Poisson Downward Continuation Solution by the Jacobi Method

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Abstract

Downward continuation is a continuing problem in geodesy and geophysics. Inversion of the discrete form of the Poisson integration process provides a numerical solution to the problem, but because the **B** matrix that defines the discrete Poisson integration is not always well conditioned the solution may be noisy in situations where the discretization step is small and in areas containing large heights. We provide two remedies, both in the context of the Jacobi iterative solution to the Poisson downward continuation problem. First, we suggest testing according to the upward continued result from each solution, rather then testing between successive solutions on the geoid, so that choice of a tolerance for the convergence of the iterative method is more meaningful and intuitive. Second, we show how a tolerance that reflects the conditioning of the **B** matrix can regularize the solution, and suggest an approximate way of choosing such a tolerance. Using these methods, we are able to calculate a solution that appears regular in an area of Papua New Guinea having heights over 3200 m, over a grid with 1 arc-minute spacing, based on a very poorly conditioned **B** matrix.

Keywords: downward continuation, Poisson integration, Jacobi method, regularization

1. Introduction

A continuous plague in geodesy and geophysics is the problem of downward continuation of gravity. That is, the determination of the values of the field below the surface of the Earth from observations of gravity field on one surface. While various methods may be used for this process, we will discuss only the theoretically exact solution by the inversion of Poisson integration.

Downward continuation can be considered the inverse operation of Poisson integration, which itself is a solution to the first boundary problem of potential theory: given values of a potential field on a sphere, that is harmonic outside the sphere, Poisson integration provides values of the field anywhere outside the sphere (MacMillan, 1930). In our case, the field will be the anomalous gravity field in the Helmert space, as described by Vaníček et al. (1996), multiplied by the geocentric radius. Considering the geoid as a spherical surface allows us to apply Poisson integration, valid for computation points external to the geoid, according to the formula (Heiskanen and Moritz, 1967):

$$\Delta g^h(r,\varphi,\lambda) = \frac{R^2(r^2 - R^2)}{4\pi r} \int_{\varphi'=-\pi/2}^{\pi/2} \int_{\lambda'=0}^{2\pi} \frac{\Delta g^h(R,\varphi',\lambda')}{\ell^3(r,\varphi,\lambda;R,\varphi',\lambda')} \cos\varphi' \, d\varphi' \, d\lambda' , \tag{1}$$

where $\Delta g^h(r,\varphi,\lambda)$ is the Helmert gravity anomaly at a point with geocentric spherical latitude φ, longitude λ and radius r. The symbol R stands for the mean radius of the Earth, used here to approximate the radius of the geoid in the geocentric spherical coordinate system where the calculation is performed. We assume the value to be R=6371008.7714 m, i.e., equal to the mean radius of the GRS-80 ellipsoid (Moritz, 1980). $\Delta g^h(R,\varphi,\lambda)$ is the Helmert gravity anomaly at an integration point on the geoid with geocentric spherical latitude φ and longitude λ. Finally, $\ell(r,\varphi,\lambda;R,\varphi',\lambda')$ is the distance between the computation and integration points, also calculated in a geocentric spherical coordinate system.

The integral in Eq. (1) is a Fredholm integral of the 1st kind (Fredholm, 1900). It can be evaluated numerically in a discrete form for a set of N points, each point P_i having coordinates $(r_i, \varphi_i, \lambda_i)$, by:

$$\forall i \in 1,2,\ldots,N : \Delta g_i^h(r_i,\varphi_i,\lambda_i) = \frac{R^2(r_i^2 - R^2)}{4\pi r_i} \sum_{j=1}^{M} \frac{\Delta g_j^h(R,\varphi_j,\lambda_j)}{\ell^3(r_i,\varphi_i,\lambda_i;R,\varphi_j,\lambda_j)} \cos\varphi_j \Delta\varphi\Delta\lambda , \tag{2}$$

where M is the number of points used for discrete representation of the gravity anomalies on the geoid, $\Delta g_j^h(R,\varphi_j,\lambda_j)$ is the gravity anomaly for the point P_j on the geoid having latitude φ_j and longitude λ_j, $\Delta\varphi$ is the step size in the latitudinal direction and $\Delta\lambda$ is the step size in the longitudinal direction. In matrix-vector notation, Eq. (2) can be written as (Vaníček et al [1996]):

$$\mathbf{\Delta g^t} = \mathbf{B}\ \mathbf{\Delta g^g}, \tag{3}$$

where $\mathbf{\Delta g^t}$ is a vector of gravity anomalies of length N, containing all point anomalies Δg_i^h on the terrain; $\mathbf{\Delta g^g}$ is a vector of gravity anomalies of length M,containing all point anomalies Δg_j^h on the geoid, and the N by M matrix $\mathbf{B}$ is composed of elements b_{ij} given by (cf. Vaníček et al. [1996]):

$$b_{ij} = \frac{R^2(r_i^2 - R^2)}{4\pi r_i} \frac{1}{\ell^3(r_i,\varphi_i,\lambda_i;R,\varphi_j,\lambda_j)} \cos\varphi_j \Delta\varphi\Delta\lambda . \tag{4}$$

Usually, $N = M$, and the horizontal coordinates of the locations of the $\mathbf{\Delta g^g}$ vector correspond to the locations of the $\mathbf{\Delta g^t}$ vector. This results in a square $\mathbf{B}$ matrix with the maximum value in each row and column being along the main diagonal, an arrangement which will be assumed in the following developments.

Note that the discretization given by Eq. (2), which relates point values on the geoid to point values on the Earth surface, is not the only possible way to discretize the Poisson integration. In addition to the "point-point" approach given by Eq. (2), there are the "point-mean", "mean-mean", and "mean-point" discretizations, according to what representation of the field is used on each surface where it is considered – the meaning of these descriptors should be self-evident. These alternative approaches result in different formulations of the elements of the **B** matrix and a different interpretation of the elements of $\mathbf{\Delta g^t}$ and $\mathbf{\Delta g^g}$. Eq. (3) and the developments presented here are valid for any of the four formulations, and for the calculations in this paper a point-mean scheme will be used, where the output is mean Helmert gravity anomalies, averaged over each cell on the geoid, and the elements of the B matrix are given by (cf. Vaníček et al. [1996]):

$$b_{ij} = \frac{R^2(r_i^2 - R^2)}{4\pi r_i}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\ell^3(r_i,\varphi_i,\lambda_i;R,\varphi_{j,k},\lambda_{j,k})}\cos\varphi_{j,k}\Delta\varphi\Delta\lambda\,, \tag{5}$$

which represents an average of n kernel values calculated at each point P_k with coordinates $\varphi_{j,k}$, $\lambda_{j,k}$. The averaging points are usually given on a regular grid within each cell, and the cell dimensions are equal to $\Delta\varphi$ and $\Delta\lambda$. The other 3 formulations for the **B** matrix are discussed in **(**Vaníček and Santos, 2010).

While Eq. (3) allows us to calculate $\mathbf{\Delta g^t}$ given $\mathbf{\Delta g^g}$, we really want to perform the inverse operation. We want gravity anomalies on the geoid based on values at the Earth's surface, i.e., we want to use the following equation:

$$\mathbf{\Delta g^g = B^{-1}\, \Delta g^t}. \tag{6}$$

Fredholm's integral equations of first kind, such as Eq.(1), are inherently unstable. Even though the problem of downward continuation is "well posed" in the Hadamard sense (Wong, 2002), the system of linear equations formed as a numerical equivalent to the Fredholm integral may be ill-conditioned (see, e.g., Martinec, 1996). Under some circumstances, the system may be almost singular, meaning that the inverse $\mathbf{B}^{-1}$ may be very difficult to obtain, a situation that some people solve by adopting one regularization scheme or another (e.g. Schwarz [1978], Goli et al. [2010]). Under these circumstances, no exact iterative solution can be found but a non-iterative solution can. The LU decomposition is the most efficient non-iterative numerical method for calculating $\mathbf{B^{-1}\, \Delta g^t}$ exactly, since it does not require a complete inversion of the **B** matrix (Press, 2002). However, it is still very time consuming because it goes through the decomposition of a potentially very large **B** matrix. For a typical example, using gravity anomalies given at 5 arc-minute spacing over a 2º by 2º region requires an **B** matrix with 576 rows and 576 columns (one for each integration/computation point).

2. The theory

An approximate iterative solution seems to be the direction to choose. The most intuitively attractive iterative approach is the Jacobi iterative method (e.g. Young [1971]), which may be formulated as (Vaníček et al. [1996]):

$$\mathbf{\Delta g^{g(k)} = \Delta g^t + (I\text{-}B)\, \Delta g^{g(k-1)}}, \tag{7}$$

where $\mathbf{\Delta g}^{g(k)}$ is the k-th estimate of $\mathbf{\Delta g}^{g}$, $\mathbf{\Delta g}^{g(k-1)}$ is the previous, $(k-1)$-st estimate, and the usual initial estimate of $\mathbf{\Delta g}^{g}$ is $\mathbf{\Delta g}^{g(0)} = \mathbf{\Delta g}^{t}$. Our formulation here is slightly different from the standard formulation by Jacobi which uses the diagonal elements of the **B** matrix instead of **I** and therefore has a slightly faster rate of convergence (Press, 2002). We have chosen the above form (Eq. (7)) since it is conducive to testing the convergence of the solution using the methods we prescribe.

The ill-conditioning of **B** increases with the maximum height of the evaluation points in the area of interest, and with decreasing the step size of the grid of integration points on the geoid. An upper bound, κ_{max}, of the condition number of the **B** matrix is given, according to Martinec (1996), by:

$$\kappa_{\max} = \left(\frac{r_{\max}}{R}\right)^{\pi/\Delta\Omega} = \left(1+\frac{H_{\max}}{R}\right)^{\pi/\Delta\Omega} \approx 1+\frac{\pi}{\Delta\Omega}\frac{H_{\max}}{R}, \tag{8}$$

where H_{max} is the maximum value of height for $i = 1,2,\ldots,N$, and $\Delta\Omega$ is the step size of the grid of input gravity anomalies, considered equal in both the latitudinal and longitudinal directions. Thus, for the area with heights over 3200 m in Papua-New Guinea (used for our computations here) and input Helmert anomalies on the topography spaced 1 arc-minute apart, the upper bound on the condition number is over 227. The practical result of such a high condition number is that the numerical precision of the solution is worse than that of the input. Note also that the condition number may be higher than that predicted by Eq. (8), which relies on a definition of the condition number that uses the eigenvalues of the **B** matrix rather than its singular values. The definition that uses eigenvalues is only valid when the **B** matrix is a normal matrix, which is rarely the case.

As an approximate rule, the loss of precision is equivalent to the base 10 logarithm of the condition number (e.g. Cheney and Kincaid [2008]), so that for a condition number of 1 there is no loss of precision, while for a condition number of 227 the loss of precision is up to 2 or 3 digits. So for gravity anomalies on the topography known with a precision of 0.01 mGal, the output gravity anomalies on the geoid may have a precision as low as 10 mGal, and any attempt to increase the precision of the output, for example by further iterations in the Jacobi method, will only add spurious noise to the result. Thus, it is reasonable to terminate the iterative process as soon as the best attainable result is reached.

A traditional approach to test whether an iterative solution has converged is to test the difference between the solutions from two successive iterations (e.g. Young [1971]). So, indicating the k-th difference being tested by the vector $\boldsymbol{\delta}^{g(k)}$, we would have:

$$\boldsymbol{\delta}^{g(k)} = \mathbf{\Delta g}^{g(k)} - \mathbf{\Delta g}^{g(k-1)}. \tag{9}$$

Since the process generating $\mathbf{\Delta g}^{g(k)}$ is convergent in the L_1 norm (Vaníček et al., 1996), we know that the accuracy improvement from any subsequent iterations will be less than $\boldsymbol{\delta}^{g(k)}$, and the process can be considered to have converged in L_1 when $\|\boldsymbol{\delta}^{g(k)}\|_{L1}$ is smaller than some prescribed tolerance. If the L_1 norm is used, a statement like "the errors will not exceed 0.1 mGal" can be substantiated. If the maximum value in $\boldsymbol{\delta}^{g(k)}$ has a magnitude less than 0.1 mGal, it is presumed that including additional iterations would not change any value in $\boldsymbol{\Delta g}^{g}$ by more than 0.1 mGal.

The difficulty in applying Eq. (9) is with choosing an adequate tolerance. We might assign a tolerance prescribing the desired accuracy for our $\boldsymbol{\Delta g}^{g\,(k)}$ determination. However, we may choose a desired accuracy higher than what the numerical apparatus is able to deliver. Also, even if the tolerance we set can be achieved by the numerical apparatus, there is no guarantee that our result will achieve the desired accuracy. In an ill-conditioned system, an exact solution will be very noisy, and as such will be very difficult to interpret.

An alternative to using Eq. (9) is, at each step, to compare "iterations of gravity anomalies on the topography", $\boldsymbol{\Delta g}^{t(k)}$, using the already calculated $\mathbf{B}$ matrix:

$$\boldsymbol{\Delta g}^{t(k)} = \mathbf{B}\, \boldsymbol{\Delta g}^{g(k)}. \tag{10}$$

The testing is then done on the difference between the gravity anomalies on the topographical surface determined from the *k*-th solution ($\boldsymbol{\Delta g}^{t(k)}$), and the input anomalies on the topographical surface ($\boldsymbol{\Delta g}^{t}$):

$$\boldsymbol{\delta}^{t(k)} = \boldsymbol{\Delta g}^{t(k)} - \boldsymbol{\Delta g}^{t}. \tag{11}$$

If we use $\boldsymbol{\delta}^{t(k)}$ as defined by Eq. (11), then we are no longer testing for convergence based on the desired accuracy of the result. Instead, we are testing whether the gravity anomalies arising from the *k*-th solution are distinguishable from those used as input. Assuming the system is well conditioned, this is done by comparing the L1 norm of $\boldsymbol{\delta}^{t\,(k)}$ to a tolerance reflecting the precision of the input data. Thus, if the input data has a precision of 0.01 mGal, we might suppose that we have derived results of the best possible quality when they produce input values less than 0.005 mGal different from the originals. In an ill conditioned system, where $\boldsymbol{\Delta g}^{t(k)}$ is imprecise because of imprecise values of $\boldsymbol{\Delta g}^{g(k)}$, it may be impossible to reproduce the input gravity anomalies to such a precision (Wong, 2002). However, the choice of half the precision of the input data as a tolerance will still produce a result compatible with an exact solution of Eq. (6), since exact solution methods are likewise blind to the effect of the ill-conditioned $\mathbf{B}$ matrix (e.g. Young [1971]).

This change in approach is important mainly because it gives a context for choosing tolerances. It is only expected to have a significant effect on results in cases where the convergence is very rapid. In fact:

$$\boldsymbol{\delta}^{g(k)} = \boldsymbol{\delta}^{t(k-1)}, \tag{12}$$

so that, for any given tolerance, testing using the geoid gravity anomalies always requires one additional iteration.

If we use a $\boldsymbol{\delta}^{t(k)}$ calculated by Eq. (11) or (12), and a tolerance consistent with the precision of our input data, we will be able to produce an iterative solution as close to the exact solution of the system of equations as the precision of our input data allows. However, if the system of equations is especially ill-conditioned then we will still produce noisy results. This can be avoided by taking the possible error in $\mathbf{\Delta g}^{g(k)}$ into account in choosing our tolerance for $\boldsymbol{\delta}^{t(k)}$.

If the tolerance chosen is consistent with the conditioning of the system of equations, the best possible solution under the circumstances may be reached. Thus, the choice of tolerance may be regarded as a regularization of the solution. If the result seems unwarrantedly rough, the tolerance applied may be too stringent, and can be varied until a reasonably smooth result is obtained. This refinement of the tolerance choice is equivalent to tuning the tolerance so that it properly reflects what the system of equations and input data are actually able to deliver.

As a rough attempt at defining a suitable tolerance a-priori, we can apply the rule described above, which says that the order of magnitude of the loss of precision due to ill conditioning is roughly equal to the logarithm of the condition number. If we take p as the L_1 norm of the vector of precision of the input data (e.g. 0.01 mGal), we can choose an approximate tolerance, T, based on:

$$T = p\kappa, \tag{13}$$

where κ is the condition number of the system of equations. If the system is well conditioned, then Eq. (13) provides a tolerance close to half of the precision of the input, suitable for testing $\|\boldsymbol{\delta}^{t(k)}\|_{L1}$. If the conditioning is poor, Eq. (13) scales the tolerance according to the expected loss of precision in the result.

In the following, we will test the methods discussed above on “real data” from an area of high mountains in Indonesia, based on a portion of the AusSEGM synthetic gravity field **(**Baran et al., 2006**)**. The next section will describe these results.

3. Results and discussion

3.1 Test area and data sets

We have performed three experiments to illustrate the points above. All cover the same area, from 147.5º to 148.5º longitude and -9º to -10º degrees latitude, using data from 147º to 149º longitude and -8.5º to -10.5º latitude, to avoid any edge effects (as described by Sun and Vaníček [1996]). This involves 14,400 input data points on a grid with1 arc-minute spacing. The large number of data points magnifies the differences in computation time for different solution methods. The maximum point height in the test area, which includes some large mountains in Indonesia, is above 3,200 metres. This corresponds, according to Eq. (8) to an upper bound for the condition number of the **B** matrix of about 227, although we have estimated the actual condition number of the **B** matrix, based on the L_1 norms of $\mathbf{B}$ and $\mathbf{B}^{-1}$, to be 1347. We therefore expect significant numerical instability in the downward continuation process, allowing us to assess how well different solutions deal with the numerical instability.

The input gravity data for the tests will be Helmertized gravity anomalies based on the AusSEGM synthetic data set (Baran et al., 2006). The input Helmert anomalies are given in Figure 1.

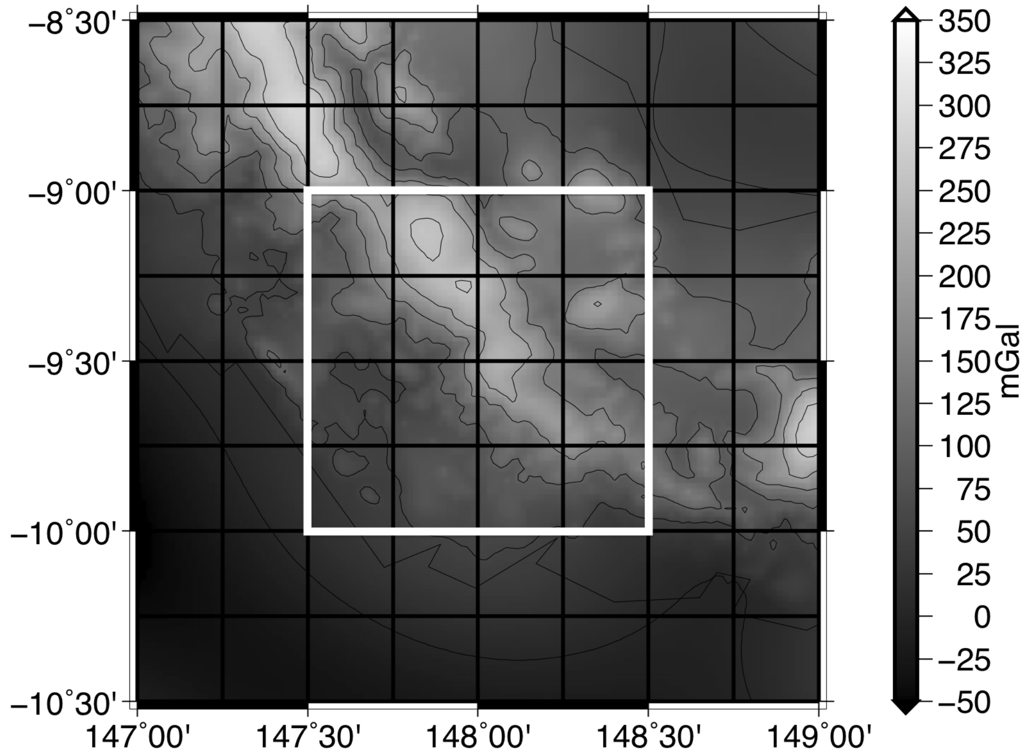


Figure 1. Helmert gravity anomalies over the study area.

While Figure 1 shows gravity anomaly data over the whole input area, the white square indicates the computation area. All input gravity anomalies are given in mGal to two decimal places, and lacking other information about their accuracy we assume they have a precision of 0.005 mGal. The values in this area range from -54.0 mGal to 290.0 mGal. Heights are taken from the digital elevation model (DEM) accompanying the AusSEGM data set, and are shown in Figure 2. They range from 0 m to over 3200 m.

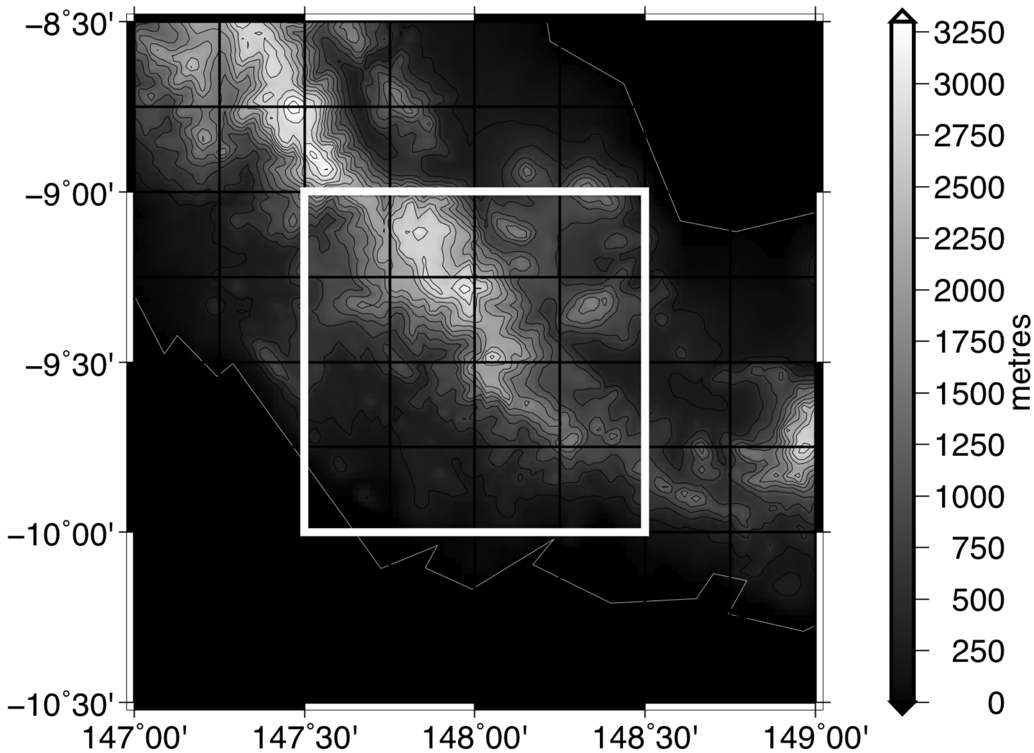


Figure 2. Heights of topography over the study area.

By using synthetic data, we will ensure that the comparisons of solution methods are affected as little as possible by errors in the input data. The influence of data errors on downward continuation is not a focus of this study.

3.2 Comparison of three methods for solving the downward continuation

First, we test whether our use of the $\boldsymbol{\delta}^{t(k)}$ vector, given by Eq. (11), allows us to properly assess convergence of the Jacobi iterative process, so as to obtain a result congruous with an exact solution. To do this, we have performed downward continuation over our test area using LU decomposition, which provides an algebraically exact solution, and the Jacobi iterative method with convergence testing based on $\|\boldsymbol{\delta}^{t(k)}\|_{L1}$, and also $\|\boldsymbol{\delta}^{g(k)}\|_{L1}$.
In this test, the $\|\boldsymbol{\delta}^{t(k)}\|_{L1}$ will be required to meet a tolerance of 0.005 mGal, equivalent to half the nominal precision of the input values. In other words, the iterative process will cease when the downward continued anomalies generate surface gravity anomalies indistinguishable in the L_1 norm from the input data on the level of one half of the maximum data error. In this case, the result should be very close to the solution by LU decomposition.
The $\|\boldsymbol{\delta}^{g(k)}\|_{L1}$ will be required to meet the same tolerance, allowing us to verify Eq. (12), and examine the additional computational cost of using $\boldsymbol{\delta}^{t(k)}$ instead of $\boldsymbol{\delta}^{g(k)}$, which requires an extra multiplication by the **B** matrix for each iteration. If Eq. (12) is correct, the extra multiplication need not be carried out in normal calculations, since $\boldsymbol{\delta}^{t(k)}$ can just

be calculated based on $\boldsymbol{\delta}^{g(k+1)}$ – and thus the computational cost of each method would be almost the same.

The result for the calculation using the Jacobi method and testing $||\boldsymbol{\delta}^{t(k)}||_{L1}$ is given in Figure 3. The results from LU decomposition and from testing $||\boldsymbol{\delta}^{g(k)}||_{L1}$ are almost identical to Figure 3, and so they are not plotted. Additionally, Table 1 provides statistics and computation times for each result. All computation times indicated below are for running computations on the ACENet grid computing system.

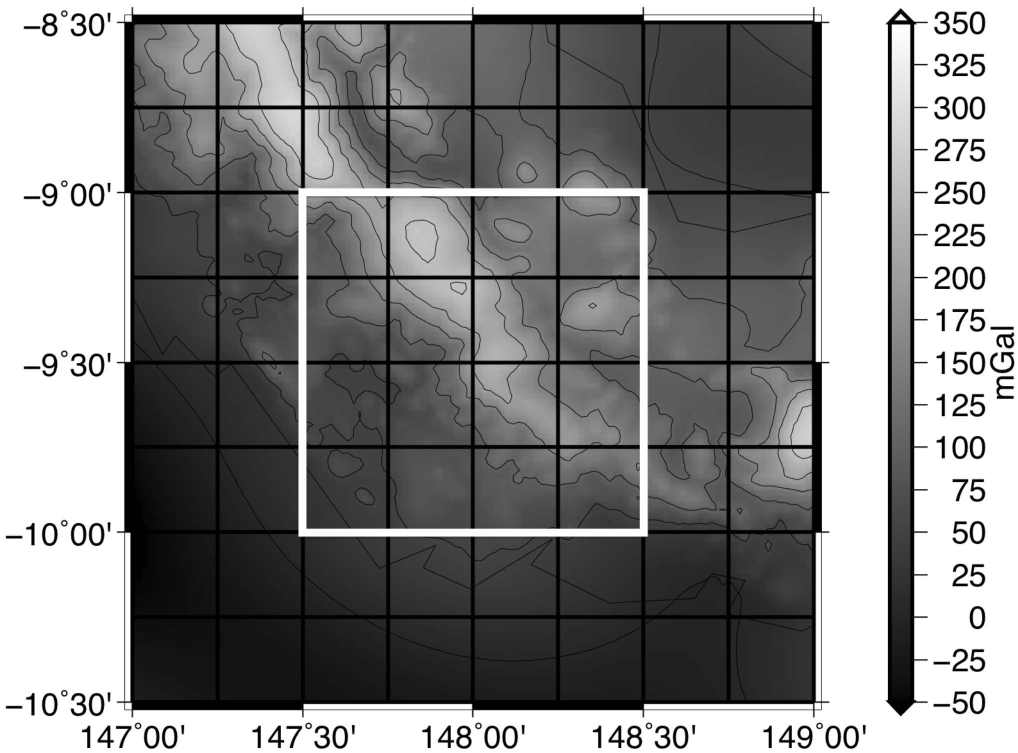


Figure 3: Result for Δg^{g} from Jacobi iterative approach based on norm of the δ^{t} vector.

Table 1: Statistics of results from three computation methods.

Method	**LU decomposition**	**Jacobi testing δ^{g}**	**Jacobi testing δ^{t}**
Minimum [mGal]	-291.7	-291.5	-291.5
Maximum [mGal]	1331.3	1331.2	1331.2
Mean [mGal]	127.4	127.4	127.4
Std. deviation [mGal]	79.0	79.0	79.0
No. of iterations	n/a	4684	4683
Processing time [mm:ss]	208:08	36:11	52:09

The similarity of all three results indicates that the tolerance of 0.005 mGal does lead to an iterative result commensurate with the exact result. Furthermore Eq. (12) is verified since the method using $\boldsymbol{\delta}^{g(k)}$ required exactly one more iteration than that using $\boldsymbol{\delta}^{t(k)}$, to meet the same tolerance. The time required per iteration, was about 0.46 seconds per iteration for the $\boldsymbol{\delta}^{g(k)}$ method and 0.67 seconds for the $\boldsymbol{\delta}^{t(k)}$ method. Thus the $\boldsymbol{\delta}^{t(k)}$ method, implemented as it was here, is relatively slow, and values of $\boldsymbol{\delta}^{t(k)}$ should be calculated by Eq. (12) in normal practice.

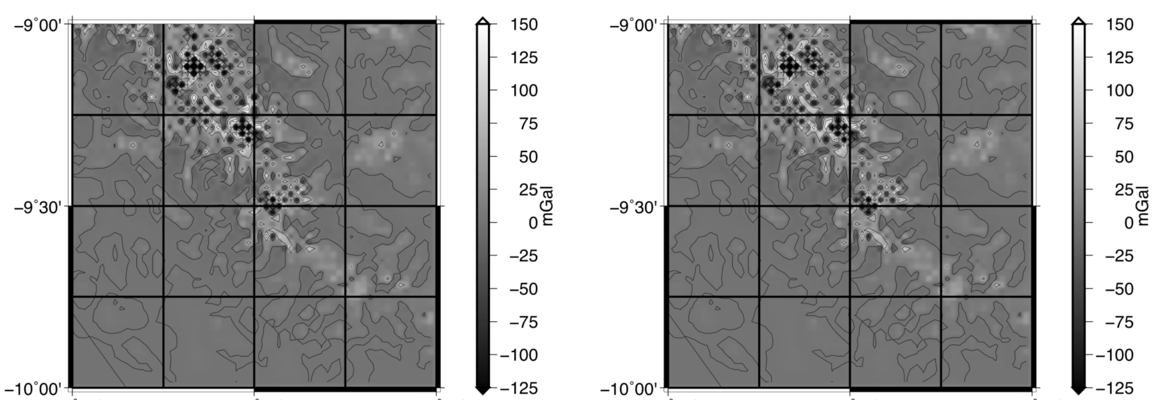
As expected, all three results are very spiky. This does not mean the iterative solutions were unsuccessful – they achieved their purpose by converging to a result very close to the LU decomposition result. However, because of the ill conditioning of the **B** matrix , we have tried to achieve a better agreement between the generated surface anomalies and the input surface anomalies than we possibly can.

3.3 Comparison of Jacobi results using varying tolerances

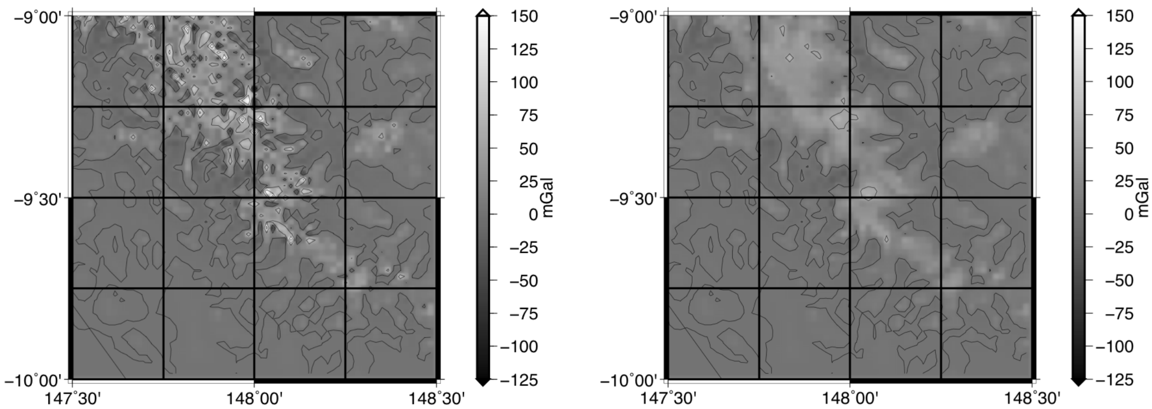
Next, we show how by varying the tolerances for the L_1 norm, both when testing the surface gravity anomalies and geoid gravity anomalies, the computational noise in the downward continuation solution can be reduced. The results given are for the "downward continuation effect", or the difference between the surface (as shown in Figure 1) and downward continued gravity anomalies, given by:

$$\boldsymbol{\varepsilon}^{(k)} = \boldsymbol{\Delta g}^{g(k)} - \boldsymbol{\Delta g}^{t}, \tag{14}$$

This presentation is used to best indicate the noisiness of the different results and their characteristic features. Figure 4 and Table 2 give four results based on tolerances of 0.05 mGal, 0.5 mGal, 5 mGal and 50 mGal, as well as the statistics of the surface gravity anomalies in the area for comparison.



(a) **(b)**



(c) **(d)**

Figure 4: Downward continuation effect from the Jacobi iterative method with testing of the δ^t vector with (a) 0.05 mGal, (b) 0.5 mGal, (c) 5 mGal and (d) 50 mGal tolerances.

Table 2: Statistics of the downward continuation effect for the Jacobi iterative method with testing of the δ^t vector using various tolerances.

	Jacobi results for anomalies on geoid				**Surface anomalies**
Tolerance [mGal]	**0.05**	**0.5**	**5**	**50**	**n/a**
Minimum [mGal]	-291.2	-268.9	15.5	15.5	15.5
Maximum [mGal]	1330.8	1303.1	558.8	332.6	261.4
Range [mGal]	1662.0	1572.0	543.3	317.1	245.9
Std. deviation [mGal]	79.0	78.3	68.8	66.4	57.2
No. of iterations	2745	1066	47	3	n/a
Processing time [mm:ss]	32:35	15:37	5:19	4:56	n/a

In all cases we have used the method of testing based on the L_1 norm of the δ^t vector. The figures and statistics confirm the smoothing effect of choosing a less stringent tolerance. The spikes evident in the solution for a 0.05 mGal tolerance, which has a standard deviation of 79.0 mGal and range of 1662 mGal, is diminished in the solutions that use a 0.5 mGal and 5 mGal tolerance. It is absent in the solution using the 50 mGal tolerance, where the standard deviation has dropped to 66.4 mGal and the range to 317.1 mGal. Since the extreme positive and negative spikes are surely spurious, this suggests that a suitable tolerance will be somewhere between 5 mGal and 50 mGal – since the 5 mGal solution shows questionable spikes, which disappear in the 50 mGal solution.
As expected, the number of iterations and time consumed drop significantly as the tolerance is relaxed, although the drop is less pronounced once the tolerance exceeds about 1 mGal. The relationship between the choice of the tolerance and the number of iterations, for our computation area, is shown in a semi-logarithmic graph in Figure 5.

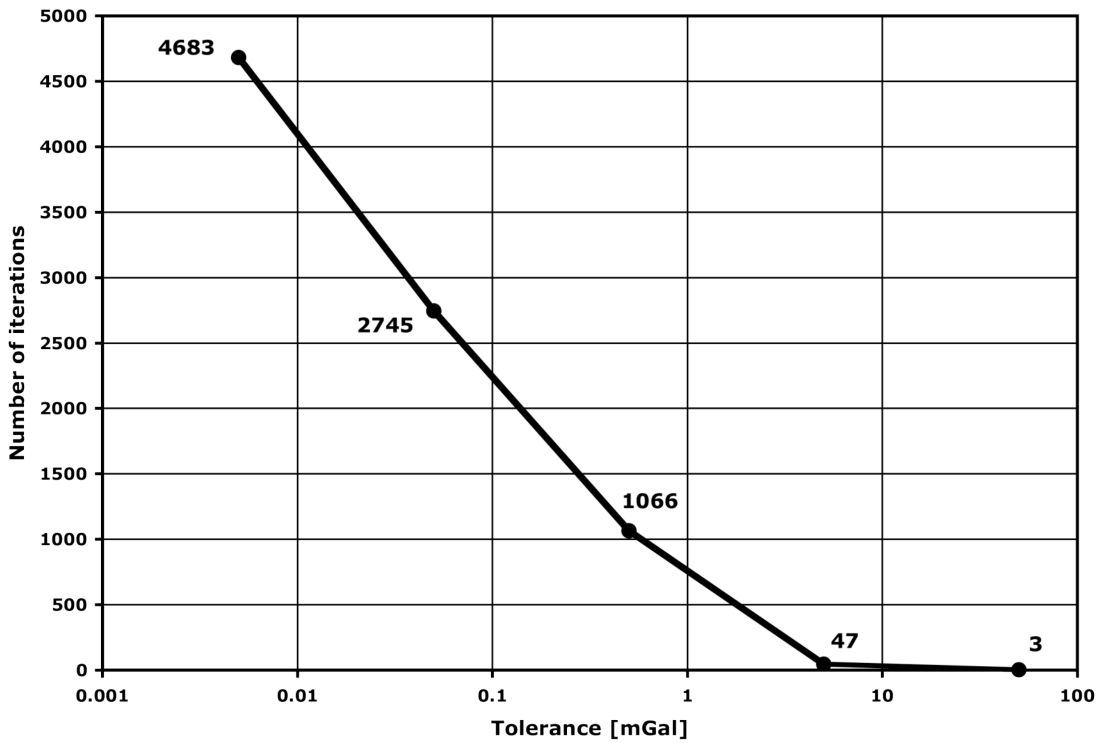


Figure 5: Number of iterations of the Jacobi method vs. tolerance chosen for convergence testing.

The impressive decrease in computation time with the relaxation of the Jacobi tolerance shows that in addition to regularizing the solution and producing a better result, using a suitable tolerance will also significantly improve computation speed.

3.4 Testing the choice of tolerance based on the condition number

Finally, we have applied Eq. (13) to our situation, using our estimate of the condition number based on the L_1 norms of 1347, and an assumed precision of the input data of 0.005 mGal, and come up with a tolerance of 6.7 mGal. This number is in the range we might guess for the best tolerance, based on the results in section 3.3. If we use this tolerance in our calculations, we find the result shown in Figure 6.

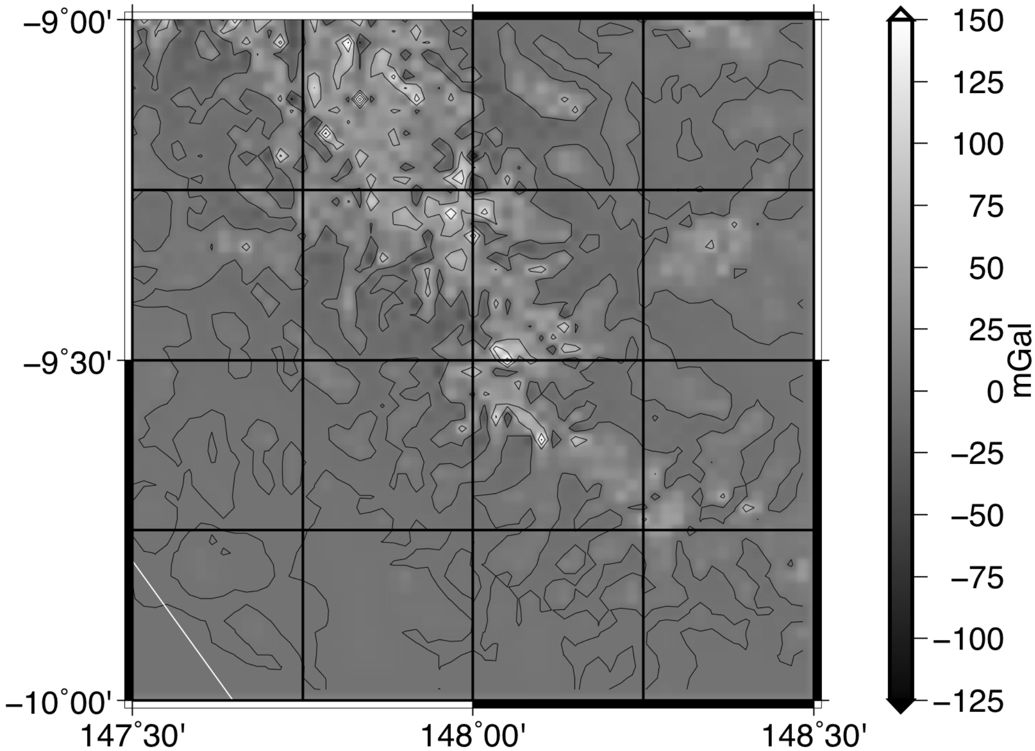


Figure 6: Downward continuation effect of Jacobi method using a tolerance of 6.7 mGal.

As with any attempt at regularization, we can do little to verify this result, apart from saying that it appears to behave as we expect it to. The gravity anomalies on the geoid are rougher than those at the topographical surface, but do not have any significant spikes, and do not have extreme ranges such as we have seen when using tolerances of 0.5 mGal or less. Thus, the tolerance chosen according to Eq. (13) at least produces a reasonable result. There are still some smaller spikes in the solution, and so Eq. (13) may provide too stringent of a tolerance, at least when based on the L_1 norm when estimating of the condition number. If the condition number is estimated according to the L_2 or L_∞ norm, or determined exactly by singular value decomposition, it will be higher and the tolerance provided by Eq. (13) will be less stringent. Also, Eq. (13) would be slow to implement in practical computations due to the time consumption involved in finding the condition number, or even one of the more specific estimates of it. Thus, a better method of finding the most suitable tolerance would help in practical applications.

4. Conclusions

The results have shown first that testing downward continuation according to the L_1 norm of gravity anomalies (the largest absolute value in the vector of values) at the topographical surface, rather than on the geoid, provides a solution commensurate with the theoretically exact LU decomposition solution and also with the traditional method of testing gravity anomalies on the geoid. Testing of anomalies on the surface should be preferred because it allows a more meaningful choice of tolerance criteria and because in cases where convergence is rapid, testing the fit of anomalies at the geoid level leads to truncating the iterations one iteration too soon and this may result in a significant difference in the results. The vector of differences for testing, however, should be determined by Eq. (12), to provide a faster calculation than Eq. (11).

We have also shown that by relaxing the L_1 tolerances in the Jacobi method from those required to get the exact solution, we can regularize the downward continuation solution and significantly decrease computation time. While a better prescription can probably be found, our proposal for the choice of a relaxed tolerance based on the condition number of the **B** matrix, according to Eq. (13), yields a reasonable result. It certainly provides a better result than the solution with a higher tolerance or the “exact” solution by LU decomposition.

It seems to us, that the described approach to downward continuation of gravity anomalies from the Earth surface to the geoid makes a good physical sense. It also produces results that appear more transparent from the mathematical point of view, and that would be evaluated faster than by using some regularization methods.

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