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Evaluation of topographical effects in precise geoid computation from densely sampled heights

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Abstract

In this paper we investigate the behaviour of Newton's kernel in the integration for topographical effects needed for solving the boundary value problem of geodesy. We follow the standard procedure and develop the kernel into a Taylor series in height and look at the convergence of this series when the integral is evaluated numerically on a geographical grid, as is always the case in practice. We show that the Taylor series converges very rapidly for the integration over the "distant zone", i.e., the zone well removed from the point of interest. We also show that the series diverges in the vicinity of the point of interest when the grid becomes too dense. Generally, when the grid step is smaller than either the height of the point of interest, or the difference between its height and those of the neighbouring points. Thus we claim that the Taylor series version of Newton's kernel cannot be used for evaluating topographical effects on too dense a topographical mesh.

1. Introduction

The strong gravitational field induced by topographical masses poses a difficulty in solving the geodetic boundary value problem for geoid determination. One possible way of considering this field is as follows (Moritz (1966, 1968); Wichiencharoen (1982); Vaníček and Kleusberg (1987); Wang and Rapp (1990); Sideris and Forsberg (1990); Heck (1992); Martinec and Vaníček (1993a,b)): first, the Newton integral is formulated for the potential of topographical masses. Then its kernel is expanded by means of Taylor's series with respect to the radial coordinate at a point on the geoid. Then integration over the vertical coordinate is carried out analytically. Finally, the singularity of each individual term of Taylor series is removed. The zero-degree term of the Taylor expansion corresponds to the potential of a single material layer (in geodetic literature called the condensation layer). It describes the behaviour of the bulk of the potential of the topographical masses and is usually included into an unknown anomalous gravitational potential (Martinec et al., 1993). The potential generated by higher order terms of the Taylor series can therefore be viewed as corrections to the condensation layer potential; provided that the density of the topographical masses is known, the Newton integral can be used to compute this residual potential. This is the basic idea of Helmert's second condensation technique (Helmert, 1884).

A questionable point of the above procedure, pointed out, e.g., by Heck (1992), is whether the Taylor series converges or not, and if so, how many terms of the series should be taken into consideration to describe the gravitational potential with a prescribed accuracy. It is the usual practice in geodesy (Moritz, 1968; Vaníček and Kleusberg, 1987; Sideris and Forsberg, 1990; Forsberg and Sideris, 1993) to take only a few first terms of the Taylor series (most often only the first three) and assume that the rest of the series may be neglected. This seems to be a good enough approximation for a flat terrain when the topographical heights can be taken on a grid of a large step size (e.g. 0.5 degree). Then the dummy point in the numerical integration (of the Newton integral) never comes too close to the computation point and the magnitudes of higher order terms of the Taylor series remain small.

The problems appear when the gravitational potential of the topographical masses is computed in a rugged terrain. For such a case, a grid of topographical heights has to be fairly dense to express the ruggedness of the terrain. The dummy point in the Newton integration than comes close to the computation point and the magnitudes of higher order Taylor terms increase faster than the magnitudes of lower order terms. As a result, higher order terms become dominant and the series no longer converges.

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residuals would indicate that our observations contain distortions that have not been appropriately modelled.

The Case Involving Two Datums

Sometimes, geodetic work is done in an area referred to two geodetic datums. The transformation of positions $\mathbf{r}^{G_1}$ referred to the first datum to positions $\mathbf{r}^{G_2}$ referred to the second datum has to go through the CT-system. We thus need two sets of transformation parameters: one set to transform from G_1 to CT, and another to transform from G_2 to CT, including their covariance matrices, and the two network distortion models. The transformation $\mathbf{r}^{G_1} \rightarrow \mathbf{r}^{G_2}$ should be carried out in the following steps:

(1) The modelled systematic distortions of the network referred to G_1 should be subtracted from the distorted positions $\mathbf{r}^{G_1}$ of base points to give undistorted positions $(\phi, \lambda)_{G_1}$. This step will not be applicable if the network distortions are not known.

(2) Horizontal undistorted (corrected for distortions) positions $(\phi, \lambda)_{G_1}$ are then transformed to Cartesian coordinates $\tilde{\mathbf{r}}^{G_1}$ (for h = 0) in the G_1-system, using equation (12), including their covariance matrices.

(3) Cartesian coordinates $\mathbf{r}^{G_1}$ in the G_1-system are transformed into the CT-system using equation (10) with the first set of transformation parameters $(\omega_0, \mathbf{t})_1$ and their covariance matrix as well as the covariance matrix of $\mathbf{r}^{G_1}$.

(4) If the two involved geodetic datums have different shapes and sizes, $(a, b)_1 \neq (a, b)_2$, then the Cartesian coordinates $\tilde{\mathbf{r}}^{CT}$ (and their covariance matrices) must be transformed onto the second ellipsoid $(a, b)_2$ by the following transformations:

(5) $$\tilde{\mathbf{r}}^{CT} \rightarrow \begin{bmatrix} \phi \\ \lambda \\ h \end{bmatrix}_{G_2} \rightarrow \begin{bmatrix} \phi \\ \lambda \\ 0 \end{bmatrix}_{G_2} \rightarrow \tilde{\tilde{\mathbf{r}}}^{CT} \quad (17)$$

(6) If the two datums have the same size and shape, then $\tilde{\mathbf{r}}^{CT} = \tilde{\tilde{\mathbf{r}}}^{CT}$ and no such transformation is required.

(7) Cartesian coordinates $\tilde{\tilde{\mathbf{r}}}^{CT}$ and their covariance matrices are then transformed into the second geodetic coordinate system G_2 using equation (11) with the second set of transformation parameters $(\omega_0, \mathbf{t})_2$, taking into account their covariance matrix.

(8) Cartesian coordinates $\tilde{\mathbf{r}}^{G_2}$ and their covariance matrix are now transformed to $(\phi, \lambda)_{G_2}$ using the inverse of equation (1). We note that the resulting height h_{G_2} should automatically equal to zero.

(9) Finally, the modelled distortions of the network referred to G_2 should be added to the transformed (undistorted) positions $(\phi, \lambda)_{G_2}$ to give distorted positions compatible with the positions of the points referred to G_2 to begin with. Again, this step will not be applicable if network distortion is not known.

We note that if we neglect to model network distortions in the first, second, or both networks, we will end up with very large estimated errors for the point positions.

Conclusions

Procedures for transforming coordinates between a horizontal geodetic datum and the CT-system, and for transforming coordinates from one geodetic datum to another were given. The importance of clearly separating these transformations (between coordinate systems) from the treatment of systematic and random errors in network coordinates was pointed out.

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In this paper, instead of expanding the Newton kernel into a Taylor series and removing singularities of each terms of the series separately, we remove the singularity of the Newton kernel right in the definition of the Newton integral. By subtracting and adding the value of the Newton kernel at the computation point to the Newton integrand, the singular computation point may be left out from the integration domain, and the singularity is thus removed. This necessitates the evaluation of Newton's integral over a fixed height and a fixed mass density, which can be done analytically, resulting in the gravitational potential of a spherical Bouguer shell.

All theoretical considerations are made under the assumption that the density of the topographical masses varies only laterally. This enables us to evaluate the radial integral of the Newton kernel analytically yielding an independent tool for investigating the convergence of the Taylor series discussed above. We will see that this series has the most unfavorable behaviour when it is evaluated at a point on the geoid. Therefore, the numerical tests will be carried out for points on the geoid. This, incidently, corresponds to the evaluation of the so-called primary indirect topographical effect on potential (Heiskanen and Moritz, 1967, sect. 3-6.; Wichiencharoen (1982); Martinec and Vaníček (1993a)). This technique works, of course, the same way for any other point and can thus be used in the evaluation of the direct topographical effect as well elsewhere.

2. The gravitational potential of topographical masses

Let the topographical masses be bounded below by the the geoid with geocentric radius $r_g(\Omega)$ and above by the topographical surface with geocentric radius $r_g(\Omega)+H(\Omega)$. It means that $H(\Omega)$ is the height of the topographical surface above the geoid, reckoned along the geocentric radius. The argument Ω stands for a horizontal position given by co-latitude ϑ and longitude λ. The gravitational potential V^t induced by the topographical masses at an arbitrary point (r,Ω) is given by Newton's volume integral

$$V^t(r,\Omega) = = G\int_{\Omega'}\int_{r'=r_g(\Omega')}^{r_g(\Omega')+H(\Omega')} \varrho(r',\Omega')L^{-1}(r,\psi,r')r'^2dr'd\Omega' , \tag{1}$$

where G is Newton's gravitational constant, $\varrho(r,\Omega)$ is the density of the topographical masses, $L^{-1}(r,\psi,r')$ is the Newton kernel (reciprocal spatial distance between the dummy point (r',Ω') and the computation point (r,Ω)):

$$L^{-1}(r,\psi,r') = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\psi}} , \tag{2}$$

ψ is the angular distance between the geocentric directions Ω and Ω', and the integration over Ω' in eqn.(1) is taken over the full solid angle.

Let us now abbreviate the notation for the orthometric heights $H(\Omega)$ by dropping the argument Ω; we will use H instead of $H(\Omega)$ for the orthometric height of the topographical surface in the direction Ω and H' instead of $H(\Omega')$ for the orthometric height of the topography in the direction Ω'.

Equation (2) shows that Newton's kernel grows to infinity when the dummy point moves towards the computation point, i.e.,

$$\lim_{\psi\to 0} L^{-1}(r,\psi,r')\Big|_{r'\to r} \to \infty . \tag{3}$$

But, the Newton kernel is only **weakly singular** which means that for $r \neq 0$ (Kellogg, 1929, Chapter VI):

$$\lim_{\psi\to 0} L^{-1}(r,\psi,r')\Big|_{r'\to r} \sin\psi < \infty . \tag{4}$$

Writing the element $d\Omega'$ of the full solid angle in polar coordinates (ψ,α) as $d\Omega' = \sin\psi d\psi d\alpha$, the weak singularity property (4) is reflected also in the integral form

$$\int_{\Omega'} L^{-1}(r,\psi,r')d\Omega' < \infty , \tag{5}$$

valid for all non-zero radii r and r'.

Further, throughout the paper, the radius of the geoid $r_g(\Omega)$ is approximated by a mean radius of the earth, R, and the actual density of the topographical masses $\varrho(r,\Omega)$ by a column average value $\overline{\varrho}(\Omega)$ (the arguments for such approximations are discussed by Martinec and Vaníček (1993a)), the potential V^t takes the following form:

$$V^t(r,\Omega) \doteq G\int_{\Omega'}\overline{\varrho}(\Omega')\int_{r'=R}^{R+H'} L^{-1}(r,\psi,r')r'^2dr'd\Omega' . \tag{6}$$

The property (5) may now be utilized for removing the singularity of the Newton integral kernel (6). Subtracting and adding a term

$$V^B(r,\Omega) = G\,\overline{\varrho}(\Omega)\int_{\Omega'}\int_{r'=R}^{R+H} L^{-1}(r,\psi,r')r'^2dr'd\Omega' \tag{7}$$

to the potential $V^t(r,\Omega)$, we get

$$V^t(r,\Omega) = V^B(r,\Omega) + V^R(r,\Omega) , \tag{8}$$

where

$$V^R(r,\Omega) = G\int_{\Omega'}\left[\overline{\varrho}(\Omega')\int_{r'=R}^{R+H'} L^{-1}(r,\psi,r')r'^2dr' - \overline{\varrho}(\Omega)\int_{r'=R}^{R+H} L^{-1}(r,\psi,r')r'^2dr'\right]d\Omega' . \tag{9}$$

The quantity V^B is easily recognized as the potential of a spherical Bouguer shell of density $\overline{\varrho}(\Omega)$ and thickness

H. This potential is finite due to inequality (5) and is equal to (Wichiencharoen, 1982),

$$V^B(r,\Omega) = \begin{cases} 4\pi G\overline{\varrho}(\Omega)\frac{1}{r}\left[R^2H + RH^2 + \frac{1}{3}H^3\right], & r \geq R+H, \\ 2\pi G\overline{\varrho}(\Omega)\left[(R+H)^2 - \frac{2}{3}\frac{R^3}{r} - \frac{1}{3}r^2\right], & R \leq r \leq R+H, \\ 4\pi G\overline{\varrho}(\Omega)\left[RH + \frac{1}{2}H^2\right], & r \leq R. \end{cases} \quad (10)$$

Since the actual earth's surface deviates from the Bouguer sphere (of radius $R+H$), there are deficiencies and/or abundances of topographical masses with respect to the mass of the Bouguer shell. These contribute to the topographical potential $V^t(r,\Omega)$ by the term $V^R(r,\Omega)$ – an analogy of the terrain correction (Heiskanen and Moritz, 1967, sect. 3-3.). We will call $V^R(r,\Omega)$ the *terrain roughness term* because it appears due to the roughness of the terrain. It depends chiefly on the behaviour of the difference $H - H'$ and weakly on the lateral density variations of $\overline{\varrho}(\Omega)$.

Let us evaluate the limit $\psi \to 0$ of the subintegral function in the angular integral (9). If $\psi \to 0$ then $\overline{\varrho}(\Omega') \to \overline{\varrho}(\Omega)$ and $H' \to H$. We will assume that both the topographical density $\overline{\varrho}$ and the topographical height H are bounded (i.e., there are no mass-singularities inside the topographical masses and the heights of the earth's topography are finite). Then the limit $\psi \to 0$ of the subintegral function in the angular integral (9) reads

$$\begin{aligned} &\lim_{\psi\to 0}\left[\overline{\varrho}(\Omega')\int_{r'=R}^{R+H'} L^{-1}(r,\psi,r')r'^2dr' - \right. \\ &\left. -\overline{\varrho}(\Omega)\int_{r'=R}^{R+H} L^{-1}(r,\psi,r')r'^2dr'\right]\sin\psi = \\ &= \overline{\varrho}(\Omega)\int_{r'=R}^{R+H}\lim_{\psi\to 0}\left[L^{-1}(r,\psi,r')\sin\psi\right]r'^2dr' - \\ &- \overline{\varrho}(\Omega)\int_{r'=R}^{R+H}\lim_{\psi\to 0}\left[L^{-1}(r,\psi,r')\sin\psi\right]r'^2dr' = 0\,. \end{aligned} \quad (11)$$

Since both functions $\overline{\varrho}(\Omega)$ and H are bounded and the Newton kernel is weakly singular, see property (4), both the integrals on the right-hand side of eqn.(11) are finite and take the same value; their difference is thus equal to zero. This means that the point $\psi = 0$ may be left out of the integration domain Ω' and the singularity of the Newton kernel at the point $\psi = 0$ is removed. This fact is important for the numerical computation of the topographical potential $V^t(r,\Omega)$ because the modified formulae (8) and (9) ensure that the numerical algorithm is not forced to evaluate the undefined expression of the type 0/0 occurring in the original Newton integral (6).

3. The radial integral of Newton's kernel

Analytical form

The indefinite radial integral of the Newton kernel may be evaluated analytically (Gradshteyn and Ryzhik, 1980, pars. 2.261, 2.264) as follows:

$$\int_{r'} L^{-1}(r,\psi,r')r'^2dr' = \widetilde{L^{-1}}(r,\psi,r') + C\,, \quad (12)$$

where

$$\begin{aligned} \widetilde{L^{-1}}(r,\psi,r') &= \frac{1}{2}(r' + 3r\cos\psi)L(r,\psi,r') + \\ &+ \frac{r^2}{2}(3\cos^2\psi - 1)\ln|r' - r\cos\psi + L(r,\psi,r')|\,, \end{aligned} \quad (13)$$

and the 'constant' C may depend on the variables r and ψ only. Using notation (13), the topographical potential V^t, cf. eqns.(8) and (9), then becomes

$$\begin{aligned} V^t(r,\Omega) &= V^B(r,\Omega) + \\ &+ G\int_{\Omega'}\left[\overline{\varrho}(\Omega')\,\widetilde{L^{-1}}(r,\psi,r')\Big|_{r'=R}^{R+H'} - \right. \\ &\left. -\overline{\varrho}(\Omega)\,\widetilde{L^{-1}}(r,\psi,r')\Big|_{r'=R}^{R+H}\right]d\Omega'\,. \end{aligned} \quad (14)$$

Taylor series expansion

There is another way of evaluating the radial integral of the Newton kernel normally used in computing the gravitational potential of topographical masses in Helmert's second condensation technique (Heiskanen and Moritz, 1967, sect. 3-7.; Vaníček and Kleusberg (1987); Martinec and Vaníček (1993 a,b)). The product of the Newton kernel $L^{-1}(r,\psi,r')$ with r'^2 as a function of the variable r' is expanded into a Taylor series at a point on the geoid. Martinec and Vaníček (1993b) showed that the results can be written in the following forms

$$r'^2L^{-1}(r,\psi,r')\Big|_{r'=R+H'} = R^2\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{H'}{R}\right)^i M_i(r,\psi)\,, \quad (15)$$

and

$$\widetilde{L^{-1}}(r,\psi,r')\Big|_{r'=R}^{R+H'} = \sum_{i=0}^{\infty}K_i(r,\psi,H')\,, \quad (16)$$

where the Taylor series kernels K_i read

$$K_i(r,\psi,H') = \frac{R^3}{(i+1)!}\left(\frac{H'}{R}\right)^{i+1} M_i(r,\psi)\,. \quad (17)$$

The kernels $M_i(r,\psi)$ for the first few degrees are:

$$M_0(r,\psi) = \frac{1}{\ell}\,, \quad (18)$$

$$M_1(r,\psi) = \frac{1}{\ell} + \frac{rz}{\ell^3} , \tag{19}$$

$$M_2(r,\psi) = -\frac{r^2}{\ell^3} + \frac{3r^2 z^2}{\ell^5} , \tag{20}$$

$$M_3(r,\psi) = \frac{3r^2}{\ell^3}\left(1 - \frac{3rz}{\ell^2} - \frac{3z^2}{\ell^2} + \frac{5rz^3}{\ell^4}\right) , \tag{21}$$

$$M_4(r,\psi) = \frac{3r^2}{\ell^3}\left(-4 + \frac{3r^2}{\ell^2} + \frac{24rz}{\ell^2} + \frac{12z^2}{\ell^2} - \right.$$

$$\left. - \frac{40rz^3}{\ell^4} - \frac{30r^2z^2}{\ell^4} + \frac{35r^2z^4}{\ell^6}\right) , \tag{22}$$

where

$$z = r - R\cos\psi , \tag{23}$$

and the symbol ℓ denotes the spatial distance between points (r,Ω) and (R,Ω'), i.e.,

$$\ell \equiv L(r,\psi,R) = \sqrt{r^2 + R^2 - 2rR\cos\psi} . \tag{24}$$

We note that $M_i(r,\psi)$ are singular at $\psi = 0$ if $r = R$, i.e., on the geoid.

Martinec and Vaníček (1993b) also derived the general formula for the higher degree ($i > 1$) integration kernels $M_i(r,\psi)$:

$$M_i(r,\psi) = \frac{1}{\ell}\sum_{s=1}^{i-1} \frac{i!(i-2)!}{(i-s-1)!(s-1)!}\left(\frac{r}{\ell}\right)^{i+1-s} \times$$

$$\times \sum_{t=0}^{i+1-s} (-1)^{\frac{3i+1-s+t}{2}} \frac{(i+2-s-t)!!(i-s+t)!!}{(i+2-s-t)!\,t!}\left(\frac{z}{\ell}\right)^t . \tag{25}$$

The summation in eqn.(25) must be taken over such t's for which $i - s + t + 1$ is an even number.

4. Potential of the condensation layer

As discussed in the next section, condensation of topographical masses onto the geoid has a fundamental role in Helmert's second condensation technique. Provided that the condensation density $\sigma(\Omega)$ is chosen to be equal to the product of $\overline{\varrho}(\Omega)$ and H and the condensation layer has the radius R, the condensation potential V^c is given by formula (Heiskanen and Moritz, 1967, sect. 1.-3):

$$V^c(r,\Omega) = GR^2 \int_{\Omega'} \frac{\overline{\varrho}(\Omega')\,H'}{\ell} d\Omega' . \tag{26}$$

To remove the singularity of the reciprocal distance $1/\ell$ in the potential V^c, we may proceed in a way analogous to that for the potential V^t. Let us rewrite eqn.(26) as

$$V^c(r,\Omega) = V^\ell(r,\Omega) + GR^2 \int_{\Omega'} \frac{\overline{\varrho}(\Omega')\,H' - \overline{\varrho}(\Omega)\,H}{\ell} d\Omega' , \tag{27}$$

where $V^\ell(r,\Omega)$ is the potential of a spherical material layer with density $\overline{\varrho}(\Omega)\,H$ and radius R,

$$V^\ell(r,\Omega) = GR^2\overline{\varrho}(\Omega)\,H \int_{\Omega'} \frac{d\Omega'}{\ell} . \tag{28}$$

The last integral may be readily evaluated yielding

$$V^\ell(r,\Omega) = \begin{cases} 4\pi G\overline{\varrho}(\Omega)H\frac{R^2}{r} , & r > R , \\ 4\pi G\overline{\varrho}(\Omega)HR , & r \le R . \end{cases} \tag{29}$$

5. Topographical effects

If the gravitational potential V^t of the actual topographical masses is approximated by the potential V^c of the condensation layer, as it is done in Helmert's second condensation technique (Helmert, 1884), i.e., when

$$V^t = V^c + \delta V , \tag{30}$$

where δV is the residual topographical potential, then the gravity measured on the earth's surface is to be corrected by both the direct topographical effect on gravity, $\delta A(\Omega)$:

$$\delta A(\Omega) = \left.\frac{\partial\,\delta V(r,\Omega)}{\partial r}\right|_{r=R+H} , \tag{31}$$

and the secondary indirect topographical effect on gravity, $\delta P^{(2)}(\Omega)$:

$$\delta P^{(2)}(\Omega) = \frac{2}{R}\delta V(R,\Omega) . \tag{32}$$

Moreover, the approximate geoid (called co-geoid) differs from the actual geoid by the quantity $\delta N_{pri}(\Omega)$ as:

$$\delta N_{pri}(\Omega) = \frac{\delta V(R,\Omega)}{\gamma} , \tag{33}$$

where γ is the normal gravity on the reference ellipsoid. In this context, the residual potential δV taken on the geoid is called the primary indirect topographical effect on potential.

6. The primary indirect topographical effect on potential

Martinec and Vaníček (1993b) showed that the Taylor kernels $K_i(r,\psi,H')$ are weakly singular at points on the earth's surface. For computing the direct topographical effect $\delta A(\Omega)$ is thus possible to use both the Taylor series (16) and the analytical form (13) for defining the radial integral of the Newton kernel, i.e., $\widetilde{L^{-1}}(r,\psi,r')$.

Unfortunately, we will see later that the kernels $K_i(r,\psi,H')$ become strongly singular on the geoid, i.e., when $r = R$. This may cause serious difficulties when the primary and secondary indirect effects are evaluated using the Taylor series (16). On the contrary, we will show that integral (13) is only weakly singular on the geoid. Therefore, we will use it to illustrate the problem arising from the use of the Taylor series (16) for computing the indirect topographical effects.

The residual topographical potential $\delta V = V^t - V^c$ at a geoidal point can be rewritten by means of eqns. (14) and (27), where r is replaced by R. We obtain

$$\delta V(R,\Omega) = \delta V^B(R,\Omega) + \delta V^R(R,\Omega) \ , \qquad (34)$$

where the "Bouguer term" $\delta V^B(R,\Omega)$ is given as

$$\delta V^B(R,\Omega) = V^B(R,\Omega) - V^\ell(R,\Omega) \ , \qquad (35)$$

and the "terrain roughness term" $\delta V^R(R,\Omega)$ is equal to

$$\delta V^R(R,\Omega) = G\int_{\Omega'}\left\{\overline{\varrho}(\Omega')\,\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H'} - \right.$$

$$-\overline{\varrho}(\Omega)\,\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H} -$$

$$\left. -R^2\left[\overline{\varrho}(\Omega')H' - \overline{\varrho}(\Omega)H\right]L^{-1}(R,\psi,R)\right\}d\Omega' \ . \qquad (36)$$

Using the last of eqns.(10) and (29) valid for $r = R$, the Bouguer term $\delta V^B(\Omega)$ becomes

$$\delta V^B(R,\Omega) = 2\pi G\overline{\varrho}(\Omega)H^2 \ . \qquad (37)$$

Even though the Bouguer term $\delta V^B(R,\Omega)$ is the dominant one in the potential $\delta V(R,\Omega)$, in this paper we are only interested in computing the terrain roughness term $\delta V^R(R,\Omega)$. We are thus not going to discuss the Bouguer term any further here.

Substituting for $R^2L^{-1}(R,\psi,R)$ from eqn.(15), and for $\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H'}$ from eqn.(16), the terrain roughness term $\delta V^R(R,\Omega)$, cf. eqn.(36), can alternatively be expressed by means of the kernels M_0 and K_i as

$$\delta V^R(R,\Omega) = G\int_{\Omega'}\left\{\sum_{i=0}^{\infty}\left[\overline{\varrho}(\Omega')K_i(R,\psi,H') - \right.\right.$$

$$-\overline{\varrho}(\Omega)K_i(R,\psi,H)\right] -$$

$$\left. -R^2\left[\overline{\varrho}(\Omega')H' - \overline{\varrho}(\Omega)H\right]M_0(R,\psi)\right\}d\Omega' \ . \qquad (38)$$

By eqn.(17), the kernel K_0 can be expressed by means of the kernel M_0, and the expression (38) can be further simplified as

$$\delta V^R(R,\Omega) = G\int_{\Omega'}\sum_{i=1}^{\infty}\left[\overline{\varrho}(\Omega')K_i(R,\psi,H') - \right.$$

$$\left. -\overline{\varrho}(\Omega)K_i(R,\psi,H)\right]d\Omega' \ . \qquad (39)$$

In practice, the infinite series in the last equation is truncated at a degree i_{max} and the correct value of $\delta V^R(R,\Omega)$ is approximated by its estimate $\widehat{\delta V^R}(R,\Omega)$:

$$\widehat{\delta V^R}(R,\Omega) = G\int_{\Omega'}\sum_{i=1}^{i_{max}}\left[\overline{\varrho}(\Omega')K_i(R,\psi,H') - \right.$$

$$\left. -\overline{\varrho}(\Omega)K_i(R,\psi,H)\right]d\Omega' \ . \qquad (40)$$

For example, Wichiencharoen (1982), Vaníček and Kleusberg (1987), and Wang and Rapp (1990) only consider the terms with $i = 2$.

The correct analytical formula (36) for $\delta V^R(R,\Omega)$ provides us with a tool for testing the accuracy of the estimate $\widehat{\delta V^R}(R,\Omega)$. But, before using this tool, we will find the condition under which the infinite series of the Taylor series kernels K_i actually converges.

7. The problem of the convergence of the Taylor series expansion

For computing the primary indirect topographical effect on potential, the topographical potential V^t, and therefore the integration kernels K_i are to be evaluated on the geoid, i.e., at points (R,Ω). As already noted, in this case all the kernels $K_i(R,\psi,H')$ are singular at the point $\psi = 0$. Fortunately, we have learned by analysing eqn.(9) that there is no need to evaluate the integration kernel $K_i(R,\psi,H')$ at the point $\psi = 0$. Nevertheless, the kernels have to be evaluated in the immediate neighbourhood of the computation point $\psi = 0$, and therefore, we have to ask about the type of singularity of the kernels $K_i(R,\psi)$ at that point.

To answer this question, we employ eqn.(25); for $r = R$ it yields

$$M_i(R,\psi) = \frac{1}{\ell_0}\sum_{s=1}^{i-1}\frac{i!(i-2)!}{(i-s-1)!(s-1)!}\left(\frac{R}{\ell_0}\right)^{i+1-s}\times$$

$$\times\sum_{t=0}^{i+1-s}(-1)^{\frac{3i+1-s+t}{2}}\frac{(i+2-s-t)!!(i-s+t)!!}{(i+2-s-t)!\,t!}\left(\frac{\ell_0}{2R}\right)^t , \qquad (41)$$

where ℓ_0 stands for the distance between points (R,Ω) and (R,Ω'), i.e.,

$$\ell_0 \equiv L(R,\psi,R) = 2R\sin\frac{\psi}{2} \ . \qquad (42)$$

The type of the singularity of the kernel $M_i(R,\psi)$ at the point $\psi = 0$ is given by the highest power of the reciprocal distance $1/\ell_0$. Inspecting eqn.(41), we can see that the highest power of $1/\ell_0$ occurs when summation indices s and t are small, i.e., $s = 1$, and $t = 0$ or $t = 1$ (subject to the constraint that $i - s + t + 1$ be an even number). Considering $s = 1$ in this constraint, we can see that t must be of the same parity as i, i.e., $t = 0$ when i is an even number, and $t = 1$ when i is an odd number.

To begin with, let us consider the case when i is an *even* number. From eqn.(41) we can see that in the vicinity of the point $\psi = 0$, (i.e., when $\psi \le \varepsilon$, and ε is a small positive number), the kernel $M_i(R,\psi)$ behaves as

$$M_i(R,\psi\le\varepsilon) \approx (-1)^{\frac{3i}{2}}\left[(i-1)!!\right]^2\frac{R^i}{\ell_0^{i+1}} \ . \qquad (43)$$

This formula shows that the higher the degree of the kernel $M_i(r,\psi)$, the stronger its singularity at the point $\psi = 0$. Substituting eqn.(43) into (17), we find that in the vicinity of the point $\psi = 0$ the Taylor series kernel K_i behaves as

$$K_i(R,\psi \le \varepsilon, H') \approx R^2(-1)^{\frac{3i}{2}} \frac{[(i-1)!!]^2}{(i+1)!} \left(\frac{H'}{\ell_0}\right)^{i+1} . \tag{44}$$

To find a condition under which the Taylor series (16) converges for $\psi \le \varepsilon$, let us evaluate the ratio (i is still an even number)

$$\frac{K_{i+2}(R,\psi \le \varepsilon, H')}{K_i(R,\psi \le \varepsilon, H')} = -\frac{(i+1)^2}{(i+2)(i+3)} \left(\frac{H'}{\ell_0}\right)^2 , \tag{45}$$

where we have used eqn.(44). Since

$$\lim_{i\to\infty} \frac{(i+1)^2}{(i+2)(i+3)} = 1 , \tag{46}$$

the convergence of the Taylor series (16) depends on the ratio H'/ℓ_0. The same criterion may be found for odd degrees i.

In the case when the kernels $K_i(R,\psi,H')$ are evaluated for ψ's for which $\ell_0 > H'$, the magnitude of the kernels K_i decreases with increasing order i, and the Taylor series (16) converges. On the other hand, computing the kernels $K_i(R,\psi,H')$ at a smaller distance ψ, such that $\ell_0 < H'$, the magnitude of the kernels K_i increases with increasing order i and the Taylor series (16) diverges.

As a consequence, special care has to be paid to numerical computation of the Newton integral for V^t when the Taylor series is employed. The topographical heights are usually given on a regular grid $(R\Delta\phi, R\cos\phi\Delta\lambda)$, where $\Delta\phi$ and $\Delta\lambda$ are steps in latitude and longitude, respectively. If both the steps $R\Delta\phi$ and $R\cos\phi\Delta\lambda$ are larger than the largest topographical height in the area of interest, then the values of the distance ℓ_0 over the topographical grid are always larger than H', and the Taylor series (16) converges at all the points on the topographical grid. Such a situation usually occurs in a flat terrain.

The problem arises when the topographical potential $V^t(r,\Omega)$ is to be evaluated at mountainous terrain. In this case, the sampling of the topographical heights has to be fairly dense (e.g., with a step of 1 km) to express the ruggedness of the terrain; the dummy point in the Newton integral than may come very close to the computation point so that $\ell_0 < H'$. At this point the higher order terms of the Taylor series become larger than the low degree terms and the series (16) diverges. Moreover, as eqn.(43) shows, individual terms of Taylor series change the sign. To use such an oscillating and divergent series for the evaluation of the Newton integral becomes impossible; we will see later that it cannot be used for an accurate geoid determination.

For comparison, let us now have a look at the behaviour of the vertically integrated form (13) of the kernel $\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H'}$, in the vicinity of the point $\psi = 0$. Using eqns.(13) and (42), we have

$$\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H'} =$$

$$= \frac{1}{2}(R+H'+3R\cos\psi)L(R,\psi,R+H') - \frac{R}{2}(1+3\cos\psi)\ell_0 + $$
$$+ \frac{R^2}{2}(3\cos^2\psi - 1)\ln\frac{H' + \frac{\ell_0^2}{2R} + L(R,\psi,R+H')}{\ell_0 + \frac{\ell_0^2}{2R}} , \tag{47}$$

where

$$L(R,\psi,R+H') =$$
$$= \sqrt{(R+H')^2 + R^2 - 2R(R+H')\cos\psi} . \tag{48}$$

Again by using eqn.(42), the last relation reads

$$L(R,\psi,R+H') = \sqrt{\ell_0^2 + \frac{H'}{R}\ell_0^2 + H'^2} . \tag{49}$$

When a dummy point of integration comes close to the computation point ($\psi \to 0$ or $\ell_0 \to 0$), the first two terms on the right-hand side of eqn.(47) come to a finite number $(4R+H')H'/2$; the magnitude of the last term grows to infinity since this term behaves like a function $\ln\ell_0$ (we assume that $H' > 0$). Because the following limit is valid

$$\lim_{\ell_0\to 0} \frac{\ln\ell_0}{\frac{1}{\ell_0}} = 0 , \tag{50}$$

the magnitude of the kernel $\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H'}$ grows to infinity even slower than the reciprocal distance $1/\ell_0$ when ψ approaches zero. Therefore, the numerical procedure of computing the Newton integral (36) based on eqn.(13) is very stable even near the computation point ($\psi = 0$) and is not limited by the restriction that the discretization step has to be greater than the highest topographical elevation.

8. Numerical computations

The Taylor kernels K_i

To demonstrate the problem with the Taylor series convergence, we computed the definite integral $\widetilde{L^{-1}}(R+H,\psi,r')\Big|_{r=R}^{R+H'}$ using the analytical expression (13) and compared its behaviour with that of the first 5 terms of the Taylor series (16). In Figure 1, we have taken the height H' of the dummy point of integration to be 1 km, the angular distance ψ to be 0.01^o (which corresponds to the horizontal distance ℓ_0 of about 1.1 km), and the height H of the computation point has been varied from -4 km to 5 km ($H = 0$ means that the computation point lies on the geoid).

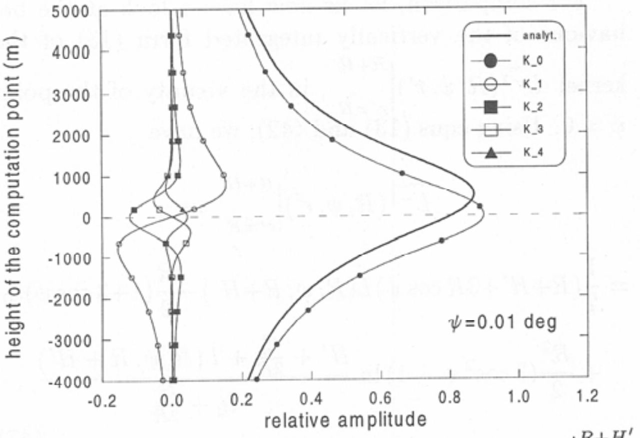

Figure 1: Integration kernels $\widetilde{L^{-1}}(R+H,\psi,r')\Big|_{r'=R}^{R+H'}$ computed by analytical formula (13) (curve denoted as 'analytical') and the Taylor series expansion kernels $K_i(R+H,\psi,H')$ for $i=0,...,4$. The height H of the computation point ranges in the interval $(-4000,5000)$ m, the height H' of an integration point is 1000 m. The angular distance ψ between the computation point and the integration point is 0.01^o.

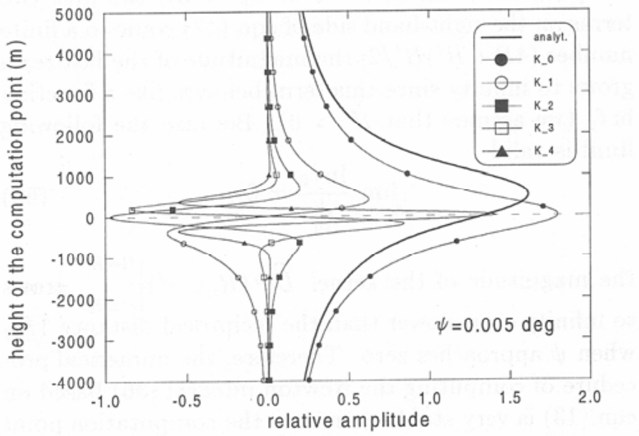

Figure 2: The same as Figure 1 for $\psi = 0.005^o$.

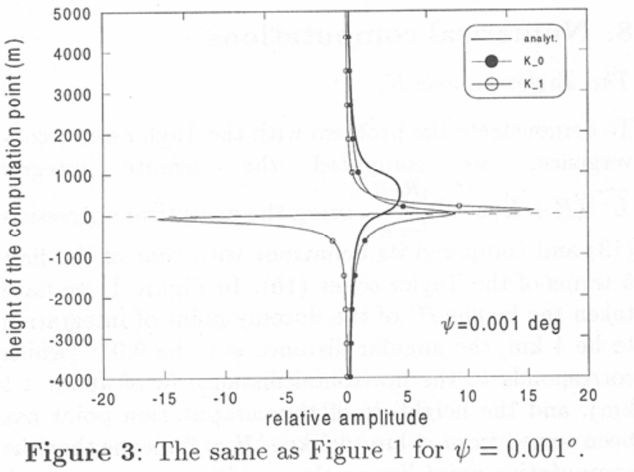

Figure 3: The same as Figure 1 for $\psi = 0.001^o$.

From Figure 1 we can observe that: (1) The magnitude of the Taylor series kernels K_i decreases with increasing order i; in other words the Taylor series (16) converges which confirms our theoretical conclusion because $\ell_0 > H'$. (2) The kernels K_i change their signs for the computation point near the geoid, which means that the Taylor series (16) oscillates for points near the geoid. (3) For a computation point on the geoid, the magnitudes of odd-degree kernels K_i are very small (but not equal to zero) compared with those of even-degree kernels K_i. This means that the primary indirect topographical effect on potential can be reasonably modelled by even degree kernels K_i only.

Figure 2 shows a similar situation as Figure 1, where the only change is that the angular distance ψ between the computation and dummy points is one half of the above, i.e., the horizontal distance ℓ_0 is 0.55 km. As already shown theoretically, the kernels K_i grow with increasing order i at the point on the geoid because $\ell_0 < H'$, and, consequently, the Taylor series (16) does not converge. Figure 2 demonstrates this fact very clearly.

An even more drastic example of the divergence of the series (16) is plotted in Figure 3. Here the angular distance ψ is equal to 0.001^o which corresponds to the horizontal distance $\ell_0 = 0.11$ km. We have plotted only the first two kernels K_0 and K_1 because the magnitude of the higher order kernels grows too fast.

The primary indirect topographical effect on potential

Particular contributions of the Taylor series kernels K_i, to the terrain roughness term $\widehat{\delta V^R}(\Omega)$ were studied over an area of $1.65^o \times 1.25^o$, $(51.20^o \leq \phi \leq 52.85^o;$ $242.7 \leq \lambda \leq 243.95^o)$ in Western Canada. The area covers a particularly rugged part of the Rocky Mountains, the chain of Columbia Mountains. The topographical heights range from 1 to 3573 m, and were sampled as means of $30'' \times 60''$ cells.

Table 1 shows the extreme values of corrections to geoidal heights due to the terrain roughness term $\widehat{\delta V^R}(\Omega)$, cf. eqn.(40), induced by kernels K_i, i=1,...,4. The correct values, computed by means of the analytical formula (36), are shown in Figure 4; they fall within the interval (-0.09, 0.20) m.

For the sake of completeness, we remark that the Bouguer term $\delta V^B(\Omega)$ for the same region gives values ranging from -1.46m to 0m for the $30'' \times 60''$ gridded heights and from -0.71m to 0m for the $5' \times 5'$ gridded heights. This demonstrates how sensitive the results are to the selected grid size, but this problem is considered beyond the scope of this paper.

Inspecting Table 1 and Figure 4, we can draw the following conclusions:

Table 1: Corrections to the geoidal heights (in metres) due to the terrain roughness term $\widehat{\delta V^R}(\Omega)$ expressed by Taylor series kernels K_i, i=1,...4.

term	min	max
K_1	-0.006	0.003
K_2	-0.222	0.394
K_3	0.000	0.000
K_4	-1.14	0.906

(i) The magnitude of $\widehat{\delta V^R}(\Omega)$ induced by kernels K_i with odd orders are small enough to be neglected for a 1 cm accuracy geoid computation in Canada.

(ii) The magnitudes of $\widehat{\delta V^R}(\Omega)$ induced by kernels K_i with even orders increase with increasing order i; the Taylor series expansion (16) diverges at points with extreme heights.

(iii) Wang and Rapp (1990) modelled the terrain roughness term $\delta V^R(\Omega)$ by the contribution generated by the kernel K_2 using the topographical grid size 30". Comparing these values with the correct ones, we can see that the K_2 term overestimates the terrain corrections $\delta V^R(\Omega)$ about two times. This means that errors in geoidal heights due to the K_2 term approximation are about two decimeters over the area under study. These errors cannot be reduced by adding higher order terms of the Taylor series, e.g., by adding the K_4 contribution, because of the divergence of this series.

We have also computed the terrain roughness term $\delta V^R(\Omega)$ from heights averaged on a sparser topographical grid $5' \times 5'$ (not shown here). The resulting values fall within the interval (-0.03, 0.03) m. Comparing these values with those on Figure 4, we can see that the differences are more than one decimetre. This is caused by the fact that the mean $5' \times 5'$ digital terrain model is smoother than the $30'' \times 60''$ model. This fact tells us that a grid size of $5'$ in a mountainous terrain such as the Rocky Mountains is not sufficiently small to express the irregularities of the terrain and thus does not reveal properly the contribution to geoidal height due to terrain height variations, as we have already noted.

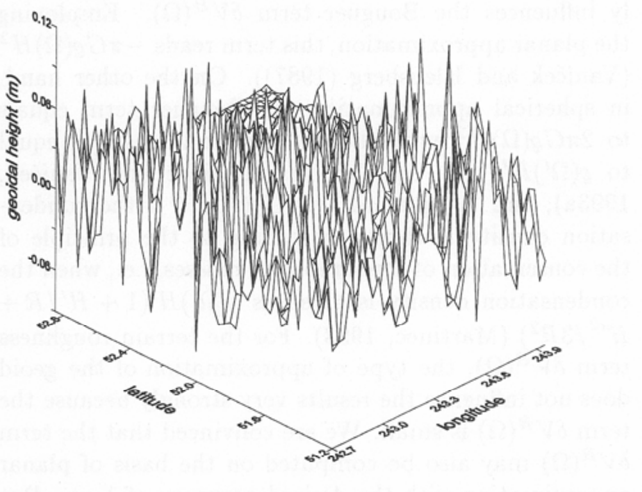

Figure 4: Geoidal heights (in metres) induced by the terrain roughness term $\delta V^R(\Omega)$ of the primary indirect topographical effect on the geoid over the area of Columbia Mountains.

9. Conclusions

Today geodesists strive to compute the geoid with an accuracy of the order of 1 cm (Vaníček and Martinec, 1993). To achieve such an accuracy, the gravity and the earth's topography must be known with a high precision. In addition in a mountainous terrain, topographical elevations must be known on a fairly dense grid.

In this paper we have concentrated on the problem of convergence of the Taylor series of the Newton gravitational kernel on a dense topographical grid used for modelling a rugged terrain. We have shown that for the primary indirect topographical effect on potential, this expansion converges if the horizontal distance between the computation point and the dummy integration point is greater than the topographical elevation. In fact, this constrains the sampling of topographical heights. On the one hand, there is a need to discretize the heights of a rugged terrain as densely as possible to model the terrain irregularities as accurately as possible and, on the other hand, the discretization step cannot be smaller than the largest topographical elevation in order for the Taylor series to converge. On a numerical example we have demonstrated that the Rocky Mountains is one of the areas where these contradicting requirements on the digitization step are not possible to be satisfied simultaneously.

If the gravitational potential of topographical masses at the point on the geoid is approximated by the zeroth and quadratic terms of the Taylor series, as it is usually done, then such an approximation is biased and the bias may easily reach decimetres. We have shown that for the Columbia Mountains these errors reach in fact two decimetres.

Our conclusion is that for a rugged terrain the approximate formulae based on the Taylor series expansion have to replaced by the correct (analytical) formula which removes the problem of divergence of the Taylor series. We have shown the very stable behaviour of this formula in the vicinity of the computation point ($\psi = 0$). However, considering this correct formula, the application of the fast Fourier technique for computing the gravitational potential of the topographical masses is not as straightforward as in the case when the Taylor series expansion is employed (see also Heck, 1992). Further investigation of this problem has to be carried out.

One thing should be emphasized at the end. Throughout the paper we have used the spherical approximation of the geoid and *not* the planar approximation (Moritz, 1966), because the type of approximation significantly influences the Bouguer term $\delta V^B(\Omega)$. Employing the planar approximation, this term reads $-\pi G\varrho(\Omega)H^2$ (Vaníček and Kleusberg (1987)). On the other hand, in spherical approximation the Bouguer term equals to $2\pi G\varrho(\Omega)H^2$ when the condensation density is equal to $\varrho(\Omega')H'$, cf. eqn.(36) in (Martinec and Vaníček, 1993a), and is equal to $-2\pi G\varrho(\Omega)H^2$ if the condensation density is chosen according to the principle of the conservation of topographical masses, i.e., when the condensation density is given as $\varrho(\Omega')H'(1 + H'/R + H'^2/3R^2)$ (Martinec, 1993). For the terrain roughness term $\delta V^R(\Omega)$, the type of approximation of the geoid does not influence the results very strongly because the term $\delta V^R(\Omega)$ is small. We are convinced that the term $\delta V^R(\Omega)$ may also be computed on the basis of planar approximation with the desired accuracy of 1 cm. But again a detailed study should be devoted to this point.

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