Chapter 4

Harmonic Continuation and Gravimetric Inversion of Gravity in Areas of Negative Geodetic Heights

P. Vajda, A. Ellmann, B. Meurers, P. Vaníček, P. Novák, and R. Tenzer

Abstract By the decomposition of the real earth's gravity potential it can be shown that the attraction of the anomalous mass density, which is sought as the unknown in gravimetric inversion (gravity data interpretation), matches exactly the gravity disturbance corrected for the attraction of topography and bathymetry (the BT disturbance), and eventually also for attractions of other known density contrasts, such as sediments, lakes, glaciers, isostatic roots, etc (the stripped BT disturbance). The involved (global) topographic correction requires the use of reference ellipsoid (RE) as the bottom interface of topographic masses. Topographic correction based on the RE introduces the attraction of "liquid topography" offshore, which is the attraction of sea water between the RE and sea level (geoid). The topo-correction onshore requires the use of reference (such as constant average crustal) topographic density for the "solid topography". The ultimate knowledge of real topo-density is avoided, since anomalous density relative to the reference topo-density is part of the interpretation (is sought). In areas of negative geodetic heights, both onshore (e.g., Dead Sea region) and offshore (negative geoidal heights), we run into the problem of evaluating the normal gravity and the problem of the legitimacy of the upward harmonic continuation of the gravity data to be interpreted (inverted). We propose to overcome these problems by a new approach based on the concept of the reference quasi-ellipsoid (RQE). The gravimetric inverse problem is first reformulated based on the RQE that replaces the RE in the decomposition of actual potential. The RQE approach enables for stations of negative heights the use of the international gravity formula (IGF) for computing normal gravity at the station, and facilitates the legitimacy of the harmonic continuation in regions of negative heights. Second, the gravity data (the RQE-based BT disturbances) are continued onto or above the RE. Third, the inverse problem is transformed back to be formulated with respect to the RE, and solved using classical known techniques.

4.1 Gravity Data Inversion/Interpretation

Gravimetric inversion or gravity data interpretation, the objective of which is to find subsurface anomalous density distribution by inverting (or interpreting) anomalous gravity data compiled from observed gravity, is based on the fact that we can compile such an anomalous gravity quantity that matches the attraction (vertical component of the gravitational attraction vector) of the sought anomalous density. By the decomposition of earth's gravity potential it can be shown (Vajda et al., 2006, 2008a) that the anomalous gravity quantity that is exactly equal to the attraction δA of the unknown anomalous density distribution $\delta\rho$ globally enclosed onshore by the relief and offshore by sea bottom is but the bathymetrically and topographically corrected gravity disturbance, the so called "BT disturbance",

$$\delta g^{BT}(h,\Omega) = \delta A(h,\Omega), \qquad (1)$$

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defined as follows:

$$\delta g^{BT}(h,\Omega) = g(h,\Omega) - \gamma(h,\Omega) - A^{ET}(h,\Omega) - \delta A^{EW}(h,\Omega), \tag{2}$$

where g is actual (observed) gravity, γ is normal gravity, $(-A^{ET})$ is the (global) topographic correction based on the *reference ellipsoid* (RE), and $(-\delta A^{EW})$ is the (global) bathymetric correction based on the RE. Both the corrections are computed at the observation point (station) using the Newtonian volume integral (Vajda et al., 2006, 2008a and references therein). Points are referred and positioned in geodetic (Gauss-ellipsoidal) coordinates, h is height above the RE, Ω is the pair of geodetic latitude and longitude. The RE is geocentric, properly oriented, biaxial (rotational), and equipotential (level), thus identical with the normal ellipsoid generating the normal gravity.

The gravity disturbance is defined based on normal gravity generated by the RE. Consequently, the decomposition of actual potential resulting in Eqs. (1) and (2) must be based on the RE, which implies that topographic and bathymetric corrections are defined based on the RE instead of geoid/quasigeoid ("sea level" as vertical datum). The use of the RE means that in terms of the topo-correction we have the attraction of the so called "reference solid topography" and the attraction of the so called "liquid topography". The "reference solid topography" is the reference rock density distribution $\rho_R(h,\Omega)$ (such as constant average crustal density ρ_0) globally enclosed onshore between the RE and the relief. The "liquid topography" is the (constant) density of sea water ρ_W enclosed between the RE and the geoid (positive above and negative below the RE), cf. Fig. 4.1. The bathymetric correction removes the attraction of the sea water density contrast $(\rho_w - \rho_0)$ offshore, globally enclosed between the sea bottom and the RE.

While in geodesy the normal density distribution of the normal ellipsoid (the RE) generating the normal potential and normal gravity may remain unspecified (normal potential is selected as a mathematical prescription) and even zero to some depth below the surface of the RE, in geophysics we must find a model normal density distribution $\rho_N(h,\Omega)$ (such as that of the Preliminary Reference Earth Model, PREM), cf. (Vajda et al., 2006, and references therein), which serves as the background density defining the (unknown and sought) density contrast $\delta\rho(h,\Omega)$ inside the RE (below the surface of the RE onshore and below the sea bottom offshore). Within the solid topography (between the RE and relief onshore), the sought $\delta\rho(h,\Omega)$ is defined relative to the ρ_0 background density.

Normal gravity in Eq. (2) is computed by the international gravity formula (IGF) which includes the height term (e.g., Heiskanen and Moritz, 1967).

The gravimetric inverse problem, formulated based on Eq. (1), is solved typically either by direct inversion or by forward modeling techniques, or eventually the gravity data are interpreted in terms of pattern recognition (e.g., Blakely, 1995). In either case the problem may be formulated for observation points at their natural positions (on relief, on sea surface, at flight trajectories, in boreholes, at sea bottom, etc.). However, in the case of forward modeling it is often required to have the gravity data on the surface of a half-space (planar approximation) or a sphere (spherical approximation), when the modeling software does not consider the relief of the earth. In such a case, the upward continuation of gravity data (BT gravity disturbances) to a reference surface is needed.

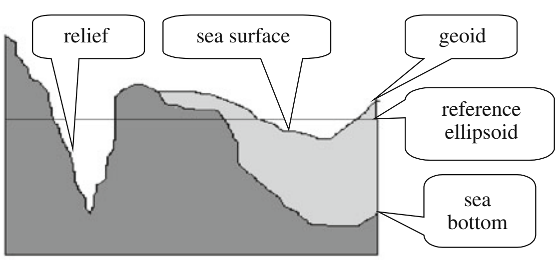

Fig. 4.1 Regions of negative geodetic heights

4.2 Regions of Negative Heights

Complications arise with the interpretation or inversion of gravity data in areas, where observation points have negative geodetic heights, such as in the Dead Sea region onshore, or offshore where geoidal heights are negative (Fig. 4.1). A point with negative height lies inside (below the surface of) the RE. The presence of a non-zero model normal density distribution $\rho_N(h,\Omega)$ there (1) violates the use of the IGF for normal gravity computation (which must be there evaluated by the Newtonian volume integral over the ρ_N inside the RE), and (2) violates the harmonic continuation of the BT disturbance given by Eq. (2) in the "free-air" region between the relief (of negative geodetic heights) and the surface of the RE. Normal masses ρ_N are present in this "free-air" region, thus the δg^{BT} is not harmonic there.

We propose to overcome these two problems by means of the so called reference quasi-ellipsoid (RQE) approach, which makes use of a remove-restore technique for the upper layer of the model normal masses of the RE.

4.3 RQE Approach

The decomposition of the actual potential, which has lead to Eqs. (1) and (2) that are the basis for formulating the inverse problem (Vajda et al., 2006, 2008a), was performed based on the RE. Let us now replace the RE by the so called reference quasi-ellipsoid (RQE) in the decomposition of the actual potential, cf. Fig. 4.2.

The RQE is defined as the surface the depth of which h^* (reckoned along the ellipsoidal normal) below the surface of the RE is constant. As such, it is no longer an ellipsoidal surface, hence the name. The value of h^* is chosen so, that it is just greater than the maximum dip of the relief below the RE elsewhere over the entire globe, e.g., as 500 m. We let the RQE serve as both the lower boundary of the "topographic masses" and the upper boundary of the "normal masses". This implies that we have to define a

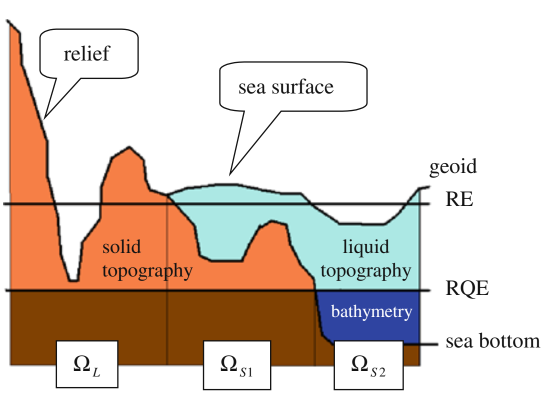

Fig. 4.2 The reference quasi-ellipsoid (RQE) approach

new model normal density distribution $\rho_N^*(h,\Omega)$ bound by the surface of the RQE, which generates the same normal potential and normal gravity in the exterior of the RE as is generated by $\rho_N(h,\Omega)$ bound by the surface of the RE. In our approach it is vital that the $\rho_N(h,\Omega)$ consists of an upper layer of constant density equal to ρ_0. We remove this upper layer of $\rho_N(h,\Omega)$ and restore it in an unspecified manner below the level of the deepest sea bottom to form an unspecified $\rho_N^*(h,\Omega)$. We do not require the $\rho_N^*(h,\Omega)$ to meet the geophysical constraints (cf. Vajda et al., 2008b, and references therein), and we even leave it unspecified. We only require that it generates the usual (IGF) normal gravity above the RE. In this approach the space between the surfaces of the RQE and of the RE is void of normal masses.

The decomposition based on the RQE leads to the following exact link (match) between the gravity and the attraction of anomalous density

$$\delta g^{BT*}(h,\Omega) = \delta A^*(h,\Omega), \tag{3}$$

where δA^* is the attraction of the sought anomalous density distribution $\delta\varrho^*(h,\Omega)$ now defined as follows: offshore below the sea bottom and onshore below the surface of the RQE relative to the background density distribution $\rho_N^*(h,\Omega)$ and onshore between the RQE and the relief relative to the average crustal density ρ_0, and where the BT disturbance is now defined with topographic and bathymetric corrections based on the RQE

$$\delta g^{BT*}(h,\Omega) = g(h,\Omega) - \gamma(h,\Omega) - A^{QET}(h,\Omega) - \delta A^{QEW}(h,\Omega). \tag{4}$$

Now the normal gravity in Eq. (4) can be computed using the IGF even at observation points of negative geodetic heights, because the space between the RQE and RE surfaces is void of normal masses. Also, now the new BT disturbance based on the RQE (in spherical approximation multiplied by geocentric distance), $(R+h)\delta g^{BT*}(h,\Omega)$, is harmonic (because even the space above the relief of negative heights is now free of any masses) and can be harmonically upward continued. The RQE approach resolves for observation points of negative heights both the problems: the problem of normal gravity computation and that of the legitimacy of harmonic continuation.

While the new normal density distribution $\rho_N^*(h,\Omega)$ inside the RQE, generating the normal gravity given by the IGF, remains unspecified, the formulation of the gravimetric inverse problem based on the RQE and $\rho_N^*(h,\Omega)$ remains meaningless, because there is no use of a solved anomalous density defined relative to an unspecified background density. We do not intend to solve the inverse problem in the RQE approach. The RQE approach is used to resolve the problem of normal gravity computation and the upward continuation of the BT disturbance at stations of negative heights. The RQE approach is used only to compile the BT disturbance in such a way that the IGF may be used for computing normal gravity, and in order to harmonically upward continue the BT disturbance (based on RQE) onto or above the surface of the RE. Once we have the BT gravity disturbance (based on the RQE), δg^{BT*}, on or above the RE, we transform the formulation of the inverse problem back to being based on the RE and $\rho_N(h,\Omega)$, which is described in the next section.

4.4 Reverting from RQE to RE Approach

After harmonic upward continuation of the RQE-based BT gravity disturbance onto or above the RE, we have the inverse problem formulated based on the RQE, but for stations on or above the RE now

$$\forall h \geq 0:\ \ \delta g^{BT*}(h,\Omega) = \delta A^*(h,\Omega). \qquad (5)$$

In the previous section we removed and restored the upper layer (between the RE and RQE surfaces) of the normal masses $\rho_N(h,\Omega)$, the upper layer being of constant density ρ_0. Now (already having all stations at which we interpret (invert) gravity data on or above the RE) we revert this step, which implies that we have to add the attraction (in spherical approximation) of this quasi-ellipsoidal layer of constant density ρ_0 and constant thickness h^*

$$A^{QELC}(h,\Omega) \approx 4\pi G \rho_0 h^*, \qquad (6)$$

G being the gravitational constant, to both sides of Eq. (5). We realize that its left-hand side becomes

$$\forall h \geq 0:\ \delta g^{BT*}(h,\Omega) + A^{QELC}(h,\Omega) = \delta g^{BT}(h,\Omega),$$

because the addition of A^{QELC} transforms the sum of the RQE based topo– and bathymetric corrections to the sum of the RE-based topo– and bathymetric corrections, thus transforming the RQE-based BT disturbance δg^{BT*} into the RE-based BT disturbance δg^{BT}. We also realize that the right-hand side of Eq. (5) becomes

$$\forall h \geq 0:\ \delta A^*(h,\Omega) + A^{QELC}(h,\Omega) = \delta A(h,\Omega),$$

because $\rho_N^*(h,\Omega)$ becomes $\rho_N(h,\Omega)$ and $\delta\varrho^*(h,\Omega)$ becomes $\delta\varrho(h,\Omega)$. Consequently, Eq. (5) becomes

$$\forall h \geq 0:\ \delta g^{BT}(h,\Omega) = \delta A(h,\Omega). \qquad (7)$$

and we have the inverse problem formulated based on the RE and $\rho_N(h,\Omega)$ using the RE-based BT gravity disturbances as observed gravity data to be interpreted or inverted.

4.5 Case Study

We have computed the RQE-based topographic and bathymetric corrections, as well as the RQE-based BT gravity disturbances in a test region comprising the NW part of North America including a portion of the Pacific. They are shown in Figs. 4.3, 4.4 and 4.5, respectively.

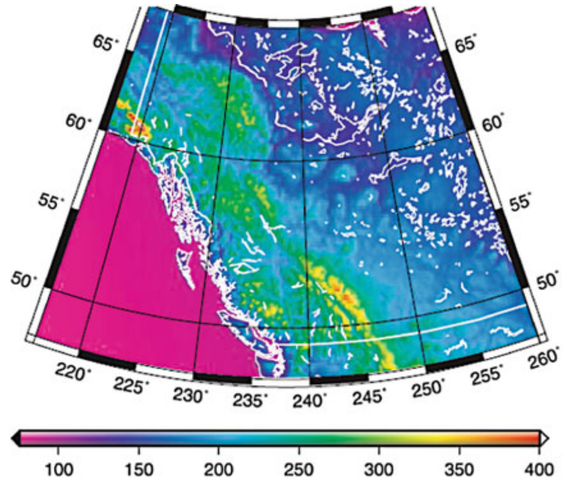

Fig. 4.3 The attraction (A^{QET}) of the RQE-based topography (both solid and liquid, cf. Fig. 4.2) evaluated on the topo-surface in our test region (mGal)

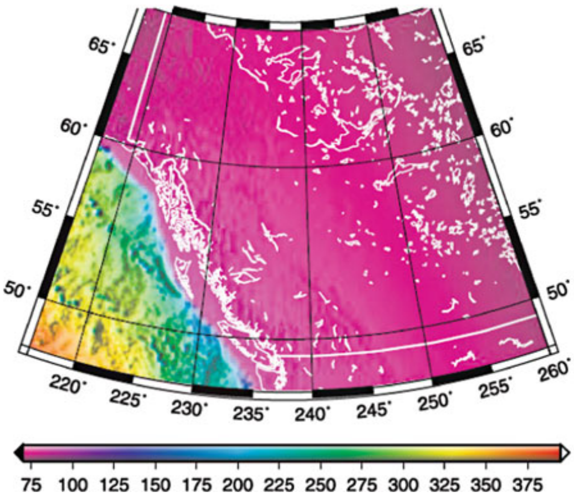

Fig. 4.4 The RQE-based bathymetric correction ($-\delta A^{QEW}$) in our test region computed on the topo-surface (mGal)

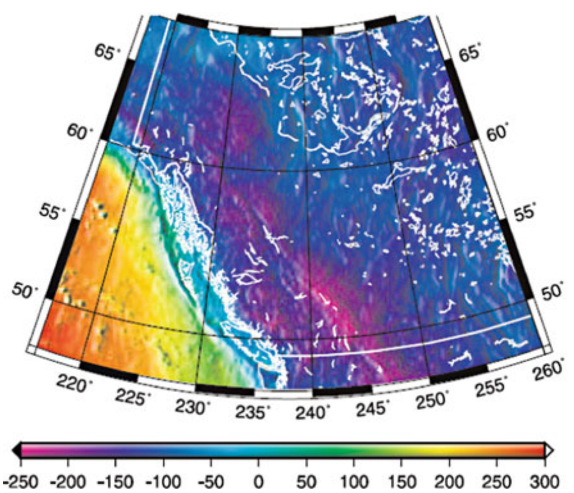

Fig. 4.5 The RQE-based BT gravity disturbance (δg^{BT*}) given on the topo-surface in our test region (mGal)

4.6 Conclusion

When the gravimetric inverse problem is formulated in terms of the attraction (vertical component of the gravitational attraction vector) of the sought unknown anomalous mass density distribution (the attraction being a Newtonian volume integral over the anomalous density distribution), the gravity data that match this attraction (Eq. (1)) and that must be compiled as input data entering the inversion or interpretation are the BT gravity disturbances (Eq. (2)). In areas of negative geodetic heights of gravity stations (observation points) the normal gravity in the definition of the BT disturbance (Eq. (2)) can no longer be computed using the IGF. Instead, it would have to be computed as inner normal gravity of the normal ellipsoid (the RE) using the Newtonian volume integral for attraction, over the normal density distribution of the RE. In "free air" areas between the topo-surface of negative geodetic heights and the surface of the RE, the BT disturbance (in spherical approximation multiplied by geocentric distance) is not harmonic due to the presence of normal masses below the surface of the RE. Consequently its harmonic continuation is not legitimate.

The above two problems complicate the compilation and inversion (interpretation) of gravity data (BT disturbances) at stations (in areas) of negative geodetic heights. They are tackled by our so called RQE approach that consists of four steps:

(1) The gravimetric inverse problem is formulated based on the RQE instead of based on the RE, as described in Sect. 4.3.
(2) The topographic and bathymetric corrections are computed and the BT disturbance compiled based on the RQE. In the RQE approach the normal gravity can be computed using the IGF even at stations of negative heights. In the RQE approach the BT disturbance (in spherical approximation multiplied by geocentric distance) is harmonic everywhere above the topo-surface even when the topo-surface is of negative heights.
(3) The RQE-based BT disturbances are harmonically upward continued onto or above the RE.
(4) The formulation of the inverse problem is now for all stations on or above the RE reverted to the RE approach and the BT disturbances are inverted or interpreted, cf. Sect. 4.4.

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