PROPOSED NEW CARTOGRAPHIC MAPPING FOR IRAN

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When the new cartographic mapping system for Iran was considered, the requirements were that the system be conformal, continuous and, naturally, possess the smallest possible scale distortion over the Iranian national territory. In the design of the map projection, these requirements have been met by an oblique conical projection with two secant oblique parallels. The proposed mapping is really a conglomerate of two conformal mappings. Firstly, mapping the reference ellipsoid to a Gaussian sphere, and secondly, relating the Gaussian sphere to a Lambert cone in an oblique position.

Both of these mappings are formulated so as to ensure minimal distortion over the territory of Iran.

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INTRODUCTION

Some seven or eight years ago, the National Cartographic Center (NCC) of Iran decided to adopt a new map projection for the new 1:1 000 000 map series of Iran. The new map projection was then conceived as a double projection: first the reference ellipsoid is mapped to a conformal sphere, then the conformal sphere is mapped onto an oblique cone. In the formulation of the two projections, two additional coordinate transformations had to be introduced, namely:

- Spherical coordinates on the Gaussian sphere into oblique spherical coordinates on the same sphere;
- Oblique Cartesian coordinates on the Lambert mapping plane into final mapping coordinates (see Sec. 4).

Thus, this research had to deal really with four coordinate transformations: (Φ,λ) to (u,v), (u,v) to (u^*,v^*), (u^*,v^*) to (x^*,y^*), (x^*,y^*) to (X,Y). These four transformations can be combined at the end, to yield one combined mapping equation:

$$P = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X(\Phi,\lambda) \\ Y(\Phi,\lambda) \end{bmatrix},$$

that transforms the geodetic latitude and longitude of a point on the reference ellipsoid (horizontal datum) to Cartesian coordinates on the mapping plane (map coordinates). This mapping equation resides in software, where it is evaluated whenever a point on the horizontal datum is to be mapped onto the mapping plane.

The inverse mapping equation, transforming the map coordinates to geodetic latitude and longitude, is also formulated side by side with the direct mapping equation.

CONFORMAL MAPPING OF REFERENCE ELLIPSOID ONTO A GAUSSIAN SPHERE

Theory

The mapping of the reference ellipsoid onto a conformal (Gaussian) sphere is fairly standard in mathematical cartography. It may be useful, however, to recapitulate the theory of this mapping here. The direct and inverse mapping equations are derived in a sequence of steps as follows (Fiala, 1955):

- Select a sphere of an as yet unspecified radius R and a curvilinear orthogonal coordinate system *u,v* (spherical latitude and spherical longitude).
- To avoid meridian convergence on the Gaussian conformal sphere it is required that spherical latitude be an as yet unspecified function of only geodetic latitude and spherical longitude be an unspecified linear function of geodetic longitude, i.e.,

$$u = f(\phi), \qquad v = C_1\,\lambda \tag{1}$$

where C1 is an integration constant to be determined.

- Conformality of the mapping requires that the differential distortion (scale factor (k)) in the meridian $\frac{R\,du}{M\,d\phi}$ be the same as the differential distortion in the parallel $\frac{R\cos u\,dv}{N\cos\phi\,d\lambda}$. This requirement is equivalent to the Cauchy-Riemann conditions that can be written as:

$$k(\phi) = \frac{R\,du}{M\,d\phi} = \frac{R\cos u\,dv}{N\cos\phi\,d\lambda} = \frac{R\,C_1\cos u}{N\cos\phi} \tag{2}$$

where *M* and *N* are the meridian and the prime vertical radii of curvature of the reference ellipsoid, see e.g., [Vanicek and Krakiwsky, 1986].

- Solution of the differential equation (2) yields the expression for u as a function of :

$$\phi : tan\left(\frac{\pi}{4}+\frac{u}{2}\right) = C_2\left[tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\,sin\,\phi}{1+e\,sin\,\phi}\right)^{\frac{e}{2}}\right]^{C_1} \tag{3}$$

which is the mapping equation for *u*, where *e* represents the eccentricity of the mapped reference ellipsoid. Values of the integration constants C_1 and C_2 have to be determined from other considerations.

- As *u* is a function of ϕ only, the scale factor (k) given by equation (2) is clearly also a function of only. It is possible thus to develop the scale factor into a Taylor series with respect to the mean latitude ϕ_o of the mapped territory as follows:

$$k(\phi) = k(\phi_\circ) + \left.\frac{\partial k}{\partial\phi}\right|_{\phi_0}\Delta\phi + \left.\frac{\partial^2 k}{\partial\phi^2}\right|_{\phi_0}\frac{\Delta\phi^2}{2} + \left.\frac{\partial^3 k}{\partial\phi^3}\right|_{\phi_0}\frac{\Delta\phi^3}{6} + \cdots \tag{4}$$

where $\Delta\phi$ is the geodetic latitude increment referred to ϕ_o.

- Now, from the requirement that:

$$k(\phi_\circ) = 1 \tag{5}$$

the value for *R* can be derived as:

$$R = \sqrt{M_\circ\,N_\circ} \tag{6}$$

where M_o and N_o are the meridian and prime vertical radii of curvature of the reference ellipsoid for the mean latitude ϕ. This is the determining equation for *R*. Substitution for the radii of curvature gives the following equation for *R*:

$$R = \frac{a\sqrt{1-e^2}}{1-e^2\sin^2\phi_\circ}\ , \tag{7}$$

where *a* is the major semi-axis of the mapped reference ellipsoid.

- From the requirement that the scale factor be as close to 1 as possible, or, equivalently, from the requirement that the first two derivatives in the Taylor series expression for the scale factor disappear:

$$\left.\frac{\partial k}{\partial\phi}\right|_{\phi_0} = 0 \qquad \left.\frac{\partial^2 k}{\partial\phi^2}\right|_{\phi_0} = 0 \tag{8}$$

the following two equations are derived:

$$C_1\,\sin u_\circ = \sin\phi_\circ \tag{9}$$

and

$$C_1 = \sqrt{1+\frac{e^2}{1-e^2}cos^4\,\phi_{\circ}} \tag{10}$$

where u_o is the mean spherical latitude of the mapped territory.

Equation (9) is the determining equation for. C_1

- Enforcement of equations (8) and (9) results in the optimization of the scale factor $k\,(\phi)$ for the mapped territory, which can now be written as:

$$k(\phi)=\frac{RC_1\cos u}{N\cos\phi}=1+\left.\frac{\partial^3 k}{\partial\phi^3}\right|_{\phi_0}\frac{\Delta\phi^3}{6}+\ldots \tag{11}$$

The convergence of this series has not been investigated in detail, and neither have the partial derivatives been evaluated as part of this research. But it should be clear from the fact that C_1 is very close to 1, that the series converges quite quickly.

- Finally, the second integration constant is evaluated from equation (3) taken for the mean latitude of the mapped territory, as:

$$C_2=\tan\left(\frac{\pi}{4}+\frac{u_0}{2}\right)\left[\tan\left(\frac{\pi}{4}+\frac{\phi_0}{2}\right)\left(\frac{1-e\sin\phi_0}{1+e\sin\phi_0}\right)^{e/2}\right]^{-C_1} \tag{12}$$

where all the quantities are now known.

- The inverse mapping equations, from the Gaussian sphere onto the reference ellipsoid, are obtained from the direct mapping equations (from the reference ellipsoid onto the Gaussian sphere) as:

$$tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)=\left[\frac{1}{C_2}tan\left(\frac{\pi}{4}+\frac{u}{2}\right)\right]^{\frac{1}{C_1}}\left(\frac{1-e\,sin\,\phi}{1+e\,sin\,\phi}\right)^{e/2}$$

$$\lambda=\frac{1}{C_1}v \tag{13}$$

Here, the first equation must be solved iteratively by putting first $\phi = u$ in the corrective term $\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{e/2}$ and updating ϕ at each iterative step [Thomson et al., 1977].

Choice of constants for Iran

The evaluation of the most appropriate constants for the conformal mapping of the reference ellipsoid onto the Gaussian sphere requires:

- selection of the mean latitude ϕ_0 for Iran;
- knowledge of the size (*a*) and shape (*e*) of the reference ellipsoid used in the mapping.

A value of ϕ_0= 3.25^o was selected for the mean latitude. For an ellipsoid of flattening $f=\frac{1}{298.25}$ equation (10) yields:

$$C_1 = 1.001703510\ , \tag{14}$$

from equation(9) is derived:

$$u_0 = 32.43794656^o \tag{15}$$

Equation (12) gives:

$$C_2 = 1.001299778 \tag{16}$$

Adopting the value of *a* = 6378137.00*m*, equation (7) provides:

$$R = 6369061.1965\quad m \tag{17}$$

These are the values defining uniquely the best conformal mapping of the reference ellipsoid for Iran onto the Gaussian sphere. As a result the scale factor of this mapping for Iran ranges from 1 – 4.76x10 in the northern part of Iran (ϕ = 40^o) to 1+4.28x10^{-6} in southern part of Iran (ϕ = 25^o). Angles and azimuth are, of course, undistorted as the mapping is conformal.

SELECTION OF THE BEST POSITION FOR THE LAMBERT CONE

Theory

The next step in the development of the mapping solution is to select an appropriate position and orientation (obliquity) of the cone on which to project the Gaussian sphere in the second Lambert conical projection. This position and orientation depends on the shape of the territory to be mapped. The obvious idea behind this concept is the overall minimization of the point scale factor over the territory. This can be achieved quite easily by appropriately rotating the *u, v* coordinates system on the sphere to obtain a new (oblique) spherical coordinate system u^*, v^* in which:

- the centre point, also called the origin, (u_0, v_0) has coordinates (u_0^*=u_0 , v_0^*= v_0) and;

- the longest segment of a great circle on the mapped territory, passing through the origin (u_0, v_0) is a segment of a parallel $u^* = u_0$.

The choice of the new spherical coordinates (u_0^*, v_0^*) for the origin is somewhat arbitrary. One could choose the value of u_0^* differently, for instance in such a way as to further optimize the distortion over the mapped territory. One may even discover that a choice of $u_0^* = 0$, putting the origin at the equator of the new coordinates system, thus converting the conical projection to a cylindrical projection, gives smaller distortions than the current choice. This possibility has not been examined in detail, but this point could be investigated further. The choice of $v_0^* = 0$ is merely a matter of convenience; it places the zero meridian of the oblique coordinate system running through the origin of the mapped territory.

Denoting the azimuth of the longest great circle segment passing through (u_0, v_0) by α_0, the above rotation of the u, v coordinate system into the u^*, v^* coordinate system is given by the following equation:

$$\mathbf{r}^* = \mathbf{R}_3(-v_0^*)\mathbf{R}_2(u_0^*)\mathbf{R}_1(\frac{\pi}{2} - \alpha_0)\mathbf{R}_2(-u_0)\mathbf{R}_3(v_0)\mathbf{r} \qquad (18)$$

where

$$\mathbf{r} = (\cos u \cos v \quad, \cos u \quad \sin v \quad, \sin u)^T \qquad (19)$$

$$\mathbf{r}^* = (\cos u^* \cos v^* \quad, \cos u^* \quad \sin v^* \quad, \sin u^*)^T \qquad (20)$$

The matrices $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, are rotation matrices as defined by Vanicek and Krakiwsky (1986). Multiplying the 5 matrices (equation (18)) provides:

$$\mathbf{r}^* = \mathbf{A} \quad \mathbf{r} \qquad (21)$$

which is the final linear transformation equation between the u,v and u^*,v^* systems. The matrix **A** is the total rotation matrix with the following elements:

$$\begin{aligned}
a_{11} = {} & \cos u_0 \ \cos v_0 \ \cos u_0^* \ \cos v_0^* \ + \\
& (\sin u_0 \ \sin v_0^* \ + \sin u_0^* \ \cos v_0^* \ \sin u_0 \ \cos v_0)\sin\alpha_0 \ + \\
& (\sin u_0 \ \cos v_0 \ \sin v_0^* \ - \ \sin u_0^* \ \cos v_0^* \ \sin v_0) \ \cos\alpha_0 \\
a_{21} = {} & \cos u_0 \ \cos v_0 \ \cos u_0^* \ \sin v_0^* \ + \\
& (\sin u_0 \ \cos v_0 \sin u_0^* \ \sin v_0^* \ - \sin v_0 \ \cos v_0^*) \ \sin\alpha_0 \ - \\
& (\sin u_0 \ \cos v_0 \ \cos v_0^* \ - \ \sin u_0^* \ \sin v_0^* \ \sin v_0) \ \cos\alpha_0 \\
a_{31} = {} & \cos u_0 \ \cos v_0 \ \sin u_0^* \ - \sin u_0 \ \cos v_0 \ \cos u_0^* \ \sin\alpha_0 \ + \\
& \sin v_0 \ \cos u_0^* \ \cos\alpha_0 \\
a_{12} = {} & \cos u_0 \ \sin v_0 \ \cos u_0^* \ \cos v_0^* \ + \\
& (\sin u_0 \ \sin v_0 \sin u_0^* \ \cos v_0^* \ - \cos v_0 \ \sin v_0^*) \ \sin\alpha_0 \ - \\
& (\sin u_0 \ \sin v_0 \ \sin v_0^* \ + \ \sin u_0^* \ \cos v_0^* \ \cos v_0) \ \cos\alpha_0 \\
a_{22} = {} & \cos u_0 \ \sin v_0 \ \cos u_0^* \ \sin v_0^* \ + \\
& (\cos u_0 \ \cos v_0^* \ + \sin u_0 \ \cos v_0 \ \sin u_0^* \ \cos v_0^*)\sin\alpha_0 \ + \\
& (\sin u_0^* \ \sin v_0^* \ \cos v_0 \ - \ \sin u_0 \ \sin v_0 \ \cos v_0^*) \ \cos\alpha_0 \\
a_{32} = {} & \cos u_0 \ \sin v_0 \ \sin u_0^* \ - \sin u_0 \ \sin v_0 \ \cos u_0^* \ \sin\alpha_0 \ - \\
& \cos u_0^* \ \cos v_0 \ \cos\alpha_0 \\
a_{13} = {} & \sin u_0 \ \cos u_0^* \ \cos v_0^* \ - \sin u_0^* \ \cos v_0^* \ \cos u_0 \ \sin\alpha_0 \ - \\
& \cos u_0 \ \sin v_0^* \ \cos\alpha_0 \\
a_{23} = {} & \sin u_0 \ \cos u_0^* \ \sin v_0^* \ - \sin u_0^* \ \sin v_0^* \ \cos u_0 \ \sin\alpha_0 \ - \\
& \cos v_0^* \ \cos u_0 \ \cos\alpha_0 \\
a_{33} = {} & \sin u_0 \ \ \sin u_0^* + \ \cos u_0^* \ \ \cos u_0 \ \ \sin\alpha_0
\end{aligned} \qquad (22)$$

Note that for the choice of $v_0^* = 0$ in this research, terms containing $\sin v_0^*$ disappear.

Once $\mathbf{r}^*$ is determined from this transformation equation, u_0^*, v_0^* are evaluated as:

$$\begin{aligned}
u^* &= \arcsin\left(r_3^*\right) \\
v^* &= 2\arctan\left(\frac{r_2^*}{r_1^* + \sqrt{r_1^{*2} + r_2^{*2}}}\right)
\end{aligned} \qquad (23)$$

where r_1^*, r_2^*, r_3^* are the three components of the vector $\mathbf{r}^*$.

The inverse linear transformation equation, from the u^*, v^*, system to u, v reads:

$$\mathbf{r} = \mathbf{A}^T \mathbf{r}^* \qquad (24)$$

where $\mathbf{A}^T$ is the transpose of **A**.

The individual coordinates u and v are obtained from equations parallel to (23).

Choice of constants for Iran

In the earlier section dealing with the choice of constants for Iran, was determined to be:

$$u_0 = 32.43794656^o \qquad (25)$$

From the map of Iran, we selected $\lambda_0 = 54^\circ$ as the mean longitude of Iran. Applying equation (1) we get :

$$v_0 = 54.09198956^o \qquad (26)$$

The azimuth of the longest great circle segment passing through the origin was estimated as $\alpha_0 = 129^0$, (Figure 1).

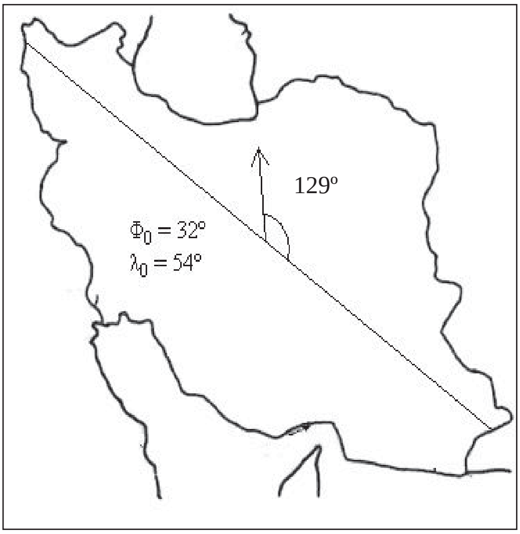

Figure 1. The longest great circle and its azimuth

Substituting these values into equations (22), the numerical values for the total rotation matrix are:

$$\mathbf{A} = \begin{pmatrix} 0.82229055 & 0.56005399 & 0.10088493 \\ -0.43148363 & 0.72919374 & -0.53112933 \\ -0.37102577 & 0.39321243 & 0.84126325 \end{pmatrix} \quad (28)$$

LAMBERT CONFORMAL PROJECTION OF OBLIQUE GAUSSIAN SPHERE

Theory

Once the u^*,v^* spherical coordinate system on the Gaussian sphere has been established, in which the longest segment in the mapped area is a segment of parallel u^*=u, the standard Lambert conformal projection (Fiala, 1955) can be used for this oblique coordinate system.

In the Lambert conformal projection the mapping equations from the oblique Gaussian sphere onto the mapping plane are:

$$\begin{aligned} x^* &= K_1 \exp\left(-K_2 q^*\right)\cos\left(K_2 \Delta v^*\right) \\ y^* &= K_1 \exp\left(-K_2 q^*\right)\sin\left(K_2 \Delta v^*\right) \end{aligned} \quad (29)$$

where K_1 and K_2 are some integration constants to be selected, $\Delta v^* = v^* - v_o^*$, and:

$$q^* = \ln\tan\left(\frac{\pi}{4} + \frac{u^*}{2}\right) \quad (30)$$

is the isometric (spherical) latitude on the rotated Gaussian sphere.

Substituting q^* from equation (30) into equations (29) yields the final direct mapping equations:

$$\begin{aligned} x^* &= K_1 \tan^{K_2}\left(\frac{\pi}{4} + \frac{u^*}{2}\right)\cos\left(K_2 \Delta v^*\right) \\ y^* &= K_1 \tan^{K_2}\left(\frac{\pi}{4} + \frac{u^*}{2}\right)\sin\left(K_2 \Delta v^*\right) \end{aligned} \quad (31)$$

where x^*, y^* are oblique Cartesian coordinates on the Lambert mapping plane.

The scale factor of the Lambert conformal mapping is given by:

$$k^L = \frac{K_1 K_2 \cot^{K_2}\left(\frac{\pi}{4} + \frac{u^*}{2}\right)}{R\cos u^*} \quad (32)$$

where R stands for the radius of the Gaussian sphere discussed previously. The superscript L is used to distinguish the scale factor (k^L) of the Lambert projection from the scale factor (k) of the conformal mapping of the reference ellipsoid onto the Gaussian conformal sphere, discussed previously.

To select the integration constants K_1 and K_2, the scale factor for the two extreme parallels (on the mapped territory) is required, namely:

$$u_1^* = u_0^* - \Delta u^* \qquad u_2^* = u_0^* + \Delta u^* \quad (33)$$

be equal to $1+\mu$, while for some parallels, close to $u^*=u_0^*$, which are referred to as $u^*=u_m^*$, the scale factor will be $1-\mu$. The result is:

$$K_2 = \frac{\ln\cos\left(u_2^*\right) - \ln\cos\left(u_1^*\right)}{\ln\tan\left(\frac{\pi}{4} + \frac{u_1^*}{2}\right) + \ln\tan\left(\frac{\pi}{4} + \frac{u_2^*}{2}\right)} \quad (34)$$

Since K_2 must also equal to u_m^*, u_m^* can be determined from K_2 as:

$$u_m^* = \arcsin K_2 \quad (35)$$

The integration constant K_1 is then evaluated from the following equation:

$$K_1 = \frac{2R\cos\ u_m^* \cos\left(u_1^*\right)}{K_2\left[\cos\left(u_m^*\right)\tan^{K_2}\left(\frac{\pi}{4} + \frac{u_1^*}{2}\right) + \cos\left(u_1^*\right)\tan^{-K_2}\left(\frac{\pi}{4} + \frac{u_m^*}{2}\right)\right]} \quad (36)$$

where R is again the radius of the Gaussian sphere.

The inverse transformation equations (to equations (31)) read:

$$u^* = 2\arctan\left[\left(\frac{x^{*2}+y^{*2}}{K_1^2}\right)^{2K_2}\right] - \frac{\pi}{2}$$
$$v^* = \frac{1}{K_2}\arctan\left(\frac{y^*}{x^*}\right) + v_0^* \tag{37}$$

Choice of constants for Iran

For the selected mean latitude u_0=32.43794656 and for Δu^*=6.5^o the maximum half width of Iran in the azimuth 39^o, the integration constants are:

$$K_1 = 13754372.9049m$$
$$K_2 = 0.537543752. \tag{38}$$

The Lambert scale factors at the mean and extreme u^* coordinates are:

$$u_m^* = 32.51658743^o \qquad k^L = 0.996774$$
$$u_1^* = 25.93794656^o \qquad k^L = 1.003226 \tag{39}$$
$$u_2^* = 38.93794656^o \qquad k^L = 1.003226$$

CHOICE OF MAPPING COORDINATES

The Cartesian coordinates x^*,y^* have the origin at the vertex of the oblique cone. In addition, the axes are also oblique. To transform this coordinate system into a practical Cartesian system the following transformations are applied.

The coordinates are first transformed to the (x'^*, y'^*) system with its origin coincident with the origin $(u_0^*,v_o^*$. The x'^* -axis points south-east at an azimuth of α_0(cf. the Lambert cone) and the y'^* -axis points north-east at a right angle to the x'^*-axis. The transformation between the two systems reads:

$$x'^* = y^*$$
$$y'^* = \rho_0 - x^* \tag{40}$$

where

$$\rho_0 = K_1 \cot^{K_2}\left(\frac{\pi}{4}+\frac{u_0^*}{2}\right) \tag{41}$$

To convert these oblique Cartesian coordinates $(x,y)$to proper mapping coordinates, the coordinate system must be rotated so that the y -axis points east and the -axis points north. This can be stated as:

$$\mathbf{r} = \mathbf{T}\left(\alpha_0 - \frac{\pi}{2}\right)\mathbf{r}'^*, \tag{42}$$

where $\mathbf{r}$ and $\mathbf{r}'^*$ are now two-dimensional vectors of Cartesian coordinates and $\mathbf{T}$ is the two-dimensional rotation matrix:

$$\mathbf{T}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{43}$$

The rotated coordinate system gives *true easting* x and *true northing* y. It is customary for a mapping to use rather a *false easting X* and a *false northing Y*, obtained from their true counterparts by adding some arbitrarily selected coordinate shifts x_0 and y_0, such that all the eastings and northings for the mapped territory will be positive (Figure 2)

$$\mathbf{P} = \mathbf{r_0} + \mathbf{r} \tag{44}$$

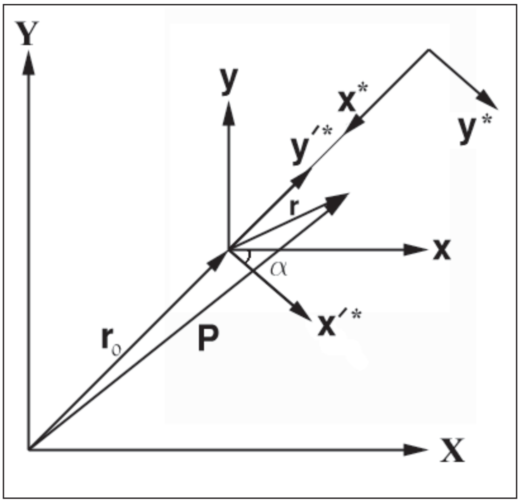

Figure 2. The four coordinate systems used in the new projection

The complete transformation from the oblique Cartesian coordinates to the mapping coordinates (false easting and northing) then reads:

$$X = x_0 - (\rho_0 - x^*)\cos\alpha_0 + y^*\sin\alpha_0$$
$$Y = y_0 - (\rho_0 - x^*)\sin\alpha_0 + y^*\cos\alpha_0 \tag{45}$$

The inverse transformation, from the mapping coordinates (false easting and northing) to the oblique Cartesian coordinates is:

$$x^* = (X - x_0)\cos\alpha_0 - (Y - y_0)\sin\alpha_0 + \rho_0$$
$$y^* = (X - x_0)\sin\alpha_0 - (Y - y_0)\cos\alpha_0 \tag{46}$$

showing the four coordinate systems used here.

Choice of constants for Iran

For the territory of Iran it seems appropriate to select the coordinate shifts as:

$$x_0 = 1000000 \quad m \qquad y_0 = 1000000 \quad m \qquad (47)$$

The azimuth α_0 was discussed previously and its value is α_0 = 129°, and the parameter ρ_0 =99967518.6644.

Substituting these particular values into equations (46) we get the appropriate transformations for Iran are obtained as:

$$\begin{aligned} X &= -0.62932039\,x^* + 0.7771459 * +7272762.7437 \\ Y &= -0.77714596\,x^* - 0.62932039\,y^* + 8746216.8758 \end{aligned} \qquad (48)$$

and the inverse transformation reads:

$$\begin{aligned} x^* &= -0.62932039X - 0.77714596Y + 11373985.0170 \\ y^* &= +0.77714596X - 0.62932039Y - 147825.5704 \end{aligned} \qquad (49)$$

DISTORTIONS OF THE PROPOSED MAP PROJECTION

In this section the mapping distortions (scale distortions, meridian convergence, T-t corrections) are determined across the mapping territory of Iran.

Scale distortion

The overall point scale factor of the double projection, mapping the ellipsoid onto the oblique cone, can be computed as a product of the two scale factors: *k* (equation (11)); and k^L (equation (32)):

$$kk^L = \frac{C_1 K_1 K_2 \cos u}{N \cos\phi \cos u^*} \cot^{K_2}\left(\frac{\pi}{4} + \frac{u^*}{2}\right) \qquad (50)$$

and as a result, the overall point distortion can be derived from the equation:

$$\varepsilon^L = kk^L - 1 \qquad (51)$$

The distortion is a function of latitude (ϕ) of a computation point. Figure 3 shows the variation of scale distortion over the mapping territory.

The distortion is mostly negative in the territory, decreasing in magnitude from the central oblique parallel.

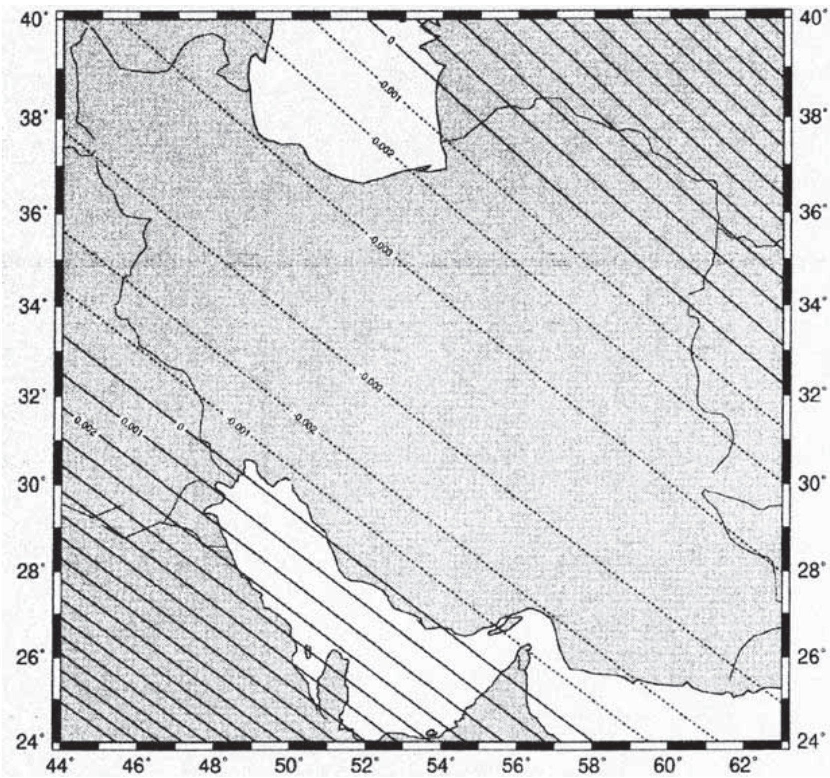

Figure 3. The point scale distortion in the new cartographic mapping of Iran.

Meridian convergence

The meridian convergence is computed from the general expression:

$$\gamma = \tan{-1}\left(\frac{\frac{\partial y}{\partial \lambda}}{\frac{\partial x}{\partial \lambda}}\right) \tag{52}$$

where x and y are the mapping coordinates; equationns (45). To derive the expression for γ, the following sets of equations are used (equationns (1)) as:

$$u = u(\phi)$$
$$v = v(\lambda)$$

equations (21) as:

$$u^* = u^*(u,v)$$
$$v^* = v^*(u,v)$$

equations (31) as:

$$x^* = x^*(u^*,v^*)$$
$$y^* = y^*(u^*,v^*)$$

and finally equations (45) as:

$$x = x(x^*,y^*)$$
$$y = y(x^*,y^*)$$

Using equations (1), the partial derivatives of u and v with respect to the variable λ read:

$$\frac{\partial u}{\partial \lambda} = 0, \qquad \frac{\partial v}{\partial \lambda} = C_1 \tag{53}$$

Taking these derivatives in the chain rule of partial derivatives applied among the four sets of equations mentioned above, one concludes with the following partial derivatives:

$$\frac{\partial x}{\partial \lambda} = C_1\left(\frac{\partial x}{\partial y^*}\frac{\partial y^*}{\partial v^*} + \frac{\partial x}{\partial x^*}\frac{\partial x^*}{\partial v^*}\right)\frac{\partial v^*}{\partial \lambda} + C_1\left(\frac{\partial x}{\partial y^*}\frac{\partial y^*}{\partial u^*} + \frac{\partial x}{\partial x^*}\frac{\partial x^*}{\partial u^*}\right)\frac{\partial u^*}{\partial \lambda}$$
$$\frac{\partial y}{\partial \lambda} = C_1\left(\frac{\partial y}{\partial y^*}\frac{\partial y^*}{\partial v^*} + \frac{\partial y}{\partial x^*}\frac{\partial x^*}{\partial v^*}\right)\frac{\partial v^*}{\partial \lambda} + C_1\left(\frac{\partial y}{\partial y^*}\frac{\partial y^*}{\partial u^*} + \frac{\partial y}{\partial x^*}\frac{\partial x^*}{\partial u^*}\right)\frac{\partial u^*}{\partial \lambda} \tag{54}$$

Deriving the partial derivatives in the right hand side from the sets of equations yields:

$$d_0 = \frac{\partial u^*}{\partial v} = \frac{1}{\cos u^*}\left(-a_{31} \cos u \sin v + a_{32} \cos u \cos v\right)$$
$$d_1 = \frac{\partial v^*}{\partial v} = \frac{cos^2 u^*}{r_1^*}\left(-a_{21} \cos u \sin v + a_{22} \cos u \cos v\right) - \frac{cos^2 v^*}{r_1^{*2}}\left(-a_{11} \cos u \sin v + a_{12} \cos u \cos v\right)$$
$$d_2 = \frac{\partial x^*}{\partial u^*} = K_2' x^*, \; K_2' = -\frac{K_2}{\cos u^*}$$
$$d_3 = \frac{\partial x^*}{\partial v^*} = -K_2 y^*$$
$$d_4 = \frac{\partial y^*}{\partial u^*} = -K_2' y^*$$
$$d_5 = \frac{\partial y^*}{\partial v^*} = -K_2 x^*$$
$$d_6 = \frac{\partial x}{\partial x^*} = \cos \alpha_0$$
$$d_7 = \frac{\partial x}{\partial y^*} = \sin \alpha_0$$
$$d_8 = \frac{\partial y}{\partial x^*} = -\sin \alpha_0$$
$$d_9 = \frac{\partial y}{\partial y^*} = \cos \alpha_0 \tag{55}$$

Substituting the derivatives in the formula for γ yields:

$$\frac{\partial u}{\partial \lambda} = 0, \qquad \frac{\partial v}{\partial \lambda} = C_1 \tag{53}$$

This equation is used to compute the meridian convergence as a function of position in a grid of points of 15 arc minute (latitude and

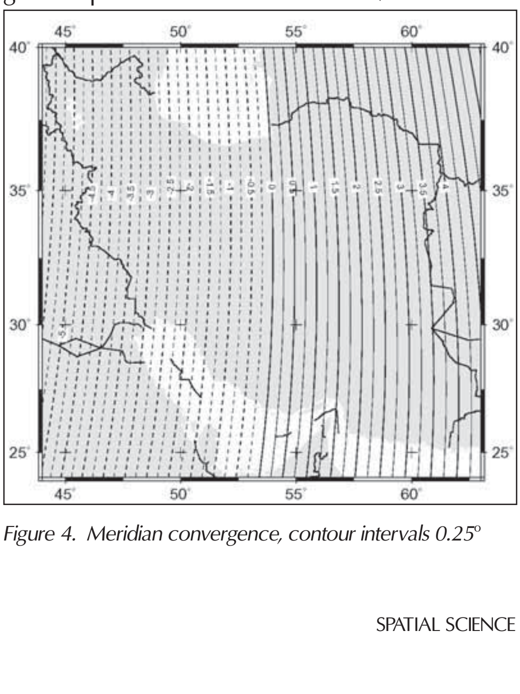

Figure 4. Meridian convergence, contour intervals 0.25°

longitude differences) spacing across the mapping territory (Iran). Contour lines of equal meridian convergences are shown in Figure 4.

The (T-t) correction

The arc to chord, or $(T-t)$, correction is a correction applied to the azimuth of a mapped geodesic (arc) at a computation point to change it to the bearing of the base line (chord). It is a function of the positions of the two end points of the base line. An approximate value for the correction can be obtained from the formula:

$$(T-t) \cong \sigma \frac{s}{2}$$
$$\sigma = \frac{1}{k^L}\left(\frac{\partial k^L}{\partial x} \cos T - \frac{\partial k^L}{\partial y} \sin T\right) \tag{57}$$

where σ is the curvature of the mapped geodesic at the computation point, s is the length of the base line, k^L is the point scale factor given by equation (32), and T is the bearing of the base line at the computation point. For the computation of the correction, the k^L needs to be expressed in terms of the mapping coordinates x and y. To do this, k^L, equation (32), is first formulated as a function of x^* Y^* and . From equations (31) the following is derived:

$$\tan^2\left(\frac{\pi}{4} + \frac{u^*}{2}\right) = \left(\frac{(K_1)^2}{x^{*2}+y^{*2}}\right)^{\frac{1}{K_2}} \tag{58}$$

Substituting this equation into equation (32), and applying the identity:

$$\frac{1}{\cos u^*} = \frac{1}{2}\cot\left(\frac{\pi}{4} + \frac{u^*}{2}\right) + \frac{1}{2}\tan\left(\frac{\pi}{4} + \frac{u^*}{2}\right) \tag{59}$$

the following expression for k^L as a function of and x^* *and* y^* is obtained:

$$k^L(x^*, y^*) = \frac{K_2}{2RK_1^{\frac{1}{K_2}}}\left(x^{*2} + y^{*2}\right)^{\frac{1+K_2}{2K_2}} + \frac{K_2 K_1^{\frac{1}{K_2}}}{2R}\left(x^{*2} + y^{*2}\right)^{\frac{K_2-1}{K_2}} \tag{60}$$

As it can be seen from equation (60), k^L is a function of the squared distance $\left(x^{*2} + y^{*2}\right)$ from the vertex of the oblique cone. Denoting this squared distance by θ^2:

$$k^L(x^*, y^*) = \frac{K_2}{2RK_1^{\frac{1}{K_2}}}\theta^{\frac{1+K_2}{K_2}} + \frac{K_2 K_1^{\frac{1}{K_2}}}{2R}\theta^{\frac{K_2-1}{K_2}} \tag{61}$$

On the other hand, $\theta^{\cdot}$ is also a function of the mapping coordinates X and Y. This function can be derived from equations (45) as:

$$\theta^2(x, y) = x^{*2} + y^{*2} = \left(X - x_0 + \rho_0 \cos\alpha_0\right)^2 + \left(Y - y_0 + \rho_0 \sin\alpha_0\right)^2 \tag{62}$$

Using the chain rule for differentiation, the partial derivatives of $\frac{\partial k^L}{\partial x}$ and $\frac{\partial k^L}{\partial y}$ are evaluated as:

$$\frac{\partial k^L}{\partial x} = \frac{\partial k^L}{\partial \theta^2}\frac{\partial \theta^2}{\partial x}, \quad \frac{\partial k^L}{\partial y} = \frac{\partial k^L}{\partial \theta^2}\frac{\partial \theta^2}{\partial y} \tag{63}$$

where the derivatives $\frac{\partial \theta^2}{\partial x}$ and $\frac{\partial \theta^2}{\partial y}$ are coming from equation (62) and the derivative $\frac{\partial k^L}{\partial \theta^2}$ is obtained from equation (61):

$$\frac{\partial k^L}{\partial \theta} = \frac{1+K_2}{4RK_1^{\frac{1}{K_2}}}\theta^{\frac{1+K_2}{2K_2}} + \frac{K_2 K_1^{\frac{1}{K_2}}}{4R}\theta^{\frac{-K_2-1}{K_2}} \tag{64}$$

Having the partial derivatives, the curvature (σ) is computed as a function of position. The $(T-t)$ correction can then be estimated, using equation (57), for a given base line length (s) and at a given bearing (T). For the study of distribution of the $(T-t)$ correction across the mapping territory, the maximum $(T-t)$ correction for a given constant base line length is determined. The maximum or minimum value of the correction is obtained at the bearing $\hat{T}$ that satisfies the condition:

$$\frac{\partial \sigma}{\partial T} = 0 \tag{65}$$

The derived $\hat{T}$ is tested for the maximum value of the correction. The maximum value of the correction is then calculated for the base line length of , on the same grid of points used for the meridian convergence computation across the mapping territory (Iran). A plot of the contour lines of equal correction are shown in Figure 5.

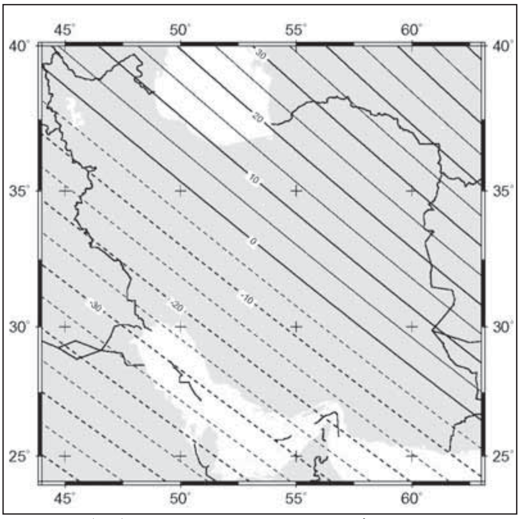

Figure 5. (T-t) correction, contour intervals 5″

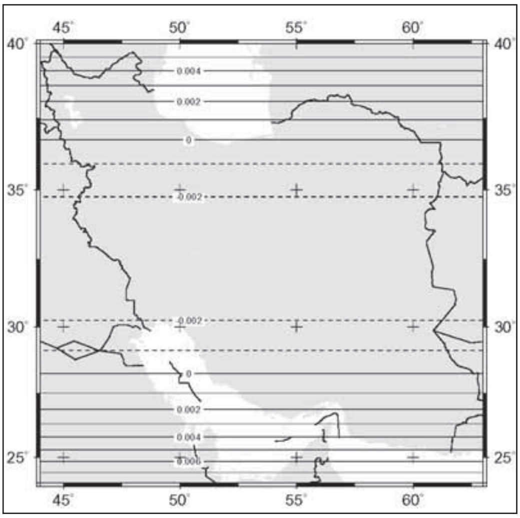

Figure 6. Scale distortion in the LCC2 map projection, contour interval 0.001

A COMPARISON WITH THE LAMBERT CONFORMAL CONIC PROJECTION

A comparison of the new cartographic mapping system is made with the Lambert Conformal Conic projection with two standard parallels (LCC2). The comparison is made in terms of the distortions determined for the two mappings. For a LCC2 to cover the mapping territory as a continuous projection, the two standard parallels would take the values:

$$\phi_1 = 28.25^\circ \qquad \phi_2 = 36.75^\circ \qquad (66)$$

implying the central latitude value of ϕ_0 = 32.5° and a half width of $\Delta\phi$ = 8.5° for the territory in the latitude direction. The central meridian is selected to be $\lambda_0 = 54^\circ$. The constant (l) for the projection (Krakiwsky, 1973) reads:

$$l = \frac{\ln N_1 - \ln N_2 + \ln\cos\phi_1 - \ln\cos\phi_2}{q_2 - q_1} \qquad (67)$$

where N_1 and N_2 are the prime vertical radii of curvature at ϕ_1 and ϕ_2 on the selected reference ellipsoid (GRS80), and q_1 and q_2 are the corresponding isometric latitudes. Introducing another constant (κ) given by:

$$\kappa = \frac{N_1 \cos\phi_1}{l} e^{lq_1} = \frac{N_2 \cos\phi_2}{l} e^{lq_2} \qquad (68)$$

the point scale factor k^{L2}, and the corresponding point scale distortion ε^{L2}, can then be derived from the following equations:

$$k^{L2} = \frac{\kappa l}{N\cos\phi} e^{-lq}$$
$$\varepsilon^{L2} = k^{L2} - 1 \qquad (69)$$

A map of scale distortion contour lines for the LCC2 projection, based on the scale distortions computed for the already used grid of points in the mapping territory, is shown in Figure 6.

A map of meridian convergence (γ) was developed (Figure 7) in the same way as a map of

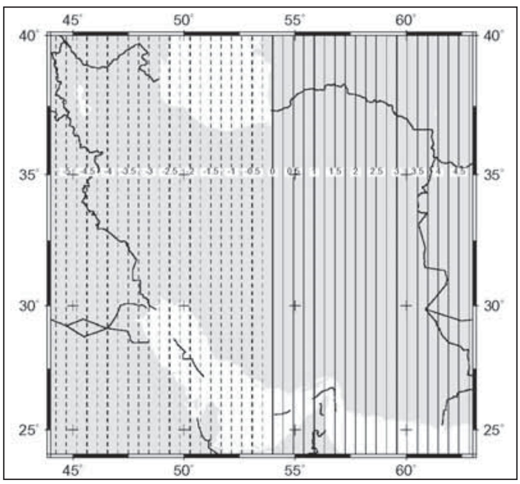

Figure 7. Meridian convergence in the LCC2 map projection, contour interval 0.25"

distortion using the formula for the convergence as (Krakiwsky, 1973):

$$\gamma = l\left(\lambda - \lambda_0\right), \tag{70}$$

A map of the $(T-t)$ correction for the LCC2 map projection has also been developed for the mapped territory. Applying equations (57), the k^{L2} and its derivatives with respect to the LCC2 Cartesian coordinates are required. Realising that the scale factor k^{L2} is a function of latitude only, and using the chain rule for derivatives, we can write:

$$\begin{aligned} \frac{\partial k^{L2}}{\partial x} &= \frac{\partial k^{L2}}{\partial \phi}\frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} \\ \frac{\partial k^{L2}}{\partial y} &= \frac{\partial k^{L2}}{\partial \phi}\frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y} \end{aligned} \tag{71}$$

where

$$\frac{\partial \phi}{\partial q} = \frac{N\cos\phi}{M} \tag{72}$$

where M and N are the ellipsoidal meridian and prime vertical radii of curvatures respectively, and:

$$\begin{aligned} q(x,y) &= \frac{ln\,\kappa - ln\,r}{l} \\ r &= \sqrt{x^2 + \left(r_0 - y\right)^2} \\ r_0 &= N_0\, cot\,\phi_0 \end{aligned} \tag{73}$$

is the inverse mapping equation, (Krakiwsky, 1973). Evaluating the partial derivatives:

$$\begin{aligned} \frac{\partial k^{L2}}{\partial x} &= \frac{lM + F}{MN\cos\phi}\frac{x}{r} \\ \frac{\partial k^{L2}}{\partial y} &= \frac{lM + F}{MN\cos\phi}\frac{y - r_0}{r} \end{aligned} \tag{74}$$

where

$$F = M\sin\phi\frac{e^2\cos^2\phi}{1 - e^2} - N\sin\phi\,. \tag{75}$$

Now the maximum $(T-t)$ correction can be again computed on the same grid of points as the other corrections. The corresponding map of contour lines is shown in Figure 8.

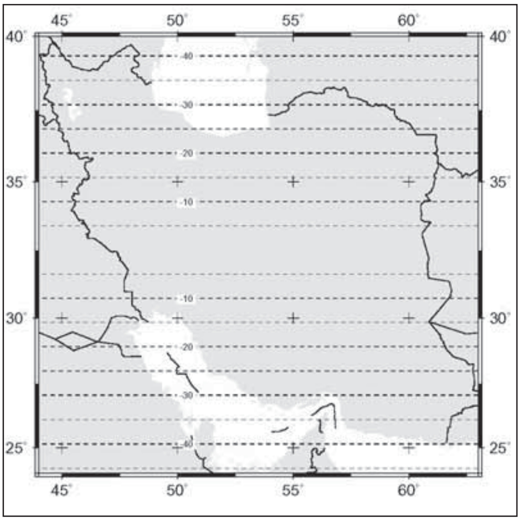

Figure 8. (T-t) correction in the LCC2 map projection, contour interval 5"

CONCLUSIONS

This paper describes the selection of the best mapping (i.e., mapping that has the smallest distortion), for the territory of Iran. It really consists of two projections: that of the reference ellipsoid onto a Gaussian conformal sphere and that of the Gaussian conformal sphere onto an oblique secant Lambert cone. The choices of the optimum constants of the two mappings for the mapped territory are also discussed, but only approximate estimates for these constants are adopted for the demonstration. These constants should be fine tuned for the actual mapping to achieve the smallest possible scale distortion.

Next, the distortions of the proposed mapping (i.e., the scale distortion), the meridian convergence and the $(T-t)$ distortion, are plotted and discussed. It is shown that the scale distortion for Iran ranges approximately between $-3.2^{*}10^{-3}$ and $3.2^{*}10^{-3}$. The meridian convergence for the proposed mapping is reasonable and the $(T-t)$ correction, standardized for a 20 km baseline, is in a reasonable range between −40″ and 40″.

For comparison, the optimal Lambert conformal (secant) conic projection for Iran is also presented and the distortions of that mapping are discussed. It transpires that while the meridian convergence and the $(T-t)$ distortion are about the same as for the proposed map projection, the scale distortion can be almost twice as large for the Lambert projection.

ACKNOWLEDGMENT

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