A note on spectral filtering of the truncated geoid

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A b s t r a c t : Producing a frequency window (wavelength bandwidth) part of the truncated geoid is understood under the term „spectral filtering of the truncated geoid" in the context of this paper. We shall furthermore focus on the high frequency (short wavelength) part of the truncated geoid. Three kinds of the high frequency part of the truncated geoid are introduced herein. All three are defined via integral transforms and referred to as high degree truncated geoids. Spectral form expressions are derived for these transforms. It is shown, that the three transforms differ from each other, and represent slightly different physical quantities.

Key words: truncated geoid, spectral windowing

1. Introduction

The truncated geoid (in the sequel often abbreviated as TG), defined in the second section as a convolution of gravity anomalies with the Stokes function (the kernel) on a spherical cap, may be interpreted eventually in terms of density anomalies, that generate the anomalous surface gravity, as suggested in *Christou et al. (1989), Vajda (1995), Vajda* and *Vaníček (1996), Vajda* and *Vaníček (1997)*.

It is sometimes useful to work with a high frequency (short wavelength) part of the truncated geoid. By the high frequency part, or the short wavelength part, we understand the sum of terms of degrees higher than some pre-selected degree in the series expansion of the TG into surface spherical

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harmonic functions. We also sometimes refer to the said series expansion as a spectral representation of the TG or a TG expressed in the spectral domain. That is also why we will refer to producing the high frequency part of the TG as high-pass filtering of the TG.

To produce such high-pass filtered truncated geoid, one can convert the TG into its spectral representation (in terms of a series of surface spherical harmonics), and cut off the unwanted low frequencies (long wavelengths). On the other hand one may wish to define the high degree truncated geoid (the high frequency part of the TG, the high-pass filtered TG) as either

1. a convolution of high degree (high frequency part of) gravity anomalies with the Stokes function on a spherical cap, or

2. a convolution of gravity anomalies with the high frequency part of the Stokes function (the spheroidal Stokes function) on a spherical cap, or even

3. a convolution of high degree (high frequency part of) gravity anomalies with the spheroidal Stokes function on a spherical cap,

which in other words may be expressed as defining the high pass filtered TG either

1. by high pass filtering the gravity anomalies, or

2. by high pass filtering the kernel, or even

3. by high pass filtering both the kernel and the gravity anomalies

in the convolution integral defining the TG.

Below, we shall examine all of these four mentioned objects, compare them, and address the differences. For the sake of the comparison, the spectral forms of the high degree truncated geoids will be derived. We will show, in the spectral domain, that the three kinds of the high degree TG differ from each other.

2. Truncated geoid

The truncated geoid is defined by the *truncated Stokes integral* (e.g., *Molodenskij et al., 1962; Vaníček et al., 1987, Vajda* and *Vaníček, 1998)*

$$N^{\psi_0}(P) = \frac{R}{4\pi\gamma} \iint\limits_{C(\psi_0)} \Delta g\ (Q) S(P,Q) \mathrm{d}\sigma(Q). \tag{1}$$

The truncated geoidal height N^{ψ_0} is evaluated at the computation point P as a convolution of the gravity anomalies Δg, free of harmonics of degree zero and one, on a spherical cap $C(\psi_0)$ of radius ψ_0 centred at P, with the Stokes function S *(Stokes, 1849)* being the kernel. Point Q is the integration point. R is the radius of the boundary sphere (e.g., mean earth) and γ is the normal gravity. The differential $\mathrm{d}\sigma$ is the surface element. Eq. (1) can be expressed in the local polar coordinates of point P as (cf. Fig. 1):

$$N^{\psi_0}(P) = \frac{R}{4\pi\gamma} \int\limits_0^{\psi_0} \int\limits_0^{2\pi} \Delta g\ (\psi,\alpha) S(\psi) \sin(\psi) \mathrm{d}\alpha \mathrm{d}\psi, \tag{2}$$

where ψ is the spherical distance between points P and Q, and α is azimuth of point Q. The Stokes function in closed form reads (ibid.) as

$$S(\psi) = 1 + \frac{1}{\sin\left(\frac{\psi}{2}\right)} - 6\sin\left(\frac{\psi}{2}\right) - 5\cos(\psi) - \\ - 3\cos(\psi)\ln\left[\sin\left(\frac{\psi}{2}\right) + \sin^2\left(\frac{\psi}{2}\right)\right], \tag{3}$$

or in spectral form as

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos(\psi)), \tag{4}$$

with $P_n(\cos(\psi))$ being the Legendre polynomials (e.g., *Vaníček* and *Krakiwsky, 1986*).

Radius ψ_0, called the truncation radius, is a free parameter of the TG. This is why we usually refer to it as the „truncation parameter".

Let us express the truncated geoid in a spectral form. To achieve this, the gravity anomalies are expressed in spectral form, i.e., in spherical harmonic function series on the boundary sphere:

$$\Delta g(\varphi,\lambda) = \sum_{n=2}^{\infty} \Delta g_n(\varphi,\lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[A_{nm}^{\Delta g} Y_{nm}^{c}(\varphi,\lambda) + B_{nm}^{\Delta g} Y_{nm}^{s}(\varphi,\lambda)\right], \tag{5}$$

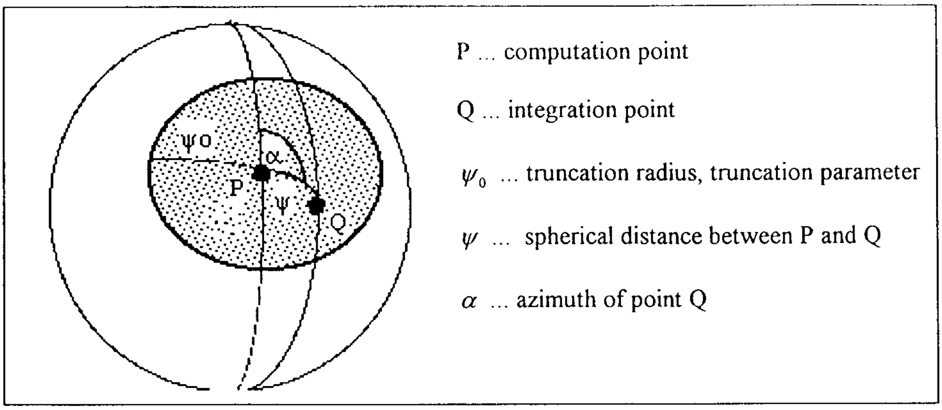

Fig. 1. The truncated Stokes integration.

where $Y^c_{nm}(\varphi,\lambda) = \cos(m\lambda)P_{nm}(\sin(\varphi))$, $Y^s_{nm}(\varphi,\lambda) = \sin(m\lambda)P_{nm}(\sin(\varphi))$ are (surface) spherical harmonic functions with $P_{nm}(\sin(\varphi))$ being the associated Legendre functions [ibid.]. The truncated Stokes transform (2) is now rewritten with a new kernel,

$$\begin{aligned} S^*(\psi_0,\psi) &= S(\psi) && \text{on } \langle 0;\psi_0\rangle \\ S^*(\psi_0,\psi) &= 0 && \text{on } (\psi_0;\pi\rangle \end{aligned} \tag{6}$$

so that the integration can be extended to the whole sphere (e.g., *Vaníček et al., 1987*):

$$N^{\psi_0}(P) = \frac{R}{4\pi\gamma}\int_0^{\pi}\int_0^{2\pi} \Delta g(\psi,\alpha)S^*(\psi_0,\psi)\sin(\psi)\mathrm{d}\psi\mathrm{d}\alpha. \tag{7}$$

The new kernel $S^*(\psi_0,\psi)$ can be also developed into Legendre polynomial series as:

$$S^*(\psi_0,\psi) = \sum_{n=2}^{\infty}\frac{2n+1}{2}\alpha_n(\psi_0)P_n(\cos(\psi)), \tag{8}$$

with

$$\alpha_n(\psi) = \frac{2}{n-1} - Q_n(\psi_0) \qquad n = 2,3,..., \tag{9}$$

where $Q_n(\psi_0)$ are the Molodenskij truncation coefficients *(Molodenskij et al., 1962; Vaníček et al., 1987*, Eq. (5.19))

$$Q_n(\psi_0) = \int\limits_{\psi_0}^{\pi} S(\psi) P_n(\cos(\psi)) \sin(\psi) \mathrm{d}\psi. \tag{10}$$

These coefficients can be numerically evaluated by Paul's algorithm *(Paul, 1973)*. Substituting Eqs (5) and (8) into Eq. (7), utilising the Legendre decomposition formula [e.g., *Vaníček* and *Krakiwsky, 1986*, Eq. (20.51)], and making use of the orthogonality of the spherical harmonic functions, we arrive at the spectral representation of the truncated geoid

$$N^{\psi_0}(P) = \frac{R}{2\gamma} \sum_{n=2}^{\infty} \alpha_n(\psi_0) \Delta g_n(\varphi_P, \lambda_P) = \frac{R}{2\gamma} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{2}{n-1} - Q_n(\psi_0) \right) \cdot \\ \cdot \left[A_{nm}^{\Delta g} Y_{nm}^{c}(\varphi_P, \lambda_P) + B_{nm}^{\Delta g} Y_{nm}^{s}(\varphi_P, \lambda_P) \right]. \tag{11}$$

3. High-pass filtered truncated geoid – the high degree truncated geoid

When we cut off the low frequency part (spherical harmonics up to degree ℓ) from the TG represented by Eq. (11), we arrive at the spectral form of the high pass filtered TG

$$\delta N_{\ell}^{\psi_0}(P) = \frac{R}{2\gamma} \sum_{n=\ell+1}^{\infty} \alpha_n(\psi_0) \Delta g_n(\varphi_P, \lambda_P) = \frac{R}{2\gamma} \sum_{n=\ell+1}^{\infty} \sum_{m=0}^{n} \left(\frac{2}{n-1} - Q_n(\psi_0) \right) \cdot \\ \cdot \left[A_{nm}^{\Delta g} Y_{nm}^{c}(\varphi_P, \lambda_P) + B_{nm}^{\Delta g} Y_{nm}^{s}(\varphi_P, \lambda_P) \right]. \tag{12}$$

Let us forget about Eq. (12), for the moment, and define three kinds of the high degree truncated geoid via integral transforms as follows:

High degree truncated geoid of the first kind, which differs from the original TG by replacing the gravity anomalies in Eq. (1) with high frequency gravity anomalies $\delta(\Delta g)^{\ell}$ of degree ℓ:

$${}^{1}\delta N_{\ell}^{\psi_0}(P) = \frac{R}{4\pi\gamma} \iint\limits_{C(\psi_0)} \delta(\Delta g)^{\ell}(Q) S(P,Q) \mathrm{d}\sigma(Q) = \\ = \frac{R}{2\gamma} \int\limits_{0}^{\psi_0} \delta(\Delta \overline{g}(\psi))^{\ell} S(\psi) \sin(\psi) \mathrm{d}\psi, \tag{13}$$

where

$$\delta\left(\Delta\overline{g}(\psi)\right)^{\ell} = \frac{1}{2\pi}\int\limits_{0}^{2\pi}\delta\left(\Delta g(\psi,\alpha)\right)^{\ell}\mathrm{d}\alpha$$

is the azimuthal average of the high degree (high-pass) gravity anomaly (of degree ℓ)

$$\delta(\Delta g(\psi,\alpha))^{\ell} = \sum_{n=\ell+1}^{\infty}\sum_{m=0}^{n}\left(A_{nm}^{\Delta g}Y_{nm}^{c}(\varphi,\lambda) + B_{nm}^{\Delta g}Y_{nm}^{s}(\varphi,\lambda)\right). \tag{14}$$

High degree truncated geoid of the second kind, which differs from the original TG by replacing the kernel in the integral transform (1) with high-pass filtered Stokes's function of degree ℓ – the spheroidal Stokes function$S^{\ell}(P,Q)$

$$\begin{aligned} {}^{2}\delta N_{\ell}^{\psi_0}(P) &= \frac{R}{4\pi\gamma}\iint\limits_{C(\psi_0)}\Delta g(Q)S^{\ell}(P,Q)\mathrm{d}\sigma(Q) = \\ &= \frac{R}{2\gamma}\int\limits_{0}^{\psi_0}\Delta\overline{g}(\psi)S^{\ell}(\psi)\sin(\psi)\mathrm{d}\psi, \end{aligned} \tag{15}$$

where

$$\Delta\overline{g}(\psi) = \frac{1}{2\pi}\int\limits_{0}^{2\pi}\Delta g(\psi,\alpha)\mathrm{d}\alpha,$$

and

$$S^{\ell}(\psi) = S(\psi) - \sum_{n=2}^{\ell}\frac{2n+1}{n-1}P_n(\cos(\psi)) = \sum_{n=\ell+1}^{\infty}\frac{2n+1}{n-1}P_n(\cos(\psi)). \tag{16}$$

High degree truncated geoid of the third kind where we replace both the kernel and the gravity anomalies in the integral transform (1) by their high-pass filtered (high frequency) counterparts of degree ℓ:

$$\begin{aligned} {}^{3}\delta N_{\ell}^{\psi_0}(P) &= \frac{R}{4\pi\gamma}\iint\limits_{C(\psi_0)}\delta(\Delta g)^{\ell}(Q)S^{\ell}(P,Q)\mathrm{d}\sigma(Q) = \\ &= \frac{R}{2\gamma}\int\limits_{0}^{\psi_0}\delta(\Delta\overline{g}(\psi))^{\ell}S^{\ell}(\psi)\sin(\psi)\mathrm{d}\psi, \end{aligned} \tag{17}$$

Equation (14) describes a high degree gravity anomaly at an integration point Q, given in polar coordinates (ψ, α) on the left-hand side, and in geographical coordinates (φ, λ) on the right-hand side. $S^{\ell}(\psi)$ is the spheroidal Stokes kernel *(Vaníček et al., 1987)*.

For $\psi_0 = \pi$, all three kinds of the high degree truncated geoid become identical, due to the orthogonality of Legendre polynomials on $\langle 0; \pi \rangle$, and amount to the high degree (short wavelength) geoid :

$$\begin{aligned}
\delta N_{\ell}(P) &= \frac{R}{4\pi\gamma} \oiint \delta(\Delta g)^{\ell}(Q) S^{\ell}(P, Q) \mathrm{d}\sigma(Q) = \\
&= \frac{R}{2\gamma} \int_0^{\pi} \delta(\Delta \overline{g}(\psi))^{\ell} S^{\ell}(\psi) \sin(\psi) \mathrm{d}\psi = \\
&= \frac{R}{4\pi\gamma} \oiint (\Delta g)(Q) S^{\ell}(P, Q) \mathrm{d}\sigma(Q) = \\
&= \frac{R}{2\gamma} \int_0^{\pi} \Delta \overline{g}(\psi) S^{\ell}(\psi) \sin(\psi) \mathrm{d}\psi = \\
&= \frac{R}{4\pi\gamma} \oiint \delta(\Delta g)^{\ell}(Q) S(P, Q) \mathrm{d}\sigma(Q) = \\
&= \frac{R}{2\gamma} \int_0^{\pi} \delta(\Delta \overline{g}(\psi))^{\ell} S(\psi) \sin(\psi) \mathrm{d}\psi.
\end{aligned} \tag{18}$$

When $\psi_0 < \pi$, the three Eqs (13), (15), and (17) give different results.

All three kinds of the high degree truncated geoid have two free parameters, the truncation parameter ψ_0 and the cut-off spherical harmonic degree (spheroidal degree) ℓ.

4. Spectral form of the high degree truncated geoid

Following the same procedure as in section 2, we can derive the spectral forms of all three kinds of the high degree truncated geoid defined in section 3. We will need to make use of the $S^*_{\ell}(\psi_0, \psi)$ kernel defined analogously to the $S^*(\psi_0, \psi)$ kernel as:

$$\begin{aligned}
S^*_{\ell}(\psi_0, \psi) &= S^{\ell}(\psi) \qquad \text{on } \langle 0; \psi_0 \rangle \\
S^*_{\ell}(\psi_0, \psi) &= 0 \qquad \text{on } (\psi_0; \pi \rangle \, ,
\end{aligned} \tag{19}$$

which reads in spectral form as *(Vaníček et al., 1987)*

$$S_\ell^*(\psi_0,\psi)=\sum_{n=0}^{\infty}k_n(\psi_0)P_n(\cos(\psi)). \tag{20}$$

The spectral harmonic coefficients of the $S_\ell^*(\psi_0,\psi)$ kernel are:

$$\begin{aligned} k_n(\psi_0) &= -Q_n^\ell(\psi_0) && \text{for } n\le\ell,\\ k_n(\psi_0) &= \frac{2}{n-1}-Q_n^\ell(\psi_0) && \text{for } n>\ell, \end{aligned} \tag{21}$$

where $Q_n^\ell(\psi_0)$ are the Molodenskij truncation coefficients of the spheroidal Stokes function (e.g., *Vaníček et al., 1987; Martinec, 1993*)

$$Q_n^\ell(\psi_0)=\int_{\psi_0}^{\pi}S^\ell(\psi)P_n(\cos(\psi))\sin(\psi)\mathrm{d}\psi. \tag{22}$$

After performing the required mathematical development similar to this in section 3, we arrive at the spectral form expressions for the three kinds of the high degree TG:

$${}^1\delta N_\ell^{\psi_0}(P)=\frac{R}{2\gamma}\sum_{n=\ell+1}^{\infty}\left(\frac{2}{n-1}-Q_n(\psi_0)\right)\Delta g_n(\varphi_P,\lambda_P), \tag{23}$$

$$\begin{aligned} {}^2\delta N_\ell^{\psi_0}(P)=&\frac{R}{2\gamma}\sum_{n=2}^{\ell}\left(-Q_n^\ell(\psi)\right)\Delta g_n(\varphi_P,\lambda_P)+\\ &+\frac{R}{2\gamma}\sum_{n=\ell+1}^{\infty}\left(\frac{2}{n-1}-Q_n^\ell(\psi_0)\right)\Delta g_n(\varphi_P,\lambda_P) \end{aligned} \tag{24}$$

$${}^3\delta N_\ell^{\psi_0}(P)=\frac{R}{2\gamma}\sum_{n=\ell+1}^{\infty}\left(\frac{2}{n-1}-Q_n^\ell(\psi_0)\right)\Delta g_n(\varphi_P,\lambda_P), \tag{25}$$

where the n-th degree spherical harmonic of the gravity anomaly Δg_n is given in Eq.(5).

Let us now recall Eq. (12). Note that the result of Eq. (12) is identical with the result of Eq. (23). This means that it is the high degree TG of the first kind that is the true high frequency part of the TG. The other two kinds of the high degree TG are introduced sort of artificially, which does not mean that they cannot be used and interpreted in some applied investigations). The cause for the differences among the three kinds of the high degree TG (cf. Eqs (23), (24), and (25)) is the loss of orthogonality of the spherical harmonic functions when truncating the integration domain (of the defining integral transform) from the full sphere to a spherical cap.

5. Conclusions

We have introduced three kinds of the high degree truncated geoid, whereby either gravity anomalies or the kernel of the integral transform defining the truncated geoid, or both, are high-pass filtered. These three are defined by Eqs (13), (15) and (17). Similarly, low-pass or band-pass filtered truncated geoids could be introduced. The frequency domain representations, or spectral forms, of all three kinds of the high degree truncated geoid are given by Eq. (23) to (25). Each kind represents a slightly different high-pass filtered truncated geoid, due to the loss of orthogonality of the associated Legendre functions when truncating the integration domain from a full sphere to a spherical cap of radius ψ_0. Each high degree truncated geoid has two free parameters: the truncation radius ψ_0 and the spheroidal cut-off degree ℓ. Thus the high degree TG may be understood as an object resulting from a combination of spatial and spectral filtering of surface gravity data (the gravity anomalies). The spatial filtering is represented by truncation, which is nothing else but weighted spherical windowing, cf. Eq. (1). The spectral filtering is represented by cutting off the unwanted wavelengths of the gravity field. This makes the high degree truncated geoid a promising tool in gravity inversion, which is currently under investigation.

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