twist in suspensions. Note, that if we had also considered the influence of the bend of the suspensions, the formulae for k_4, k_5 would have looked different while eq. (27) would remain unchanged.

4. A FIRST APPROXIMATION TO THE SOLUTION

Let us, for the first approximation, neglect $\mathcal{O}$ completely. Then we can seek a solution in the form $\psi = \psi_0 + \Delta\psi$, where ψ_0 is a constant term representing the angular displacement of the beam with respect to the xz plane at some arbitrary time when, for instance, period measurements are commenced. Then $\Delta\psi$ represents the swing of the beam. Denoting the left hand side of (27) by $L_4(\psi)$ we can write:

$$(37) \qquad L_4(\psi) \doteq Qk_2 Jg + k_1 k_5.$$

Making use of (36) we obtain for ψ_0:

$$(38) \qquad \psi_0 \doteq \left(Qk_2 Jg + k_1 k_5\right)/\left(Qk_2 Ig + k_1 k_4\right).$$

To find $\Delta\psi$ we shall need to solve the equation:

$$(39) \qquad L_4(\Delta\psi) = 0 .$$

It yields following approximate solution:

$$(40) \qquad \Delta\psi \doteq a_1 \cos\left(\mu_1 t - \Phi_1\right) + a_2 \cos\left(\mu_2 t - \Phi_2\right)$$

where

$$(41) \quad \mu_1 = \sqrt{\left[(Qk_2 Ig + k_1 k_4)/(k_1 + Qk_2 x_0 + k_4)\right]}, \quad \mu_2 = \sqrt{\left[k_1 + Qk_2 x_0 + k_4\right]}$$

and a_1, a_2, Φ_1, Φ_2 are some integration constants.

The characteristic polynomial of eq. (39) is:

$$(42) \quad \lambda^4 + k_3\lambda^3 + \left(k_1 + Qk_2 x_0 + k_4\right)\lambda^2 + k_1 k_3 \lambda + Qk_2 Ig + k_1 k_4 = 0.$$

The roots of this polynomial could be found numerically if we knew the numerical values of all the coefficients involved. Unfortunately, this is not the case since the value of constant K in k_3 is usually not available. To express the roots as functions of K would clearly be very cumbersome. Let us then content ourselves with the assumption that k_3 is much smaller than the other coefficients and can therefore be disregarded in the first approximation. Thus, (42) can be reduced to bi-quadratic form:

$$(43) \qquad \lambda^4 + \mu_2^2 \lambda^2 + \varkappa^2 = 0 ,$$

where μ_2 is given by (41) and $\varkappa^2 = Qk_2 Ig + k_1 k_4$. Since $\mu_2 \gg \varkappa$ we can evaluate the roots with sufficient precision as follows:

$$(44), (45) \qquad \lambda_{1,2} \doteq \pm i\varkappa/\mu_2 \doteq \pm i\mu_1 , \quad \lambda_{3,4} \doteq \pm i\mu_2 .$$

Hence $\Delta\psi$ can be written, in a first approximation, as a pair of purely periodic motions with frequencies μ_1, μ_2 as expressed by eq. (40).

Let us now examine the frequencies μ_1, μ_2. Period $\mathscr{T}_1$ defined by μ_1 can be calculated from the following expression:

$$(46) \qquad \mathscr{T}_1 = 2\pi/\mu_1 .$$

If we denote $I' = I + Qk_1 k_4 g^{-1} k_2^{-2}$ and neglect k_4 comparing with $k_1 + Qk_2 x_0$ we shall obtain:

$$(46') \qquad \mathscr{T}_1 \doteq 2\pi \sqrt{\left[(k_1 + Qk_2 x_0)/(Qk_2 I'g)\right]} .$$

Substituting k_2/k_1 for x_0 from (36) we can write:

$$(46'') \qquad \mathscr{T}_1 \doteq 2\pi \sqrt{\left[(\mathscr{C}_\zeta \mathscr{M}^{-1} x_0^{-1} + x_0)/(I'g)\right]} .$$

We know from the theory of the vertical pendulum that the period of swing $\mathscr{T}_v$ is given by formula:

$$(47) \qquad \mathscr{T}_v = 2\pi \sqrt{(\mathrm{L}/g)} ,$$

where L is the "reduced" length of pendulum defined by:

$$(48) \qquad \mathrm{L} = \mathscr{C}/(\mathscr{M} x_0)$$

with $\mathscr{C}$ denoting the moment of inertia towards the axis of swing. Using Steiner's theorem relating $\mathscr{C}$ and $\mathscr{C}_\zeta$:

$$(49) \qquad \mathscr{C} = \mathscr{C}_\zeta + \mathscr{M} x_0^2$$

we can rewrite (46″)

$$(46''') \qquad \mathscr{T}_1 \doteq 2\pi \sqrt{[L/(I'g)]} = \sqrt{(\mathscr{T}_v^2/I')} \, .$$

Thus we finally find for the reduced angle I'

$$(50) \qquad I' \doteq \mathscr{T}_v^2 \mathscr{T}_1^{-2} \, ,$$

where $\mathscr{T}_v$ is the free period of the pendulum mounted vertically and $\mathscr{T}_1$ the operational period of the pendulum mounted horizontally. The axis of rotation in both cases must be, of course, identical with respect to the pendulum beam.

It remains to be mentioned that the second frequency μ_2, represents the well known tremble with a period in the region of 0·2^s. We shall not take any special interest in it since we are investigating more particularly those laws governing pendulum behaviour which have a bearing upon the establishment of the calibration factor for tilt. When measuring the free oscillatory periods of a pendulum in its operational position, the initiation of the tremble is in any case usually avoided. Even so, without further excitation, any tremble which does occur is inevitably attenuated very quickly.

5. DAMPING OF AMPLITUDE AND ITS INFLUENCE ON THE MAIN FREQUENCY

The complete solution of eq. (39) can be written as follows:

$$(51) \qquad \Delta\psi \doteq a_1 \exp\left(-\alpha_1 t\right) \cos\left(\tilde{\mu}_1 t - \Phi_1\right) + a_2 \exp\left(-\alpha_2 t\right) \cos\left(\tilde{\mu}_2 t - \Phi_2\right) ,$$

where

$$(52) \qquad \tilde{\mu}_1 = \mu_1 + d\mu_1 \, , \quad \tilde{\mu}_2 = \mu_2 + d\mu_2 \, .$$

The change of the mean frequency μ_1 due to the damping of amplitude is given by equation

$$(53) \qquad d\mu_2 = -\alpha_1^2/\mu_1 = -(2\pi)^{-1} \mathscr{T}_1 \alpha_1^2 \, .$$

It typically attains a value of the order of $1 \, . \, 10^{-6} \ \mathrm{s}^{-1}$ (for metal pendulums) and can therefore be disregarded altogether. $d\mu_2$ does not interest us here.

We shall show that the damping of amplitude, known from experiments, is due to the presence of non-zero coefficients in the derivatives of odd order in (39). In showing their influence let us assume that the first approximation of the roots $(\lambda_{1,2}, \lambda_{3,4})$ is adequate so that their increments, after taking account of odd order derivatives, can be expressed in the form of linear differences. Let us write for the new roots $\tilde{\lambda}$:

$$(54) \qquad \tilde{\lambda} = \lambda + d\lambda$$

and neglect the higher powers of $d\lambda$. Thus we can write approximately the characteristic equation of (39) as follows:

$$(55) \qquad \lambda^4 + 4\lambda^3\,d\lambda + k_3(\lambda^3 + 3\lambda^2\,d\lambda) + \mu_2^2(\lambda^2 + 2\lambda\,d\lambda) + \\ + k_1 k_3(\lambda + d\lambda) + \varkappa^2 = 0\,.$$

Subtracting (43) from (55) we obtain for $d\lambda$:

$$(56) \qquad d\lambda = -k_3(\lambda^3 + k_1\lambda)/(4\lambda^3 + 3k_3\lambda^2 + 2\mu_2^2\lambda + k_1 k_3)\,.$$

Substituting $\lambda_{1,2}$ from (44) for λ we obtain the complex increment in $\lambda_{1,2}$:

$$(57) \qquad d\lambda_{1,2} = -k_3(\mp i\mu_1^3 \pm ik_1\mu_1)/(\mp i4\mu_1^3 - 3k_3\mu_1^2 \pm i2\mu_2^2\mu_1 + k_1 k_3)\,.$$

Since obviously $4\mu_1^3 \ll 2\mu_2^2\mu_1$, $3\mu_1^2 \ll k_1$ we then obtain:

$$(57') \qquad d\lambda_{1,2} \doteq i(\mp k_1 k_3\mu_1)/(k_1 k_3 \pm i2\mu_2^2\mu_1)$$

and subsequently:

$$(57'') \qquad d\lambda_{1,2} \doteq -2k_1 k_3\mu_2^2\mu_1^2/(k_1^2 k_3^2 + 4\mu_1^2\mu_2^4) \mp ik_1^2 k_3^2\mu_1/(k_1^2 k_3^2 + 4\mu_1^2\mu_2^4)\,.$$

Similarly, we could obtain a complex increment $d\lambda_{3,4}$ in which we are not interested.

Thus it has been shown that the roots of (42) make a couple of pairs of complex conjugates. If we denote now the real parts of $d\lambda$ by $-\alpha_1$, $-\alpha_2$ (both α_1 and α_2 are positive) and imaginary parts by $d\mu_1$, $d\mu_2$ then the complete solution $\Delta\psi$ of (39) can be written in the form of (51). Eq. (51) represents the second approximation, a pair of purely periodic, damped motions.

Let us try now to establish the influence $d\mu_1$ of the damping effect on the main frequency μ_1. We can write from (57''):

$$(58) \qquad d\mu_1 = -k_1 k_3\alpha_1/(2\mu_1\mu_2^2)\,.$$

The real part of (57'') is giving a quadratic equation for k_3 (since K in the formula (28) for k_3 is not known, k_3 cannot be calculated directly) from which only the smaller root

$$(59) \qquad k_3 = 2\alpha_1\mu_2^2/k_1$$

has physical meaning. Substituting (59) to (58) we obtain finally eq. (53).

6. DAMPING OF THE MAIN PERIOD

Hitherto we have been neglecting the terms of higher order $\mathcal{O}$ in (27) given by (29). This is appropriate only when $\Delta\psi$ is very small. As this is not generally the case in practice, when $\Delta\psi$ often reaches several tens of minutes of arc, $\mathcal{O}$ cannot be disregarded. We are going to show that the non-linear terms are responsible for the damping of the main period, which again is a phenomenon well known and established empirically by many authors of papers on earth-tide techniques. The main period $\mathcal{T}$ varies according to formula:

$$(60) \qquad \mathcal{T} \doteq \mathcal{T}_1 + c\exp(-2\alpha_1 t)$$

where

$$(61) \qquad c = \left(5J^2I^{-2} + 3\right) \mathscr{T}_1 a_1^2 / 48 .$$

The eq. (60) has also been derived by Schneider [3] purely from experimental data. A similar formula, namely

$$(60') \qquad \mathscr{T} = \mathscr{T}_1 + c_1 \exp\left(-c_2 t\right), \quad c_1, c_2 = \text{const.}$$

has been derived in [4].

In order to trace the influence of higher order terms we can take $\psi_0 = 0$ without any loss of generality. This is permissible because our choice of the system $\mathscr{B}$ was purely arbitrary in the sense that the plane xz is freely directed. Therefore, the result cannot be expected to be linked with any particular direction. Thus we can rotate the system $\mathscr{B}$ around z so that the plane xz will fall with the plane $\xi\zeta$ and ψ_0 becomes zero. Exactly the same effect can be achieved by varying J in such a way that $Qk_2 Jg = -k_1 k_5$.

Since we are interested in the influence on the main frequency only, we shall deal with the following equation only:

$$(62) \qquad \tilde{L}_4(\psi) = \psi^{IV} + \mu_2^2 \psi^{II} + \varkappa^2 \psi = 0 ,$$

where the order of derivative is given by roman numerals. This again is a permissible simplification because the coefficients of odd ordered derivatives are very small indeed. It can be shown that if we consider them at all the result will change very little.

No finite method is known which might solve the equation (62). We shall use a variant of an iterative method attributed to Lindström and Poincaré described for instance in [12, p. 109—112].

Let us seek the solution ψ of (62) in the form of a series

$$(63) \qquad \psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \ldots$$

with the main frequency ω given by a similar series:

$$(64) \qquad \omega = \mu_1 + d\omega_1 + d\omega_2 + \ldots$$

Let us consider

$$(65) \qquad \psi^{(1)} = a_1 \cos \omega t + a_2 \cos \mu_2 t$$

and take, for simplicity, $a_2 = 0$. This is allowed since Φ_1, Φ_2 can be made zero by transforming the time base, and a_2 by suppressing the tremble. The presence of the second periodic term in $\psi^{(1)}$ is irrelevant in the present context.

Let us rewrite (62), neglecting terms higher than second order:

$$(62') \qquad \mu_1^4 \omega^{-4} \psi^{IV} + \left(\mu_1^2 \mu_2^2 + \mu_1^4\right) \omega^{-2} \psi^{II} + \mu_1^2 \mu_2^2 \psi = -\mathscr{A}_1 \psi^2 -$$
$$- \left(1 - \mu_1^4 \omega^{-4}\right) \psi^{IV} - \left[\mu_2^2 - \left(\mu_1^2 \mu_2^2 + \mu_1^4\right) \omega^{-2}\right] \psi^{II},$$

where $\mathscr{A}_1 = \frac{1}{2} Q k_2 Jg$ from (29). Taking $\omega^4 \doteq \mu_1^4 + 4\mu_1^3 d\omega_1$, $\omega^2 \doteq \mu_1^2 + 2\mu_1 d\omega_1$ and considering that $\mu_1 \ll \mu_2$ we can write approximately:

$$(62'') \qquad \tilde{L}_4(\psi) \doteq -\mathscr{A}_1\left(\psi^{(1)}\right)^2 + \left(2\mu_1 \mu_2^2 \, d\omega_1 + \mu_1^4\right) \psi^{(1)} .$$

Here μ_1^4 is a rudiment from the abbreviated form of $\widetilde{L}_4(\psi)$ and must be disregarded completely. The fundamental condition of the present method is that there cannot be any resonance terms on the right hand side because we are dealing with the resultant frequency ω in $\psi^{(1)}$ from the very beginning. Thus

$$(66) \qquad\qquad 2\mu_1\mu_2^2 \, d\omega_1 \,=\, 0$$

and therefore $d\omega_1 = 0$. Solving the equation:

$$(67) \qquad \widetilde{L}_4(\psi^{(2)}) \,=\, -\mathscr{A}_1 a_1^2 \cos^2 \omega t \,=\, -\tfrac{1}{2}\mathscr{A}_1 a_1^2 - \tfrac{1}{2}\mathscr{A}_1 a_1^2 \cos 2\omega t$$

we obtain:

$$(68) \qquad \psi^{(2)} \doteq -\tfrac{1}{2}\mathscr{A}_1 a_1^2 \mu_1^{-2} \mu_2^{-2} - \tfrac{1}{2}\mathscr{A}_1 a_1^2 \cos 2\omega t \left(16\mu_1^4 - 3\mu_1^2 \mu_2^2\right)^{-1} \doteq$$
$$\doteq \tfrac{1}{2}\mathscr{A}_1 a_1^2 \mu_1^{-2} \mu_2^{-2} \left(-1 + \tfrac{1}{3}\cos 2\omega t\right).$$

Considering one more term in (63), (64) and neglecting terms of higher than third order we can write:

$$(69) \qquad \widetilde{L}_4(\psi^{(3)}) \doteq -2\mathscr{A}_1 \psi^{(1)} \psi^{(2)} + \mathscr{A}_2(\psi^{(1)})^3 + 2\mu_1\mu_2^2 \, d\omega_2 \psi^{(1)} \,,$$

where $\mathscr{A}_2 = \tfrac{1}{6}k_2 IgQ$ from (29). Substituting for $\psi^{(1)}$, $\psi^{(2)}$ we obtain:

$$(69') \qquad \widetilde{L}_4(\psi^{(3)}) \doteq \left(\mathscr{A}_1^2 a_1^3 \mu_1^{-2} \mu_2^{-2} + 2\mu_1\mu_2^2 \, d\omega_2 \, a_1\right) \cos \omega t -$$
$$- \tfrac{1}{3}\mathscr{A}_1^2 a_1^3 \mu_1^{-2} \mu_2^{-2} \cos \omega t \cos 2\omega t + \mathscr{A}_2 a_1^3 \cos^3 \omega t \,.$$

The last equation can be rewritten as follows:

$$(69'') \qquad \widetilde{L}_4(\psi^{(3)}) \doteq \left(\mathscr{A}_1^2 a_1^3 \mu_1^{-2} \mu_2^{-2} + 2\mu_1\mu_2^2 a_1 \, d\omega_2 - \tfrac{1}{6}\mathscr{A}_1^2 a_1^3 \mu_1^{-2} \mu_2^{-2} + \right.$$
$$\left. + \tfrac{3}{4}\mathscr{A}_2 a_1^3\right) \cos \omega t + \left(\tfrac{1}{4}\mathscr{A}_2 a_1^3 - \tfrac{1}{6}\mathscr{A}_1^2 a_1^3 \mu_1^{-2} \mu_2^{-2}\right) \cos 3\omega t \,.$$

From the above mentioned condition concerning the resonance term we obtain:

$$(70) \qquad d\omega_2 \doteq -\left(\tfrac{5}{12}\mathscr{A}_1^2 \mu_1^{-3} \mu_2^{-4} + \tfrac{3}{8}\mathscr{A}_2 \mu_1^{-1} \mu_2^{-2}\right) a_1^2 \,.$$

Solving (69″) we derive:

$$(71) \qquad \psi^{(3)} \doteq -a_1^3 2^{-4} \mu_1^{-2} \mu_2^{-2} \left(\tfrac{1}{3}\mathscr{A}_1^2 \mu_1^{-2} \mu_2^{-2} + \tfrac{1}{2}\mathscr{A}_2\right) \cos 3\omega t \,.$$

The terms of higher order would not contribute to this result to any important extent and we shall not, therefore, proceed any further with our iterations.

In this way we have established that the non-linear terms in the eq. (27) can account for the damping of the main period. The main frequency ω can be expressed as

$$(64') \qquad\qquad \omega \,=\, \mu_1 + d\omega_2$$

where $d\omega_2$ increases with the amplitude squared. In order to see the influence on the main period $\mathscr{T}$ let us write:

$$(72) \qquad\qquad \mathscr{T} \,=\, 2\pi/\omega \doteq \mathscr{T}_1 - \mathscr{T}_1^2/(2\pi) \, d\omega_2 \,.$$

Substituting for $d\omega_2$ from (70) and for $\mathscr{A}_1$, $\mathscr{A}_2$ then we obtain:

$$(72') \qquad \mathscr{T} \doteq \mathscr{T}_1 + \mathscr{T}_1^2 / (2\pi\mu_1\mu_2^2) \left(\tfrac{5}{48} Q^2 k_2^2 J^2 g^2 \mu_1^{-2} \mu_2^{-2} + \tfrac{3}{48} Q k_2 I g\right) a_1^2 \,.$$

Since $Q k_2 I g \doteq \mu_1^2 \mu_2^2$ (see eqs. (44), (45)) we obtain after some development:

$$(72'') \qquad\qquad \mathscr{T} \doteq \mathscr{T}_1 + \mathscr{T}_1^2 \mu_1 / (2\pi \cdot 48) \left(5 J^2 I^{-2} + 3\right) a_1^2 \,.$$

Taking here $a_1^2 \exp(-2\alpha_1 t)$ instead of a_1^2 we obtain immediately formulae (60), (61).

Examining more closely (61), we obtain from (38):

$$(73) \qquad\qquad J/I = \left(Q k_2 g I \psi_0 + k_1 k_4 \psi_0 - k_1 k_5\right) / (Q k_2 g I) \,.$$

Taking $I \doteq \mathscr{T}_v^2 / \mathscr{T}_1^2 - k_4 / (Q x_0 g)$ we can write:

$$(73') \qquad J/I \doteq \left(\mathscr{M} g x_0 \psi_0 \mathscr{T}_v^2 - \mathscr{C}_\zeta \mathscr{T}_1^2 k_5\right) / \left(\mathscr{M} g x_0 \mathscr{T}_v^2 - \mathscr{C}_\zeta \mathscr{T}_1^2 k_4\right) \,.$$

The denominator can obviously be either positive, zero or negative, corresponding to I either positive, zero or negative. A positive value of the denominator thus represents the case of a suspended pendulum, a negative value the case of an inverted pendulum. In the near proximity of $I = 0$ we can expect very abrupt changes in period damping, depending on small variations of J and I. This seems to explain the strange behaviour of some pendulums under specific circumstances when the design is under stress by reason of the high sensitivity demanded by the user. Evidently, such a critical position (given by $\mathscr{T}_1$) can be established from (73') for any pendulum.

Straightforward application of formulae (60), (61) and (73') on results obtained for some pendulums yields too high a value of the initial twist ψ_{0A}, ψ_{0B} of suspensions. This gives a hint that the inherent bend of suspensions has to be taken into account if we want to interpret the present theory quantitatively as well as qualitatively. Note that if we considered the bend, the relationship between J/I and ψ_{0A}, ψ_{0B} would no longer be mediated through k_4 and k_5 given by eq. (28), while eqs. (60), (61) and (73') would still hold true.

It should be pointed out that all the formulae were derived under the assumption that the two fixing points D, E (Fig. 1) would remain stable during the swing of the pendulum. If the design of the instrument does not ensure that, we may expect very peculiar phenomena to arise. The treatment of this problem as well as that of the bend, remains unexploited and will require better equipped and more experienced researchers than the authors of this paper.

7. DETERMINATION OF SENSITIVITY

To establish the relationship between ψ_0 and J, is the ultimate aim for geophysicists studying the variations of tilt by means of horizontal pendulums. Since the interest lies in very slow time variations in J and ψ_0 we shall use the formula (38). Substituting

(50) into (38) we obtain:

$$(38') \qquad \psi_0 = J\mathscr{T}_1^2/\mathscr{T}_v^2 + k_5\mathscr{T}_1^2/(Qgx_0\mathscr{T}_v^2) \,.$$

For time variations $\delta\psi_0$, δJ we can write:

$$(74) \qquad \delta J = (\mathscr{T}_V^2/\mathscr{T}_1^2)\,\delta\psi_0 \,,$$

which is the well-known and widely used formula for establishing the required sensitivity, see for instance [13].

Thus even though $\mathscr{T}_v^2/\mathscr{T}_1^2$ is not giving the tilt along the pendulum beam, formula (74) holds. $\mathscr{T}_1$ in (74) is, of course, the zero-amplitude period determined by (60''').

Design and Installation Implications Arising from the Theory now Established

8. THE INITIAL TWIST AND BEND OF THE SUSPENSIONS

The foregoing theoretical development makes it clear that the major cause of period damping arises from the initial twist, and also implicitly from the bend, of the suspensions. These phenomena must, after all, be the fundamental reason for the existence of a residual J, the angle between g and z in the yz plane. In particular, with a certain value of ψ_0, the numerators of the right hand sides of eqs. (73) and (73') are reduced to zero and so consequently is J/I. This finding is consistent with experience in that the damping phenomenon is most marked in the behaviour of metallic pendulums where the filaments are non-integral components secured to beam and framework by clamping plates of other questionable mechanical means. It is also obvious that, other factors being equal, wire suspensions should be more prone to the fault than bands and this again is in accordance with experience. Those instruments, such as quartz pendulums, which have the advantage of integral construction, whereby all major components are connected by fusion, suffer less in this respect since the procedure of fusing the suspensions should release the major tendency to an initial twist. The quartz pendulums are not entirely free from the fault, however, for two reasons. In the first place the anchoring points for the suspensions on framework and beam seldom allow the two suspensions to lie within a single plane and thereby induce initial bends. Secondly unless great care is exercised in installing the instrument in the field, the process of bringing the pendulum into its desired azimuth will induce an initial twist. In the latter connection it is also clear that the beam deviations from the operating azimuth should be restricted so that the common arrangement involving a triplet of recording lamps, which allow for substantial deviations from the initial azimuth, should be treated with reserve. Returning to the metallic pendulum case it has been found that a condition approaching that of the quartz pendulums can be achieved after fixing the supensions. With the beam unclamped, but restrained in an operational azimuth with respect to the supporting frame, heat treatment significantly reduces initial twists and bends. Certainly with tungsten wire suspensions this procedure has been carried out effectively with no apparent alteration to the breaking strain of the filaments nor change in suspension geometry. Nevertheless, a distinct advantage would be gained at a time of suspension fitting, if instrumental design allowed the whole structure to be tilted so that the line of each suspension in turn could adopt a vertical attitude. In each case the upper anchoring point could be secured first and the suspension allowed to hang freely under the tension of a suitable weight while the

lower attachment is secured. Again construction should pay particular attention to the attitude of the suspension anchoring points which must allow the beam and its suspensions to lie as nearly as possible in a single plane.

All the above considerations suggest that when fitting the suspensions the ability to secure the pendulum beam as precisely as possible in an operational attitude, horizontal with respect to the supporting frame, parallel to the line pivot/sensitivity footscrew and normal to the line pivot/drift footscrew, is fundamental to the entire exercise. Suspensions, as free as possible from twist, should be fitted in this condition and thereby the fundamental internal azimuth of the instrument is defined. It is then desirable that installation procedures should be capable of recovering this internal azimuth and that they ensure that it coincides with the desired geographical azimuth of observation. Suggestion as to procedure would take the following form.

The instrument is set up and levelled in its operating sensitivity with the beam approximately horizontal, so that the line joining the two anchoring points of the framework is almost vertical (i.e. I is small). The beam is then secured in the fundamental internal azimuth originally used for the fitting of the suspensions. A recording lamp is arranged and the contact point of the reflected beam with a vertical plane surface is noted. The beam is then released and the sensitivity screw is adjusted so as to give the minimum convenient sensitivity. During this adjustment I is steadily increased but the beam remains in an almost vertical plane since the fundamental internal azimuth is, to all intents and purposes, normal to the line pivot/drift footscrew. With the same recording lamp, a second point of contact with the plane surface is noted. Inevitably some angular displacement of the beam is noted indicating the extent to which the secured position of the beam fails to conform to the geometry of the footscrews. This displacement should be noted and should be capable of retrieval as one of the fundamental constants of the instrument. When installing the instrument in its field station, the desired azimuth of observation may be achieved by theodolite survey of the instrument feet, allowing of course for the angular displacement previously noted. This procedure automatically takes care of any deviation of the plane of the mirror from that normal to the beam.

The importance of design and procedural considerations may be illustrated by the record of period measurement obtained with operational instruments at Bidston in the recent past, but before the implications of the present theory were fully realised. If the change in period is expressed in the form $\Delta \mathcal{T} = c \exp(-2\alpha_1 t)$ the values of the parameter, c, (obtained for the same amplitudes) which fit the observed sets are significant. A quartz pendulum, ORB 87, operating at a period of 30·46 seconds gave a value $c = 194·7$ msec. Later after the suspensions had been replaced and the instrument returned to the same station, with a similar operating period of 32·20 seconds, c was found to be 69·0 msec. The discrepancy in these figures seems to indicate inadequate control of beam attitude during the fitting of suspensions, the later exercise being more successful than the first. In the case of a second pendulum, a metallic Tomaschek/Schaffernicht instrument operating much closer to its limit than the former instrument, and fitted with tungsten wire suspensions which survived three installations, the following values were found:

Station 1	$c = 400·4$ msec.	Period 52·62 seconds
Station 2	$c = 170·1$ msec.	Period 52·74 seconds
Station 3	$c = 440·3$ msec.	Period 64·39 seconds .

Here, since the suspensions remained unchanged, the only apparent variable is installation procedure. Clearly at Station 2 the procedure happened to be more satisfactory than the other two.

9. SENSITIVITY CALCULATIONS

The conventional method of determining the sensitivity of a horizontal pendulum has been based upon the relationship $\mathscr{T}_V^2/\mathscr{T}_1^2$ given in eq. (74). Largely because of the difficulties of establishing $\mathscr{T}_1$ with precision, the practice has found less favour in recent years and in the earth tide discipline as a whole greater reliance has been placed upon calibration pulses, for example, to the quartz pendulum through the well-known "crapaudine dilatable" and even in the modern Askania vertical pendulum, where a displacement is caused by the shift of a metal ball by a known amount within the pendulum. The latter procedures lend themselves readily to automation and allow sensitivity measurements to be made from the records under laboratory conditions, thereby reducing time spent at an underground station. The question which now arises is whether, in view of the more complete understanding of the theory of motion of the horizontal pendulum, it would not be wise to return to the measurement of period as the basis of sensitivity determination.

All earth tide records and particularly those of the horizontal pendulum contain a troublesome noise level unless some mechanical or electronic damping procedure is adopted. Against this background noise, the measurement of a discrete and infrequent event such as the displacement caused by a calibration pulse must cause anxiety. By comparison the time measurement of several successive swings of an oscillating pendulum scores highly in that the multiplicity of events measured reduces the effect of random noise. It is likely therefore that sensitivities derived from period measurement will represent the condition of an instrument at an epoch more faithfully than the measurement of pulses. If instrumental sensitivity were to remain nearly constant, pulse measurements, averaged over an extended period, might produce adequate mean results for the duration of an experiment. Unfortunately, sensitivities vary considerably within each experimental period. Drift is always present in any tiltmeter record representing real or apparent tilt in the yz plane. Similar tilts occur in the xz plane which, by changing the angle I, contribute variations to sensitivity. This places a premium on the ability to determine sensitivity with precision at frequent intervals throughout the experiment if the perturbation of harmonic constants, produced by harmonic analysis of the record, is to be avoided.

Moreover all pulse generators must themselves be calibrated and in many cases this process itself is subject to significant uncertainties often of the order of 1%. There is therefore little or no latitude for the intrusion of any other uncertainty in its use. The crapaudine system is an interesting example of the difficulty involved.

(a) The crapaudine itself is first calibrated by interferometric means whereby Moire fringes are counted as a mercury reservoire is raised and lowered discretely, thus causing a dilation or contraction of the crapaudine surface in response to the pressure generated in its internal mercury-filled cavity.

(b) Later on such crapaudine is driven by a reservoire describing harmonic motion in a period of the order of 1 hour while a pendulum is mounted so that its drift screw sits upon the crapaudine. The amplitudes of swing of the pendulum, so induced, are related to period and a calibration constant of the pendulum is obtained.

(c) In addition it is possible to mount a field pendulum on a crapaudine which receives a discrete pulse, typically twice per week for a duration of 3 hours on each occasion.

The certainty of initial crapaudine calibration (a) is unknown but reference to Ducarme [14] suggests that repeated calibrations of the same crapaudine, even closely spaced in time, reveal discrepancies of the order of 1 to 2% for the older chromed crapaudines and 1% for the stainless crapaudines. Moreover there seems to be evident that the older crapaudines are temperature sensitive to an extent which is comparable with the above quoted uncertainties. The probable error of the pendulum calibration against period (b) is claimed to be of the order of 0·5% although this is an internally derived quantity and may be subject to further systematic error if, for example, the crapaudine response has a time lag and may not fully respond to a harmonic pulse of 1 hour

period. In this it is significant to note that procedures (a), (b) and (c) all rely upon different pulse periods so that a time lag in the crapaudine will produce, incompatible results. Procedure (c) must combat the noise problem previously mentioned and together with (b) relies upon the engineering ability to manufacture the instrument base to a high degree of precision. Any error in the spacing of the points of the pivot and drift screw create a proportional error in sensitivity unless this can be measured and taken into account. In use it is possible to estimate operational sensitivity by two means, firstly by measuring operating period and using calibration (b) and secondly
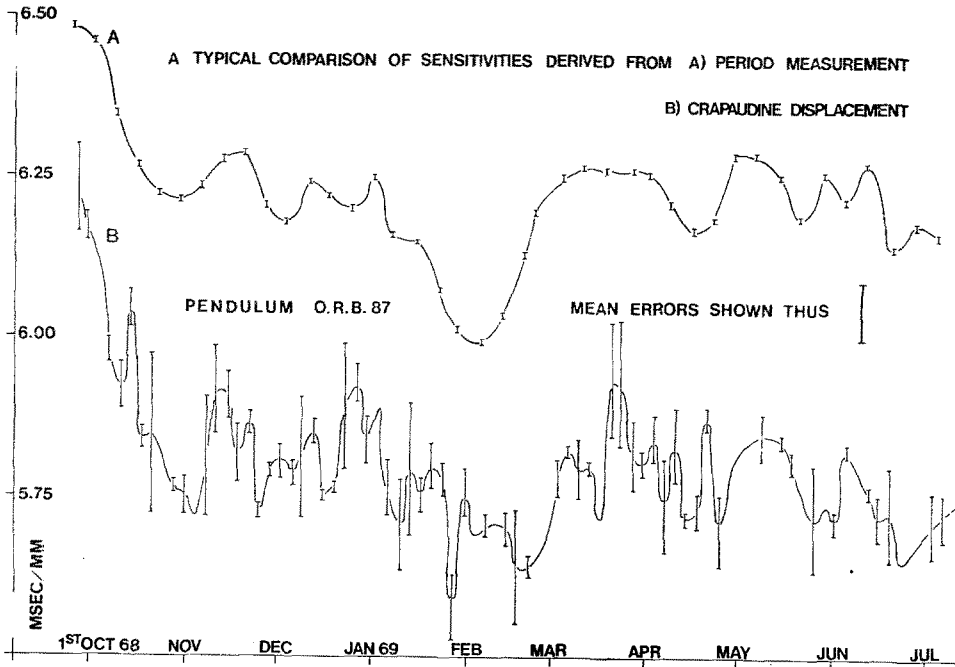

Fig. 2.

by operating the automatic crapaudine drive and using the measurement of displacement pulses calibration (c). Figure 2 shows a comparison of these two calibration procedures using a quartz pendulum in the Bidston vault. At this station the noise level is admittedly high but then so is the tidal signal so that the overall signal to noise ratio is equivalent to that anticipated at most field stations. Appropriate comments would be as follows:

1) There is a disparity between sensitivites determined by crapaudine and those determined by period measurement which rely upon a laboratory calibration (b) from a different crapaudine. Both crapaudines are of the early chromed variety. The discrepancy is of the order of 7%.

2) Although the overall pattern of variation in sensitivity is in agreement in the two systems the internal dispersion of the period measurement system is much less than that of the crapaudine pulse system. Here one should note the indication of mean errors of single observations indicated on the diagram which demonstrate quite clearly the ease with which period measurement can be determined with much greater precision than displacement measurements. Using the crapaudine system and taking pairs of observations only, that is in comparing the positive pulse with the negative pulse in each cycle of the crapaudine drive the confidence limit of sensitivity calculation in this exercise emerged as $\pm 1 \cdot 6\%$. Real confidence limits are clearly much greater than this as

shown by the plotted dispersion. In contrast, the confidence limits of sensitivity determination by period, assuming a perfect calibration of type (b) above, are of the order of $\pm 1\%$. If the exercise were to be relieved of the uncertainties of the (b) calibration by using an appropriate theory of motion as developed in this paper, then it is clear that real confidence limits must be very close to this figure.

3) It is important to note that the displacement pulse system allows little improvement in technique since the problems arise from the measurement of single displacements in the presence of noise. Pendulum damping or output damping might possibly give some improvement, although such procedures often create their own problems and uncertainties. It is possible also that sophisticated treatment of a digital output might also effect some advantage, but the basic problem remains. In contrast the data plotted in figure 2 and the confidence limits quoted above for period measurements rely upon the use of a simple manual stopwatch. More recently tests with an electronic timer using a quartz crystal have been shown to reduce confidence limits by one half, and moreover it is clear that the automation of period measurements would greatly improve the precision, probably by one order of magnitude. The use of a light spot follower with a fast recording speed is one technique which would make this possible and this is under test. The construction of a photocell sensing system for period timing is also a possibility which has been introduced elsewhere.

The conclusion which must be drawn is that a period measurement system, as a basis for sensitivity calculation, has a greater potential than any pulse displacement system and further that a theoretical basis of pendulum motion is the closest means of approach to an absolute calibration. There remain two points which require clarification. In the first place one must be assured of a precise determination of $\mathscr{T}_v$. This presents no real difficulty if pendulums are provided with a knife edge for swinging in a vertical altitude and if a simple evacuated container with a photocell counter is provided for the purpose. At Bidston a typical precision attained in this procedure is 0·1 milliseconds using 10,000—20,000 swings. In the second place it is essential that the installation of the pendulum in the horizontal altitude be accompanied by a precise measurement of the discrepancy by which the knife edge fails to lie in the line of the two anchor points C and D. This is readily achieved by theodolite observation with the assistance of a simple scale mounted parallel to, and closely behind, the pendulum beam. By such a measurement the experimentally derived value for $\mathscr{T}_v$ can be corrected, while maintaining the same order of precision.

In recommending the above procedure for calibration it is possible to quote supporting evidence from experimental work at Bidston. Here at a surface station barely 6 metres in depth and below an occupied building, three metallic pendulums were installed side by side so as to measure the same directional component of tilt over a period of three months in the summer of 1968. The analysis of their records gave the following reductions for the major lunar tidal constituent, M_2:

Amplitude in millisecs	Phase lag on equilibrium tide at Greenwich
67·06	316·8°
67·28	318·0°
66·65	317·9°

It is clear from the above that the determination of sensitivities together with operating procedures managed to achieve an experimental precision of the order of 1%. It can also be claimed that when variations in sensitivity during this period were interpreted as long period tilts in a direction normal to their recording azimuth these were consistent, one with another, and moreover fitted the long period drift shown by other instruments at the same station which had been installed in the complementary azimuth. The agreement of the above results should be assessed in the light of the peculiarly hostile environment of the Bidston vault where in particular temperature changes range over approximately 5 °C.

Other factors are worthy of comment. In the 1968 exercise, period measurements were made using 16 consecutive swings of the pendulum and with the aid of a manual stopwatch. $\mathscr{T}_1$, the zero amplitude period, was determined in each case by a simple linear fit to these observed quantities. Since this time, experience has resulted in more sophisticated techniques. In addition to the improved timers to which reference has already been made, it is now the practice to divide the period measurements into two phases, separated by a pause used for attention to another instrument. This practice allows for significant amplitude, and consequently period, damping to occur during the measuring interval. Furthermore, as a result of the greater knowledge of the pendulum theory now acquired, the $\mathscr{T}_1$ extrapolation is derived from a quadratic rather than a linear fit. It is therefore felt that an overall experimental precision significantly better than 1% can now be achieved.

10. MISCELLANEOUS NOTES

There remain several matters appertaining to instrumental design which arise from this work. Their comprehensive treatment, however, would not be possible within the scope of this single paper. It is proposed that brief references should now be made to the more apparent considerations which occur to the authors.

Mention has been made of the secondary frequency, μ_2, which may prove to be tiresome in the measurement of operating period, but is no more serious than this. It has been stated that, given due care in the initiation of pendulum swings, a_2 can be kept within negligible limits and in any case it is rapidly dissipated. The phenomenon may be more significant if it happens to be excited by some external oscillation, perhaps a common microseismic frequency or possibly by a systematic content of local noise which results from some natural resonance in the immediate vicinity of the station. In order to improve sensitivity calculations or to clean up the recorded output it may be desirable to ensure that μ_2 is not matched to these local perturbations. Again it is not inconceivable that in certain instrumental types, or with certain instrumental output devices, where there is risk of phenomena such as friction, stickiness or filter lag, it may be desirable to deliberately encourage the generation of a_2 by local phenomena. In such circumstances it may be advisable to control the μ_2 frequency by design considerations based upon eq. (41): $\mu_2 \doteq \sqrt{(k_1 + Qk_2x_0)}$.

Much has been written elsewhere on the choice of material and dimensions of the suspensions. The findings here confirm the importance of striking the optimum compromise between the modulus of torsion and mechanical strength while retaining low creep properties under stress and corrosive resistance. If the present findings make any contribution in this field, they emphasize the importance of minimizing the torsional rigidity of the suspension perhaps even at the expense of its mechanical strength. In this way the aim is to seek a minimum value for $\mathscr{J}_c$ in k_4 and k_5, see eq. (28). By this means the values of k_4 and k_5 in eq. (73') will be decreased so that J/I will depend as nearly as possible on ψ_0 only. Geary [15] investigates the problem of selecting a suspension of small cross sections, of optimum shape but with a low shear modulus and has produced an index of torsional rigidity, μ, so that suspension ligaments of identical breaking loads can be compared. So as to avoid ambiguity with other expressions in this paper this parameter will be renamed μ_g here. Tungsten wire was selected for the Bidston instruments because its μ_g value was 0·07, compared with 0·18 for platinum nickel wire, and 0·25 for both phosphor bronze wire and quartz fibre. Geary points out however that strip suspensions have many advantages over wire for this purpose since for the same cross section, and consequently for the same breaking load, strip has a lower torsional rigidity. For example phosphor bronze strip has a μ_g of 0·05, platinum nickel strip a μ_g of 0·036 and beryllium copper strip a μ_g of 0·024. The latter material will be investigated more intensively for use in future pendulum applications. In addition to the above mechanical properties, the advantages in clamping a strip, as opposed to a wire, are obvious as also is the inherent implication with regard to initial twist and bend. Having selected the materi-

al it is still necessary to study problems of instrumental geometry leading to considerations of load and breaking strain so that dimensions may be minimized. Otherwise advantages gained in the choice of material will be lost. At Bidston it has been found satisfactory merely to ensure that the suspension is unlikely to undergo a stressing within $\frac{2}{3}$ of its elastic limit.

The theory of pendulum motion developed here does suggest that certain advantages may be gained by a re-examination of the design geometry of the horizontal pendulum. For example it can be seen that increasing the length of both suspensions will effectively decrease all the coefficients k_1 to k_5. This would influence J/I in eq. (73′) and also, for that matter, μ_2 in eqs. (41) and (45). By differential changes in the suspension lengths l_1 and l_2 the k coefficients can be influenced separately and a measure of control can be established over all the appropriate parameters $\alpha_1, \mu_1, \mu_2, J/I$. Again earlier remarks have stressed the desirability of minimizing J but it should be noted that increasing I would have a similar effect. Moreover the implications of critical values of I together with small variations in J and I have already been discussed in the comments in the text upon eq. (73′). An intensive study of such design implications might result in a closer approach to an ideal instrument design but the general conclusion remains, namely that the operation of the Zöllner pendulum in its existing form is amenable to interpretation in terms of real tilts with a precision of calibration acceptable in the framework of geophysics.

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