Geodynamic applications of the gravity inversion by means of the truncated geoid: A synthetic case study

Peter Vajda [1], Ladislav Brimich [2], and Petr Vaníèek [3]

1) Geophysical Institute, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovakia
Tel.: +421 7 5941 0603, Fax: +421 7 5941 0626, Email: geofvajd@nic.savba.sk
2) Geophysical Institute, Slovak Academy of Sciences, Slovakia
3) Department of Geodesy and Geomatics Engineering, University of New Brunswick, Fredericton, N.B., Canada

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Abstract In this paper we simulate a model geodynamic event generated by a magmatic heat source represented by a point source of heat embedded in an elastic halfspace. This simplistic model, which can be considered relevant to geodynamic studies of volcanism, produces a thermoelastic displacement field generating a perturbation density distribution. Both the displacements and the perturbed density produce a gravity anomaly on the surface of the halfspace. The objective is to demonstrate that this gravity anomaly, being a change in the surface gravity caused by such a model geodynamic event, can be interpreted using the truncation methodology, i.e., by transforming the gravity anomaly into a sequence of surfaces or profiles of the truncated geoidal heights and interpreting such sequence instead, in exactly the same fashion as described in Vajda and Vaníèek [1999a]. The idea behind this simulation, demonstrated on a case of one point source, is that density perturbations caused by geodynamic events can be interpreted from temporal changes of gravity anomalies analogically as the anomalous density can be interpreted from gravity anomalies with help of the truncation methodology. In turn the interpreted density perturbations can give insights on the nature of the geodynamic events as well as estimates on the depth of their origin.

1. Introduction

In deformation modeling and prediction, the most interesting effects are those, which in principle can be detected on the surface of the earth prior to an eruption. The surface temporal change of gravity is an example of such observable piece of information. Hvoždara and Brimich [1995] approximated a magmatic intrusion in earth's crust by a point source of heat in order to model the stress and displacement distribution within an elastic homogenous halfspace, the heat flow, and the temporal change of gravity on the surface of the halfspace. Their model study will serve here as a basis for our synthetic investigations, the aim being to indicate one possible way of interpreting the temporal change of gravity in terms of the depth of the source of heat.

The objective of this paper is to demonstrate that the truncation methodology, a particular gravity inversion technique based on the interpretation of the truncated geoid [Vajda and Vaníèek, 1999a]; [1997], so far developed only for density distributions represented by point masses, can also be applied in geodynamic studies. Let us therefore briefly describe the truncation technique, as we used to refer to it, first.

Vajda and Vaníèek have studied the possibility of using the truncated geoid, which is an incomplete geoid, defined below, in the gravimetric inverse problem, or in other words, for gravity data interpreting. The truncated geoid is defined by the Truncated Stokes integral transform (cf. e.g. Vajda and Vaníèek [1999a], eqn.(1), or [1997], eqn.(1) and Fig.1)

$$N^{\psi_o}(\varphi,\lambda) = \frac{R}{4\pi\gamma} \iint_{\Re(\psi_o)} \Delta g(\varphi',\lambda') S(\varphi,\lambda,\varphi',\lambda') d\sigma \,, \tag{1}$$

where, just to recall, R is the radius of the reference boundary sphere (mean earth's radius for our purposes), γ is normal gravity, (φ,λ) and (φ',λ') are the horizontal positions of the computation point and the integration point on the reference surface respectively, ψ_0 is the spherical radius of the spherical cap being the integration domain $\Re(\psi_0)$ and is referred to as the truncation radius or the truncation parameter, Δg is the gravity anomaly, and N^{ψ_o} is the truncated geoidal height, The integral in eqn.(1), which transforms the gravity anomaly into a truncated geoid, may be understood as a linear filter - a weighted spherical window filter - with the Stokes function S [Stokes, 1849], the kernel of the convolution integral in eqn.(1), being the weight. Thus the interpretation of the gravity anomaly is transformed into the interpretation of the truncated geoid. Note that the truncated geoid has one free parameter, the truncation radius, which controls the size of the window used in the filtering procedure mathematically described by eqn.(1). This is why it would perhaps be more accurate to refer to the truncation technique as the „truncation filtering“. In the gravity interpretation by means of the truncated geoid, not just one truncated geoid (a surface or profile of the truncated geoidal heights) at a specific selected value of the truncation parameter is used, but a whole sequence of the truncated geoids with the value of the truncation parameter being varied systematically is interpreted. Such sequence is denoted as the TG sequence. A sequence of the first derivative of the truncated geoid with respect to the truncation parameter for the same sequence of the values of the truncation parameter as in the TG sequence, denoted as the DTG sequence, is interpreted along with the TG sequence, this being considered a data enhancement technique.

Vajda and Vaníèek used the computer simulation to produce synthetic gravity anomalies generated by point mass anomalies, and the TG and DTG sequences computed from such synthetic anomalies were studied with the aim of developing the interpretation know-how. The computer simulations have proved that the truncated geoid is a suitable tool in gravity inversion. A methodology for interpreting synthetic gravity data (generated by sets of point masses) with help of the truncation filtering - weighted spherical windowing (linear spatial filtering using the TG and DTG sequences) was developed and at that time denoted as the truncation technique (Vajda and Vaníèek [1997], [1999a]). It is proposed, although not yet tested, that this filtering be used on real gravity data to yield maximum depth to source estimates for pronounced isolated gravity anomalies. Work on developing an interpretation know-how based on the truncation filtering for real density distributions is in progress.

The gravity interpretation using the truncation filtering is illustrated on a case of one point mass anomaly in Fig.1, which presents the TG and DTG sequences generated by a negative point mass located at the depth of 4 km. In this figure, the profiles that are animated on the computer are drawn all in one plot, hoping that the transparency is not lost and that the reader can follow the evolution of the sequence (cf. also the more detailed comments to Fig.4 in Vajda and Vaníèek [1999a]).

Fig.1

Fig.1 *The synthetic TG (a) and DTG (b) sequences generated by one negative point mass anomaly at the depth of 4 km. The sequences are computed for the values of the truncation parameter* ψ_0 *(in degrees of arc) from 0.1123 to 2.25 10*$^{-3}$ *with a step of 2.25 10*$^{-3}$*; for the TG decreasingly, for the DTG increasingly.*

The above sequences are interpreted as follows: The horizontal position (e.g. latitude and longitude, or plane x and y coordinates) of the point mass coincides with the horizontal position of the high (positive point mass anomaly) or the low (negative point mass anomaly) of the TG or the DTG. The depth of the point mass is determined from the onset of the so called dimple event (for a negative point mass the sequence is up-side-down and there we should talk about a pimple event), that is observed in the DTG sequence (cf. Vajda and Vaníèek [1999a])

$$d[km] \doteq 135.82[km/^\circ]\psi_0^*[^\circ]\,, \qquad (2a)$$

where d is the depth of the point mass in kilometers and ψ_0^* is the instant of the dimple onset, i.e., a particular value of the truncation parameter, at which the dimple sets on in the DTG sequence, given in degrees of arc. When evaluating the TG and DTG sequence in planar approximation and using a planar truncation parameter s_0^* given in kilometers rather than in degrees of arc, cf. [Vajda and Vaníèek, 2000], the relation between the depth of the point mass and the instant of the dimple onset reads

$$d[km] = s_0^*[km]\,, \qquad (2b)$$

For more details on interpreting the gravity anomalies in terms of sets of point mass sources the reader is referred to Vajda and Vaníèek [1997].

2. Interpreting temporal changes in gravity anomalies by means of the truncation filtering

As seen above, the three dimensional position of a point mass anomaly can be determined by interpreting the TG and DTG sequences, i.e., by using the truncation filtering. Let us now assume that a gravity anomaly $\Delta g(t_1)$ given at a time instant t_1 changes over a time period of $\Delta t = t_2 - t_1$ to a new gravity anomaly of $\Delta g(t_2)$ due to some enforced geodynamic event. So the change in the gravity anomaly in time reads

$$\Delta^t g = \Delta g(t_2) - \Delta g(t_1) \ . \tag{3}$$

Note that the position argument was omitted in eqn.(3), so that this expression may represent a temporal change in the point gravity anomaly, a temporal change of a gravity profile, or a temporal change of a surface of gravity anomalies. When the geodynamic event is such, that it has impact on the mass (density) distribution below the reference surface, on which the gravity anomaly is given - measured and reduced to, or computed at in case of synthetic case studies - i.e., the mass distribution undergoes changes over the time interval specified above, the $\Delta^t g$ is different from zero and may be interpreted in the same way a gravity anomaly itself is interpreted.

By replacing the gravity anomaly in eqn.(1) with the temporal change of the gravity anomaly, we obtain a prescription for transforming the temporal change of the gravity anomaly into the space of the truncated geoid

$$\Delta^t N^{\psi_o}(\varphi,\lambda) = \frac{R}{4\pi\gamma} \iint_{\Re(\psi_o)} \Delta^t g(\varphi',\lambda') S(\varphi,\lambda,\varphi',\lambda') d\sigma \ . \tag{4}$$

We can refer to quantity $\Delta^t N^{\psi_o}$ as the temporal change of the truncated geoid at a value of the truncation parameter of ψ_0, since the integral transform of eqns. (1) and (4) is linear and non-singular (cf. e.g., Vaníèek et al. [1987]) and it holds true that (again omitting the position arguments):

$$\Delta^t N^{\psi_o} = N^{\psi_o}(t_2) - N^{\psi_o}(t_1) \ , \qquad \text{for any } \psi_0 \tag{5}$$

where

$$N^{\psi_o}(t_i) = \frac{R}{4\pi\gamma} \iint_{\Re(\psi_o)} \Delta g(t_i) S d\sigma \ , \quad \text{i = 1, 2 .} \tag{6}$$

By computing the temporal change of the truncated geoid for a sequence of values of the truncation parameter, we can construct the $\Delta^t TG$ sequence to be interpreted, similar to the construction of the TG sequence in the above section. Analogically we can construct a $\Delta^t DTG$ sequence representing the first derivative of the temporal change of the truncated geoid with respect to the truncation parameter. This sequence is composed of the $\Delta^t \left(\frac{dN^{\psi_0}}{d\psi_0} \right)$ values evaluated for a sequence of systematically varied values of the truncation parameter as follows:

$$\Delta^t \left(\frac{dN^{\psi_0}}{d\psi_0} \right) = \frac{dN^{\psi_0}}{d\psi_0}(t_2) - \frac{dN^{\psi_0}}{d\psi_0}(t_1) = \lim_{\Delta\psi_o \to 0} \frac{\Delta^t N^{\psi_o + \Delta\psi_o} - \Delta^t N^{\psi_o}}{\Delta\psi_o} \ . \tag{7}$$

Numerically the $\Delta^t DTG$ sequence is evaluated by differencing the $\Delta^t TG$ sequence and dividing each difference by the step in the truncation parameter $\Delta\psi_0$. For numerical aspects of transforming the $\Delta^t g$ to $\Delta^t N^{\psi_o}$ and thus constructing the $\Delta^t TG$ sequence, the interested reader is referred to Vajda and Vaníèek [1998]. When computing the $\Delta^t TG$ and $\Delta^t DTG$ sequences in planar coordinates, the truncation parameter ψ_0 is replaced by the planar truncation parameter s_0 , cf. [Vajda and Vaníèek, 2000]. Once computed, these two sequences are to be interpreted exactly in the same fashion as in the above section.

3. A synthetic example - a gravity anomaly caused by the thermoelastic deformation of the halfspace due to a point source of heat

The concept of applying the truncation filtering in geodynamic studies is illustrated on a synthetic case study for a gravity anomaly generated by a model geodynamic event - a thermoelastic deformation of a halfspace caused by a point source of heat.

Let us first introduce the mathematical apparatus for modeling the said geodynamic event. A static non-uniform temperature field T(x,y,z) in an elastic medium produces a thermoelastic displacement field $\vec{u}$ satisfying the Lamé equation (e.g., Nowacki [1962])

$$(\lambda+\mu)\, grad\, div\, \vec{u} + \mu\, div\, grad\, \vec{u} = \alpha_T (3\lambda + 2\mu)\, grad\, T \quad , \tag{8}$$

where λ, μ are the Lamé elastic parameters, and α_T is the thermal coefficient of the linear expansion. The term $\alpha_T(3\lambda+2\mu)\, grad\, T$ represents the force in the equation of the elastic equilibrium (eqn.(8)). Selecting a point heat source located at (0,0,*d*) with the heat power *w* in Watts, the temperature field, assumed static, satisfies in a homogeneous medium the following equation (e.g., Hvoždara and Brimich, [1995]):

$$\lambda_T\, div\, grad\, T = -w\,\delta(x)\delta(y)\delta(z-d) \quad , \tag{9}$$

where λ_T is the heat conductivity and *d* is the depth of the point source. The temperature field due to such point source is axially symmetric with respect to the z axis *T(x,y,z)=T(r,z),* where $r=\sqrt{x^2+y^2}$. By setting forth a boundary condition (Nowacki [1962])

$$T(r,z)\big|_{z=0} = 0 \quad , \tag{10}$$

this being a somewhat artificial condition, physically corresponding to having an equivalent but negative heat source at height *d* above the boundary *z*=0 (the upper halfspace having the same heat conductivity λ_T), we arrive at the expression for the temperature field

$$T(r,z) = \frac{w}{4\pi\lambda_T}(R_1^{-1} - R_2^{-1}) \quad , \tag{11}$$

where

$$R_1 = \sqrt{r^2 + (z-d)^2} \qquad , \qquad R_2 = \sqrt{r^2 + (z+d)^2} \quad . \tag{12}$$

Substituting the temperature field into eqn.(8), it yields the solution for the thermoelastic displacement $\vec{u}_T$ (subscript T stands for „thermoelastic“), cf. [Hvoždara and Brimich, 1995]

$$(u_r)_T = Ar\left\{R_1^{-1} - R_2^{-1} - 2dzR_2^{-3} + 4d(1-\nu)R_2^{-1}(R_2 + d + z)^{-1}\right\} ,$$

$$(u_z)_T = A\left\{(z-d)R_1^{-1} - (z+d)R_2^{-1} - 2d(1-2\nu)R_2^{-1} - 2zd(z+d)R_2^{-3}\right\} , \tag{13}$$

where $A = w(3\lambda + 2\mu)\alpha_T\,[8\pi\lambda_T(\lambda + 2\mu)]^{-1}$ and $\nu = \lambda / 2(\lambda + \mu)$ is the Poisson ratio.

The scenario for our modeled geodynamic event is as follows: Prior to the time instant t_1 the heat source remains switched off and the halfspace has a constant density ρ_0 producing no gravity anomaly on the boundary z=0, thus having

$$\Delta g(t_1) = 0 \quad . \tag{14}$$

Then the point heat source at depth d is switched on. At a time instant t_2 the system reaches a thermal equilibrium with the boundary condition of eqn.(10) with thermal displacements of eqn.(13) producing a new and non-uniform density distribution $\rho(x,y,z) = \rho(r,z) \neq \rho_0$, and an uplift of the halfspace. Both the uplift and the non-uniformity of the density generate a gravity anomaly at and after the instant t_2

$$\Delta g(t_2) = \Delta g_1(t_2) + \Delta g_2(t_2) \ , \tag{15}$$

where the non-uniform density generates $\Delta g_1(t_2)$ and the uplift generates $\Delta g_2(t_2)$. As was already mentioned, the non-uniformity (perturbation) of the density distribution is caused by the thermoelastic displacement, according to (cf. [ibid.])

$$\rho(r,z) = \rho_0 - \rho_0\, div\, \vec{u}_T \quad . \tag{16}$$

The thermoelastic density perturbation $-\rho_0\, div\, \vec{u}_T$ generates $\Delta g_1(t_2)$ that can be computed from the perturbation potential V_P [ibid.] satisfying the Poisson equation below the boundary, κ being the gravitational constant,

$$div\, grad\, V_P(r,z) = 4\pi\kappa\rho_0\, div\, \vec{u}_T \ , \qquad z \geq 0 \ , \tag{17}$$

and the Laplace equation above the boundary

$$div\, grad\, V_P(r,z) = 0 \ , \qquad z < 0 , \tag{18}$$

with help of the boundary conditions stipulating that the perturbation potential and its first derivatives must be continuos at the boundary z=0, as

$$\Delta g_1(t_2) = \left.\frac{\partial V_P}{\partial z}\right|_{z=0} \quad . \tag{19}$$

For that sake we need to evaluate the r.h.s. of eqn.(17), cf. [ibid.]:

$$div\,\vec{u}_T = 2A\left\{R_1^{-1} - R_2^{-1} + 2(1-2\nu)d(z+d)R_2^{-3}\right\} , \tag{20}$$

which results in

$$\Delta g_1(t_2) = -8\pi\kappa\rho_0 A(1-\nu)\frac{d}{\sqrt{r^2+2d^2}} \quad . \tag{21}$$

The $\Delta g_2(t_2)$, being a consequence of the vertical uplift, is expressed as [ibid.]

$$\Delta g_2(t_2) = \left[\frac{\partial\gamma}{\partial z} - 2\pi\kappa\rho_0\right](u_z)_T \quad , \tag{22}$$

where $\frac{\partial\gamma}{\partial z}$ is the vertical gradient of normal gravity and the displacement is given by eqn.(13).

Hence our synthetic (modeled) temporal change of the gravity anomaly reads, cf. eqns. (3) and (14)

$$\Delta^t g = \Delta g_1(t_2) + \Delta g_2(t_2) \tag{23}$$

and is evaluated by substituting from eqns. (21) and (22). Now the synthetic $\Delta^t TG$ and $\Delta^t DTG$ sequences can be computed and interpreted.

The model that is presented here was constructed with the following parameters: the point heat source is located at the depth of 4 km, its heat power w = 6.596 10^6 W, the halfspace having the following material properties: ρ_0 = 2 670 kg/m^3, λ_T = 3 Wm^{-1}K^{-1} , α_T = 10^{-6} K^{-1}, λ = 7.05 10^{10} Pa, μ = 6.075 10^{10} Pa. The resulting gravity anomaly is presented in Fig.2.

Fig.2.

Fig.2 *The temporal change of gravity anomaly caused by the model heat source*.

Fig.3.

Fig.3 *The* $\Delta^t DTG$ *sequence, caused by the model point heat source at the depth of 4 km - a detailed view. This sequence is computed in a planar approximation and the values of the truncation parameter are in kilometers. The step in the truncation parameter in the sequence is 0.5 km.*

Figure 3 presents the $\Delta^t DTG$ sequence generated by our model. Again the sequence that was animated on a computer is displayed here with all the profiles drawn in the same plot, hoping that the reader can follow the evolution of the sequence. The sequence evolves in a way resembling the evolution of a DTG sequence generated by one point mass anomaly, cf. Fig.1b. From this we conclude that the temporal variation of the gravity anomaly is generated by a temporal change (perturbation) of the density distribution that is either a „point mass perturbation“ or a radial perturbation density field. The dimple (pimple) sets on at the value of s_0^* = 5.75 km $\pm$ 0.5 km (Fig.3b), from which the depth of the „point mass perturbation“ or of the center of the radial perturbation density field is determined (cf. eqn.(2b)) as d = 5.75 km $\pm$ 0.5 km. What this means is that by direct processing of data on the temporal change of the gravity anomaly we have constructed a $\Delta^t DTG$ sequence that yielded the interpretation that the temporal change of gravity was caused by a point source located at a depth of 5.75 ($\pm$ 0.5) km.

Fig.4

Fig.4 *The perturbation density field produced by the point heat source after reaching thermal equilibrium. The horizontal axis is distance from the point heat source in meters, the vertical axis is depth in meters.*

We observe immediately, that our interpretation is precise to 500 meters and inaccurate by 1.75 km. The precision is given by the step in the planar truncation parameter. But what causes such a bias (offset) in the determination of the depth of the heat source? Since we are interpreting a synthetic case and we know the model, we can take a look at the perturbation density given by eqns. (16) and (20) - it is portrayed in Fig.4. The perturbation density field is only nearly radial, the deviation from being radial being caused by the boundary condition of eqn.(10). However, the signature of this perturbation density appears to be in the $\Delta^t DTG$ sequence alike that of a point mass perturbation, although with a shifted (offset) depth. In order to have an unbiased interpretation of the depth of the point heat source, we have to establish a new relationship between the depth of the point heat source and the instant of the dimple onset, which would replace eqn.(2b). This is accomplished by simulating the point heat source at various depths in the interval of 1 km to 30 km and determining the respective instants of the dimple onset, in other words by computer simulation. Surprisingly enough, this relationship appears to be linear and reads

$$d[km] \doteq 0.694\left(\pm 0.002\right)\, s_0^*[km] \ . \tag{24}$$

for depths between 1km and 30 km.

Conclusions

In the first section it was suggested that a temporal change of the gravity anomaly can be interpreted in terms of a density perturbation generating the temporal gravity change in the same way a gravity anomaly is interpreted in terms of an anomalous density generating the gravity anomaly. This suggestion is general and may be applied to forward and inverse modeling as well as to any other method of the gravimetric inversion. Here we have applied this suggestion to a particular gravity inversion technique, namely the truncation filtering. The second section provides the apparatus for such treatment. In the third section we have verified the said concept on a simplistic model case - a geodynamic event driven by a point source of heat embedded in an homogenous elastic halfspace.

It was shown, that although the signature of the density, perturbed due to the point heat source, in the $\Delta^t DTG$ sequence is alike the point mass anomaly signature in the DTG sequence, the interpreted depth of the point source, when using the expression relating the depth of a point mass source to the instant of the dimple onset (eqn.(2b)), is biased. This is caused by the fact that the perturbation density field is nearly radial rather than radial. A new formula for determining the depth of the point heat source from the instant of the dimple onset, eqn.(24), was established and validated by the computer simulation (modeling).

At the moment the truncation filtering methodology is limited to interpretations in terms of point sources only (cf. Vajda and Vaníèek [1997]), but work is in progress to extend its capabilities to more realistic density distributions.

It is also suggested, yet not tested, that real observed temporal variations of gravity anomalies that are well isolated and pronounced, and accompanied by heat flows, be interpreted by means of the technique presented in this paper, to provide the maximum depth to source of heat estimates (with help of eqn.(24)), in particular for estimates of maximum depth to sources of volcanic activity.

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