On Some Numerical Aspects of Primary Indirect Topographical Effect Computation in the Stokes-Helmert Theory of Geoid Determination

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Abstract

After solving the geodetic boundary value problem in the Helmert space, the co-geoid has to be transformed back into the real space to obtain finally the geoid. For this transformation, the indirect topographical and atmospheric effects on the geoidal heights have to be evaluated. *Martinec and Vaníček* (1994) discussed theoretical aspects of the primary indirect topographical effect. In this paper, some aspects of the numerical evaluation are investigated.

Resumen

Después de resolver el problema de valor geodésico de frontera en el espacio de Helmert, el cogeoide tiene que ser llevado a espacio real para obtener finalmente el geoide. Para esta transformación, los efectos topográficos y atmosféricos indirectos tienen que ser evaluados. *Martinec y Vaníček* (1994) muestran los aspectos teóricos del efecto topográfico indirecto primario. En este artículo, se investigan algunos aspectos de su evaluación numérica.

Introduction

Solving the geodetic boundary value problem in the Helmert space, according to the Stokes-Helmert approach, the co-geoid is obtained [*Vaníček and Martinec*, 1994; *Vaníček et al.*, 1999]. To find the geoid in the real space, the primary indirect topographical effect is evaluated, while the primary indirect atmospheric effect is negligible due its small size. The primary indirect topographical effect is defined as a product of the residual gravitational potential of topographical masses referred to the geoid surface and the reciprocal value of the normal gravity referred to the surface of the geocentric reference ellipsoid. Considering Helmert's second condensation method, the residual gravitational potential is specified as a difference between the gravitational potential of topographical masses and the gravitational potential of topographical masses condensed on the geoid surface.
According to *Martinec and Vaníček* (1994), the planar approximation of the geoid is not adequate for the computation of the primary indirect topographical effect. Moreover, according to *Martinec* (1993), for precise geoid determination, the density of the topographical masses must be assumed to be at least laterally non-homogeneous.

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Primary Indirect Topographical Effect

According to *Martinec* (1993), the residual gravitational potential of topographical masses $\delta V^t(r,\Omega)$ is defined as the difference between the gravitational potential of topographical masses $V^t(r,\Omega)$ and the gravitational potential of condensed topographical masses $V^{ct}(r,\Omega)$. Referred to the geoid surface $\forall\, \Omega \in \Omega_o : r_g(\Omega)$, the residual gravitational potential reads

$$\forall\, \Omega \in \Omega_o : \qquad \delta V^t[r_g(\Omega)] = V^t[r_g(\Omega)] - V^{ct}[r_g(\Omega)]. \qquad (1)$$

The geocentric position is described by the geocentric spherical coordinates ϕ and λ; $\Omega = (\phi, \lambda)$, and the geocentric radius r; $r \in \Re^+ \left(\Re^+ \in \langle 0, +\infty)\right)$. In equation (1), Ω_o denotes the total solid angle $\left[\, \phi \in \langle -\pi/2, \pi/2 \rangle, \lambda \in \langle 0, 2\pi \rangle \right]$.
The gravitational potential of topographical masses $V^t[r_g(\Omega)]$ is given by Newton's volume integral [*Martinec*, 1993]

$$\forall\, \Omega \in \Omega_o : \qquad V^t[r_g(\Omega)] = G \iint_{\Omega' \in \Omega_O} \int_{r' = r_g(\Omega')}^{r_g(\Omega') + H^O(\Omega')} \rho(r,\Omega')\, \ell^{-1}[r_g(\Omega); r', \Omega']\, r'^2\, dr'\, d\Omega' , \qquad (2)$$

where G is Newton's gravitational constant, and $\rho(r,\Omega)$ is the actual density of topographical masses. The geocentric radius of the earth surface is approximately equal to the geocentric radius of the geoid surface plus the orthometric height $H^O(\Omega)$, i.e., $\forall\, \Omega \in \Omega_o : r_t(\Omega) \cong r_g(\Omega) + H^O(\Omega)$.
The spatial distance $\ell(r,\Omega; r',\Omega')$ between two points (r,Ω) and (r',Ω') is given by [e.g., *Heiskanen and Moritz*, 1967, Eq. 1-79]

$$\forall\, \Omega, \Omega' \in \Omega_o; r, r' \in \Re^+ : \qquad \ell(r,\Omega; r',\Omega') = \sqrt{r^2 + r'^2 - 2rr'\cos\psi} , \qquad (3)$$

where the spherical distance ψ is defined by the cosine theorem

$$\forall\, \psi \in \langle\, 0, \pi \,\rangle : \qquad \cos\psi = \sin\phi' \sin\phi + \cos\phi' \cos\phi \cos(\lambda' - \lambda). \qquad (4)$$

The gravitational potential $V^{ct}[r_g(\Omega)]$ of condensed topographical masses is defined by Newton's surface integral [e.g., *Martinec*, 1998]. It reads

$$\forall\, \Omega \in \Omega_o : \qquad V^{ct}[r_g(\Omega)] = G \iint_{\Omega' \in \Omega_O} \sigma(\Omega')\, \ell^{-1}[r_g(\Omega); r_g(\Omega')]\, r_g^2(\Omega')\, d\Omega' , \qquad (5)$$

where $\sigma(\Omega)$ is the surface density of condensed topographical masses.
The primary indirect topographical effect on the geoidal height $\delta N^t(r_g,\Omega)$ is given by applying Bruns' formula [*Bruns*, 1878] to the residual gravitational potential. Thereby

$$\forall\, \Omega \in \Omega_o : \qquad \delta N^t(\Omega) = \frac{\delta V^t[r_g(\Omega)]}{\gamma_o(\phi)} , \qquad (6)$$

where $\gamma_o(\phi)$ is the normal gravity referred to the surface of the geocentric reference ellipsoid [*Somigliana*, 1929; *Heiskanen and Moritz*, 1967].

Spherical Approximation

Approximating the geoid surface by a sphere of the mean radius of the earth R, i.e., $\forall\,\Omega \in \Omega_o : r_g(\Omega) \approx R$, and the actual density of topographical masses $\rho(r,\Omega)$ by the laterally varying topographical density $\rho(\Omega)$,

$$\forall\,\Omega \in \Omega_o : \qquad \rho(\Omega) = \frac{1}{H^O(\Omega)} \int_{r=R}^{R+H^O(\Omega)} \rho(r,\Omega)\, r^2 \, dr \,, \tag{7}$$

the gravitational potential $V^t[r_g(\Omega)]$ in equation (2) becomes

$$\forall\,\Omega \in \Omega_o : \qquad V^t(R,\Omega) = G \iint_{\Omega' \in \Omega_O} \rho(\Omega') \int_{r'=R}^{R+H^O(\Omega')} \ell^{-1}(R,\Omega;r',\Omega')\, r'^2 \, dr' \, d\Omega' \,. \tag{8}$$

According to the principle of conservation of topographical masses [*Wichiencharoen*, 1982], the surface density $\sigma(\Omega)$ is given by [*Martinec and Vaníček*, 1994]

$$\forall\,\Omega \in \Omega_o : \qquad \sigma(\Omega) = \rho(\Omega) H^O(\Omega) \left[1 + \frac{H^O(\Omega)}{R} + \frac{1}{3} \left(\frac{H^O(\Omega)}{R} \right)^2 \right] = \rho(\Omega) \frac{r_t^3(\Omega) - R^3}{3R^2} \,. \tag{9}$$

Regarding equation (9), the gravitational potential of condensed topographical masses $V^{ct}[r_g(\Omega)]$ in equation (5) takes the following form

$$\begin{aligned} \forall\,\Omega \in \Omega_o : \qquad V^{ct}(R,\Omega) &= G\, R^2 \iint_{\Omega' \in \Omega_O} \sigma(\Omega')\, \ell^{-1}(R,\Omega;R,\Omega')\, d\Omega' \\ &= G \iint_{\Omega' \in \Omega_O} \rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3}\, \ell^{-1}(R,\Omega;R,\Omega')\, d\Omega' \,. \end{aligned} \tag{10}$$

Substituting equations (8) and (10) into equation (1), the residual gravitational potential of topographical masses $\delta V^t(R,\Omega)$ becomes

$$\begin{aligned} \forall\,\Omega \in \Omega_o : \qquad \delta V^t(R,\Omega) &= G \iint_{\Omega' \in \Omega_O} \rho(\Omega') \int_{r'=R}^{R+H^O(\Omega')} \ell^{-1}(R,\Omega;r',\Omega')\, r'^2 \, dr' \, d\Omega' \\ &\quad - G \iint_{\Omega' \in \Omega_O} \rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3}\, \ell^{-1}(R,\Omega;R,\Omega')\, d\Omega' \,. \end{aligned} \tag{11}$$

The radial integral of the reciprocal spatial distance multiplied by r'^2 can be found in the following analytical form [*Gradshteyn and Ryzhik*, 1980; see also *Martinec*, 1998, eq. 3.52]

$$\int_{r'} \ell^{-1}(R,\Omega;r',\Omega')\, r'^2 \, dr' = \frac{r' + 3R\cos\psi}{2\,\ell^{-1}(R,\Omega;r',\Omega')} + \frac{R^2}{2} \left(3\cos^2\psi - 1 \right) \ln \left| \, r' - R\cos\psi + \ell^{-1}(R,\Omega;r',\Omega') \, \right| . \tag{12}$$

Singularity Of Newton's Kernel

The reciprocal spatial distance $\ell^{-1}(R,\Omega;R,\Omega')$ grows to infinity when the integration point moves towards the computation point

$$\lim_{\substack{\psi \to 0^+ \\ r' \to r}} \ell^{-1}(r, \Omega; r', \Omega') = +\infty \,. \tag{13}$$

To investigate the singularity of the Newton integral kernel, the surface integration domain Ω_o in equation (11) is described in the form of the polar spherical coordinates α and ψ, where α stands for the spherical azimuth. The surface integration domain is then rewritten as

$$\iint_{\Omega \in \Omega_O} d\Omega = \int_{\lambda=0}^{2\pi} \int_{\phi=-\pi/2}^{\pi/2} \cos\phi \, d\phi \, d\lambda = \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \sin\psi \, d\alpha \, d\psi \,. \tag{14}$$

Consequently, the residual gravitational potential $\delta V^t(R, \Omega)$ takes the form

$$\forall\, \Omega \in \Omega_o : \qquad \delta V^t(R, \Omega) = G \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \rho(\Omega') \left[\int_{r'=R}^{R+H^O(\Omega')} \ell^{-1}(R, \Omega; r', \Omega') \, r'^2 \, dr' - \frac{r_t^3(\Omega') - R^3}{3} \, \ell^{-1}(R, \Omega; R, \Omega') \right] \sin\psi \, d\alpha \, d\psi \,. \tag{15}$$

Therefore, the Newton integral kernel is only weakly singular [Kellogg, 1929; Martinec, 1998]

$$\forall\, r \neq 0 : \qquad \lim_{\substack{\psi \to 0^+ \\ r' \to r}} \ell^{-1}(r, \Omega; r', \Omega') \sin\psi = \frac{1}{r} \lim_{\psi \to 0^+} \sin\psi \sum_{n=0}^{\infty} P_n(\cos\psi) = \frac{1}{r} \lim_{\psi \to 0^+} \frac{\sin\psi}{2 \sin\frac{\psi}{2}} = \frac{1}{r} < +\infty \,. \tag{16}$$

Numerical Investigation

Solving for the residual gravitational potential $\delta V^t(R, \Omega)$ the integration domain Ω_o can be divided into two integration sub-domains: the near-zone Ω_{NZ} and the far-zone Ω_{FZ}. The far-zone contribution can be solved either in the spectral form [*Novák et al.* 2001], or by methods of the numerical integration. In our computation area, which is a part of the Canadian Rocky Mountains, we have obtained the value of the far-zone contribution to the primary indirect topographical effect around –3 cm, using any of the two approaches. One-degree step-size of the numerical integration is used in the numerical integration of equation (11), while the TUG-87 [*Wieser*, 1987] global elevation model of degree and order 180 is used in the spectral approach.

The near-zone contribution is usually evaluated by numerical integration [*Martinec and Vaníček*, 1994; *Sjöberg and Nahavandchi*, 1998], in two or three dimensions. In this contribution, we used the three-dimensional integration in a local Cartesian coordinate system (x,y,z), while approximating the earth surface by a piece-wise constant function $F(x, y) = H$. This amounted to adopting a locally planar model for the geoid surface and for the topography referred to it. The Cartesian coordinate system was selected so that the z-axis coincides with the outer normal to the geoid (approximated by a sphere), and the x- and y-axes are in the horizontal plane, and point towards north and east, respectively. The integration step then divides the surface of the earth into two-dimensional cells and the topography can then be regarded as being composed of prisms delineated by the cells' borders. These prisms are all flat at the top because of the approximation used for the earth surface.

Under the planar approximation of the geoid surface the residual gravitational potential $\delta V^t(R, \Omega)$ is approximated by a series of potentials of individual prisms with a surface area equal to the area of the integration cell, a height equal to the orthometric height $H^{O}(x,y)$ of the integration dummy point, and a uniform density $\rho(x,y)$. The residual gravitational potential generated by each prism is given by

$$\delta V^{t}(x,y,z) = G\rho(x',y') \int_{z'}\int_{y'}\int_{x'} \frac{1}{\sqrt{x'^2+y'^2+z'^2}}\, dx'\,dy'dz' - G\sigma(x',y') \int_{y'}\int_{x'} \frac{1}{\sqrt{x'^2+y'^2}}\, dx'\,dy' \,. \tag{17}$$

The complete near-zone contribution to the gravitational potential of topographical masses is given by the summation over the individual prisms, where the contribution of individual prism is computed from the following closed expression [*Bronstein and Semendjajev*, 1974; *Nagy, et. al.*, 2000], which makes use of just only the first integral in equation(17)

$$\begin{aligned} V^{t}(x,y,z) = G\rho(x',y') \Bigg|\Bigg|\Bigg| & x'y'\ln\left| z' + \ell(x',y',z')\right| + x'z'\ln\left| y' + \ell(x',y',z')\right| \\ & + y'z'\ln\left| x' + \ell(x',y',z')\right| - \frac{1}{2}x'^2 \arctan\frac{y'z'}{x'\,\ell(x',y',z')} \\ & - \frac{1}{2}y'^2 \arctan\frac{x'z'}{y'\,\ell(x',y',z')} - \frac{1}{2}z'^2 \arctan\frac{x'y'}{z'\,\ell(x',y',z')} \Bigg|_{0}^{H(x',y')} \Bigg|_{y'-\frac{\Delta y'}{2}}^{y'+\frac{\Delta y'}{2}} \Bigg|_{x'-\frac{\Delta x'}{2}}^{x'+\frac{\Delta x'}{2}} . \end{aligned} \tag{18}$$

In equation (18), $\Delta x' = R\,\Delta\phi'$ and $\Delta y' = R\cos\phi'\,\Delta\lambda'$ represents steps of the analytical integration. The spatial distance is equal to $\ell(x',y',z') = \sqrt{x'^2 + y'^2 + z'^2}$, and the co-ordinates x', y', z' of the integration point are given by

$$x' = R(\phi' - \phi), \qquad y' = R(\lambda' - \lambda)\cos\frac{\phi' + \phi}{2}, \qquad z' = r(x',y') - R \,. \tag{19}$$

Similarly, the near-zone contribution to the gravitational potential of condensed topographical masses is given by a summation over the contributions of individual condensation layers within the cells, where the contribution of each individual condensation layer cell of constant surface density $\sigma(x,y)$ can be expressed in the following analytical form [*Kuhn*, 2000]

$$\begin{aligned} V^{ct}(x,y,z) &= G\sigma(x',y') \int_{x'}\int_{y'} \frac{1}{\sqrt{x'^2 + y'^2}}\, dy'dx' \\ &= G\sigma(x',y') \Bigg|\Bigg| x'\ln\left| y' + \ell(x',y')\right| + y'\ln\left| x' + \ell(x',y')\right| \Bigg|_{y'-\frac{\Delta y'}{2}}^{y'+\frac{\Delta y'}{2}} \Bigg|_{x'-\frac{\Delta x'}{2}}^{x'+\frac{\Delta x'}{2}} . \end{aligned} \tag{20}$$

The total value of primary indirect topographical effect, from both the near and far zones, in the computation area is shown in Figure 1. In this area the effect ranges between –3 cm and -36 cm.

Conclusion

The integration domain in the evaluation of the residual gravitational potential of topographical masses may be divided into the near and far integration sub-domains, where different computational techniques can be applied. One-degree grid for the numerical integration is sufficient for a few millimeters precision of the far-zone contribution to the primary indirect topographical effect.

The weak singularities of Newton's volume and surface integrals at the computation point cause the computation to be numerically unstable. The existence of the weak singularity forces one to be very careful when dealing with the immediate neighborhood of the computation point. For this reason, the residual gravitational potential of topographical masses can be evaluated using the analytical solution for the potential of the prism with constant topographical density and its corresponding condensation layer.

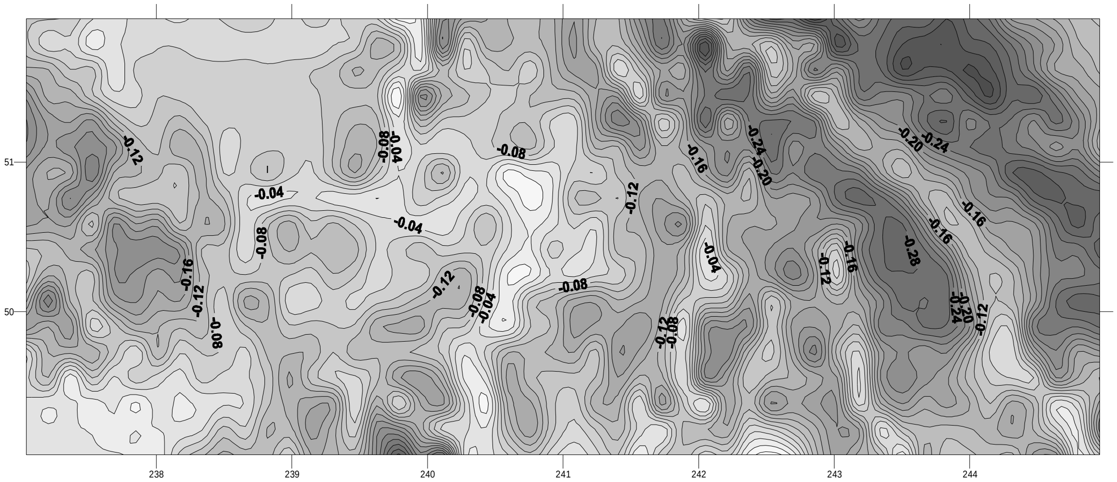

Figure 1. The primary indirect topographical effect on geoidal height.

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