manuscripta geodaetica (1994) 19:213–219

manuscripta geodaetica

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The indirect effect of topography in the Stokes-Helmert technique for a spherical approximation of the geoid

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Received June 1992; Accepted May 1994

Abstract

It has been customary in Geodesy to evaluate the indirect effect that arises from mathematical removal of the topography in solving the geodetic boundary value problem using Stokes approach, by modelling the geoid as a plane. In this contribution, we show that this planar model gives an incorrect result. Adopting a spherical model for the geoid, we derive a new expression for the indirect topographical effect on potential.

Introduction

The effect of topographical masses on geoid height computation has been discussed by many geodesists (Heiskanen and Moritz, 1967; Wichiencharoen, 1982; Vaníček et al., 1987; Vaníček and Kleusberg, 1987; Heck, 1993, etc.). The problem arises from the fact that Stokes's formula for geoid undulation assumes that there are no masses outside the geoid and that the gravity measurements are referred to the geoid. These assumptions require the real Earth's topography to be regularized. Helmert suggested that the masses outside the geoid be condensed as a surface material layer on the geoid (Helmert, 1884). This condensation implies that observed gravity on the Earth's surface is to be corrected by the so-called *direct topographical effect on gravity* (Vaníček and Kleusberg, 1987). Furthermore, by condensing the masses, the original gravitational potential of the Earth is also changed. The difference between the gravitational potential of the actual topographical masses and the gravitational potential of the condensed masses referred to a point on the geoid is usually called the *indirect topographical effect on potential* (Heiskanen and Moritz, 1967, Sec. 3-6.).

The indirect topographical effect for Helmert condensation has been intensively discussed by Heiskanen and Moritz (1967), Wichiencharoen (1982), Vaníček et al. (1987), Wang and Rapp (1990) and Heck (1993) among others. They all employed the planar approximation of the geoid and assumed a constant density of all the topographical masses. They determined the indirect effect correction terms independently and obtained the same result.

This paper also deals with the correction for the indirect topographical effect on potential, but the planar approximation of the geoid is replaced by a spherical approximation. Moreover, the density of the topographical masses is assumed to be laterally non-homogeneous. It is shown that the indirect effect term determined on the basis of planar approximation differs significantly from that resulting from the spherical approach; the differences may reach up to 0.5 m in absolute value. Thus, it is shown that the planar approximation of the geoid is not adequate for the indirect topographical effect computation when a precise geoid is to be produced.

Helmert's decomposition of gravity potential

Let us start with a decomposition of the gravitational potential generated by the Earth. The gravitational potential may be split into two parts:

$$V = V^g + V^t \,, \tag{1}$$

where V^g is the potential generated by the masses below the geoid and V^t is the potential generated by the topographical masses (masses between the geoid and the topographical surface). Helmert (1884) suggested to decompose the potential of the topographical masses as

$$V^t = V^c + \delta V \,, \tag{2}$$

where V^c is the potential of the masses condensed on the geoid and δV is the residual potential.

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The gravity potential $V^g + V^c + \Phi$, where Φ is the centrifugal potential of the Earth, may be written as a sum of a normal gravity potential U, generated by a geocentric biaxial ellipsoid spinning with the same angular velocity as the earth, and a disturbing potential T^h:

$$V^g + V^c + \Phi = U + T^h \ . \qquad (3)$$

Here we denote the disturbing potential by a superscript h and will call it the "Helmert disturbing potential" to differentiate it from the usual disturbing potential T. The important consequence of eqn.(3) is that the Helmert disturbing potential T^h is harmonic outside the geoid (Martinec et al., 1993)

Let us re-derive the Bruns formula (Heiskanen and Moritz, 1967, Sec. 2-13.) for the case when the Helmert decomposition (2) is employed. Let P_g be a point on the geoid and Q be the corresponding point on the reference ellipsoid (lying on the same geocentric radius). Then the actual gravity potential of the Earth at the point on the geoid, W_{P_g}, can be written as

$$W_{P_g} \equiv W_0 = U_{P_g} + T^h_{P_g} + \delta V_{P_g} \ . \qquad (4)$$

The normal potential on the geoid, U_{P_g}, can be expanded into a Taylor series as:

$$U_{P_g} = U_Q + \left.\frac{\partial U}{\partial r}\right|_Q N + \ldots \ , \qquad (5)$$

where N is the geoid-ellipsoid separation called geoidal height. Since the normal potential is assumed to have been chosen in such a way that it has the same value on the reference ellipsoid as the actual gravity potential has on the geoid, $U_Q = W_0$, eqns.(4) and (5) may be arranged to give the correct form of the Bruns formula as

$$N \doteq \frac{1}{\gamma_Q}\left(T^h_{P_g} + \delta V_{P_g}\right) \ , \qquad (6)$$

where γ is the normal gravity and the terms of magnitude of the order of $O(N^2)$ have been neglected. The term

$$N^h = \frac{T^h_{P_g}}{\gamma_Q} \qquad (7)$$

denotes the separation between the so-called co-geoid (Heiskanen and Moritz, 1967, Sec. 3-6.) and the reference ellipsoid and the term

$$\delta N = \frac{\delta V_{P_g}}{\gamma_Q} \qquad (8)$$

describes the separation between the geoid and co-geoid; it is the so-called primary indirect topographical effect on potential divided by γ_Q (ibid.).

Gravitational potential of topographical masses

Let the topographical masses be bounded below by the the geoid with geocentric radius $r_g(\Omega)$ and above by the topographical surface with geocentric radius $r_g(\Omega) + H(\Omega)$. This means that $H(\Omega)$ is the height of the topographical surface above the geoid, reckoned along the geocentric radius. The argument Ω stands for a horizontal position given by co-latitude ϑ and longitude λ. The gravitational potential V^t induced by the topographical masses at an arbitrary point (r, Ω) is given by Newton's volume integral

$$V^t(r,\Omega) = \\ = G\int_{\Omega'}\int_{r'=r_g(\Omega')}^{r_g(\Omega')+H(\Omega')} \varrho(r',\Omega')L^{-1}(r,\psi,r'){r'}^2 dr' d\Omega' \ , \qquad (9)$$

where G is Newton's gravitational constant, $\varrho(r,\Omega)$ is the density of the topographical masses, $L^{-1}(r,\psi,r')$ is the Newton kernel (reciprocal spatial distance between the dummy point (r', Ω') and the computation point (r, Ω)):

$$L^{-1}(r,\psi,r') = \frac{1}{\sqrt{r^2 + {r'}^2 - 2rr'\cos\psi}} \ , \qquad (10)$$

ψ is the angular distance between the geocentric directions Ω and Ω', and the integration in eqn.(9) is taken over the full solid angle Ω'.

Note that Newton's kernel grows to infinity when the dummy point (r', Ω') moves towards the computation point (r, Ω). But, the Newton kernel is only **weakly singular** which means that for $r \neq 0$ (Kellogg, 1929, Chapter VI):

$$\lim_{\substack{\psi \to 0 \\ r' \to r}} \left[\sin\psi \, L^{-1}(r,\psi,r')\right] = \frac{1}{r} < \infty \ . \qquad (11)$$

Writing the element $d\Omega'$ of the full solid angle in polar coordinates (ψ, α) as $d\Omega' = \sin\psi d\psi d\alpha$, the weak singularity property (11) is easily seen reflected also in the corresponding integral form

$$\int_{\Omega'} L^{-1}(r,\psi,r') d\Omega' < \infty \ . \qquad (12)$$

This inequality is valid for all non-zero radii r and r'.

Throughout the paper, the geoid will be approximated by a sphere of radius R:

$$r_g(\Omega) = R \ , \qquad (13)$$

where R is the mean radius of the Earth. This approximation to the geoid can be easily shown to be good to about 0.5%, which represents the accuracy limit of our results. Furthermore, the actual density of the topographical masses $\varrho(r,\Omega)$ will be approximated by a column average value $\overline{\varrho}(\Omega)$:

$$\overline{\varrho}(\Omega) = \frac{1}{H(\Omega)} \int_{r=R}^{R+H(\Omega)} \varrho(r,\Omega)dr \; . \tag{14}$$

Under these assumptions, eqn.(9) takes the following form:

$$V^t(r,\Omega) \doteq G \int_{\Omega'} \overline{\varrho}(\Omega') \int_{r'=R}^{R+H(\Omega')} L^{-1}(r,\psi,r') r'^2 dr' d\Omega' \; . \tag{15}$$

The property (12) may now be used for removing the singularity of the Newton integral (15). Subtracting and adding a term

$$V^B(r,\Omega) = G\,\overline{\varrho}(\Omega) \int_{\Omega'} \int_{r'=R}^{R+H(\Omega)} L^{-1}(r,\psi,r') r'^2 dr' d\Omega' \tag{16}$$

to the potential $V^t(r,\Omega)$, we get

$$V^t(r,\Omega) = V^B(r,\Omega) + V^R(r,\Omega) \; , \tag{17}$$

where

$$V^R(r,\Omega) = G \int_{\Omega'} \left[\overline{\varrho}(\Omega') \int_{r'=R}^{R+H(\Omega')} L^{-1}(r,\psi,r') r'^2 dr' - \right.$$

$$\left. -\overline{\varrho}(\Omega) \int_{r'=R}^{R+H(\Omega)} L^{-1}(r,\psi,r') r'^2 dr' \right] d\Omega' \; . \tag{18}$$

The quantity V^B is easily recognized as the potential of a spherical Bouguer shell of density $\overline{\varrho}(\Omega)$ and thickness $H(\Omega)$. This potential is finite - due to the inequality (12) - and is equal to (Wichiencharoen, 1982),

$$V^B(r,\Omega) = \begin{cases} 4\pi G\overline{\varrho}(\Omega)\frac{1}{r}\left[R^2H(\Omega) + RH^2(\Omega) + \frac{1}{3}H^3(\Omega)\right] \; , & r \geq R + H(\Omega) \; , \\ 2\pi G\overline{\varrho}(\Omega)\left[(R+H(\Omega))^2 - \frac{2}{3}\frac{R^3}{r} - \frac{1}{3}r^2\right] \; , & R \leq r \leq R + H(\Omega) \; , \\ 4\pi G\overline{\varrho}(\Omega)\left[RH(\Omega) + \frac{1}{2}H^2(\Omega)\right] \; , & r \leq R \; . \end{cases} \tag{19}$$

Since the actual earth's surface deviates from the Bouguer sphere (of radius $R+H(\Omega)$), there are deficiencies and/or abundances of topographical masses with respect to the mass of the Bouguer shell. These contribute to the topographical potential $V^t(r,\Omega)$ through the term $V^R(r,\Omega)$ – an analogy of the terain correction (Heiskanen and Moritz, 1967, sect. 3-3.). We will call $V^R(r,\Omega)$ the *terrain roughness term*. It depends chiefly on the behaviour of the difference $H(\Omega) - H(\Omega')$.

Let us now investigate the limit for $\psi \to 0$ of the subintegral function in the angular integral (18). When $\psi \to 0$, then $\overline{\varrho}(\Omega') \to \overline{\varrho}(\Omega)$ and $H(\Omega') \to H(\Omega)$. We will assume, reasonably, that both the topographical density $\overline{\varrho}$ and the topographical height H are bounded (i.e., that there are no mass-singularities inside the topographical masses and the heights of the earth's topography are finite). Then the limit for $\psi \to 0$ of the subintegral function in eqn.(18) reads

$$\lim_{\psi\to 0} \left[\overline{\varrho}(\Omega') \int_{r'=R}^{R+H(\Omega')} L^{-1}(r,\psi,r') r'^2 dr' - \right.$$

$$\left. -\overline{\varrho}(\Omega) \int_{r'=R}^{R+H(\Omega)} L^{-1}(r,\psi,r') r'^2 dr' \right] \sin\psi =$$

$$= \overline{\varrho}(\Omega) \int_{r'=R}^{R+H(\Omega)} \lim_{\psi\to 0} \left[L^{-1}(r,\psi,r')\sin\psi\right] r'^2 dr' -$$

$$- \overline{\varrho}(\Omega) \int_{r'=R}^{R+H(\Omega)} \lim_{\psi\to 0} \left[L^{-1}(r,\psi,r')\sin\psi\right] r'^2 dr' \; . \tag{20}$$

Since both functions $\overline{\varrho}(\Omega)$ and $H(\Omega)$ are bounded and the Newton kernel is weakly singular, see property (11), both integrals on the right-hand side of eqn.(20) are finite and have the same value; their difference is thus equal to zero. This means that the point $\psi = 0$ can be left out of the integration domain Ω' and the singularity of the Newton kernel at the point $\psi = 0$ is removed. This fact is important for the numerical computation of the topographical potential $V^t(r,\Omega)$ because the formulae (17) and (18) ensure that the numerical algorithm is not forced to evaluate the undefined expression of the type of 0/0 encountered in the original Newton integral (15).

Radial integral of Newton's kernel

The indefinite radial integral of the Newton kernel may be evaluated analytically (Gradshteyn and Ryzhik, 1980, pars. 2.261, 2.264) as follows:

$$\int_{r'} L^{-1}(r,\psi,r') r'^2 dr' = \widetilde{L^{-1}}(r,\psi,r') + C \; , \tag{21}$$

where

$$\widetilde{L^{-1}}(r,\psi,r') = \frac{1}{2}(r' + 3r\cos\psi)L(r,\psi,r') +$$

$$+\frac{r^2}{2}(3\cos^2\psi-1)\,\ln|r'-r\cos\psi+L(r,\psi,r')|\ , \quad (22)$$

and the 'constant' C may depend on the variables r and ψ only. Using notation (21), the topographical potential V^t (cf., eqns.(17) and (18)) becomes

$$V^t(r,\Omega)=V^B(r,\Omega)+$$
$$+G\int_{\Omega'}\left[\overline{\varrho}(\Omega')\,\widetilde{L^{-1}}(r,\psi,r')\Big|_{r'=R}^{R+H(\Omega')}-\right.$$
$$\left.-\overline{\varrho}(\Omega)\,\widetilde{L^{-1}}(r,\psi,r')\Big|_{r'=R}^{R+H(\Omega)}\right]d\Omega'\ . \quad (23)$$

Potential of the condensation layer

As we have seen in the Introduction, the condensation of topographical masses onto the geoid plays a fundamental role in the Helmert second condensation technique. Provided that $\sigma(\Omega)$ is a (surface) density of the condensation layer, the potential of Helmert's condensation layer may be expressed by Newton's surface integral:

$$V^c(r,\Omega)=GR^2\int_{\Omega'}\sigma(\Omega')\,L^{-1}(r,\psi,R)\,d\Omega'\ . \quad (24)$$

To remove the singularity of the reciprocal distance $1/L(r,\psi,R)$, we may proceed in a way analogous to that for the potential V^t. Let us rewrite eqn.(24) as

$$V^c(r,\Omega)=V^\ell(r,\Omega)+$$
$$+\,GR^2\int_{\Omega'}[\sigma(\Omega')-\sigma(\Omega)]\,L^{-1}(r,\psi,R)d\Omega'\ . \quad (25)$$

Here the symbol $V^\ell(r,\Omega)$ denotes the gravitational potential of a spherical layer with density $\sigma(\Omega)$ and radius R:

$$V^\ell(r,\Omega)=GR^2\sigma(\Omega)\int_{\Omega'}\frac{d\Omega'}{L(r,\psi,R)}\ . \quad (26)$$

The last integral may be readily evaluated yielding

$$V^\ell(r,\Omega)=\begin{cases}4\pi G\sigma(\Omega)\frac{R^2}{r}\ , & r>R\ ,\\ 4\pi G\sigma(\Omega)R\ , & r\le R\ .\end{cases} \quad (27)$$

The density $\sigma(\Omega)$ of condensed masses can be chosen in a variety of ways. In this paper we will choose it according to the principle of conservation of topographical masses (Wichiencharoen, 1982), i.e.,

$$\sigma(\Omega)=\overline{\varrho}(\Omega)\tau(\Omega)\ , \quad (28)$$

where

$$\tau(\Omega)=H(\Omega)\left(1+\frac{H(\Omega)}{R}+\frac{H^2(\Omega)}{3R^2}\right)\ . \quad (29)$$

The indirect topographical effect on potential

To find the expression for the indirect topographical effect on potential, the residual topographical potential $\delta V=V^t-V^c$ must be evaluated on the geoid ($r=R$). Considering eqn.(23) for the topographical potential V^t, eqn.(25) for the condensation potential V^c, and replacing r by R, we obtain

$$\delta V(R,\Omega)=\delta V^B(R,\Omega)+\delta V^R(R,\Omega)\ , \quad (30)$$

where the "Bouguer term" $\delta V^B(R,\Omega)$ is given as

$$\delta V^B(R,\Omega)=V^B(R,\Omega)-V^\ell(R,\Omega)\ , \quad (31)$$

and the "terrain roughness term" $\delta V^R(R,\Omega)$ is equal to

$$\delta V^R(R,\Omega)=G\int_{\Omega'}\left\{\overline{\varrho}(\Omega')\,\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H(\Omega')}-\right.$$
$$-\overline{\varrho}(\Omega)\,\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H(\Omega)}-$$
$$\left.-R^2\,[\sigma(\Omega')-\sigma(\Omega)]\,L^{-1}(R,\psi,R)\right\}d\Omega'\ . \quad (32)$$

Using the last of eqns.(19) and (27) valid for $r=R$, and taking the condensation density $\sigma(\Omega)$ according to eqns.(28) and (29), the Bouguer term $\delta V^B(R,\Omega)$ becomes

$$\delta V^B(R,\Omega)=-2\pi G\overline{\varrho}(\Omega)H^2(\Omega)\left(1+\frac{2}{3}\frac{H(\Omega)}{R}\right)\ . \quad (33)$$

Analysing eqn.(18), we have learned that there is no need to evaluate the vertically integrated kernel $\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H(\Omega')}$, at the computation point ($\psi=0$). Nevertheless, the kernel has to be evaluated in the immediate neighbourhood of the computation point; therefore, we have to investigate the type of singularity of the kernel $\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H(\Omega')}$ at that point. Thus, let us have a look at the behaviour of that kernel in the vicinity of the point $\psi=0$. Using eqn.(22), we get

$$\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H(\Omega')}=$$
$$=\frac{1}{2}(R+H(\Omega')+3R\cos\psi)L(R,\psi,R+H(\Omega'))-$$
$$-\frac{1}{2}(R+3R\cos\psi)\ell_0+$$
$$+\frac{R^2}{2}(3\cos^2\psi-1)\ln\frac{H(\Omega')+\frac{\ell_0^2}{2R}+L(R,\psi,R+H(\Omega'))}{\ell_0+\frac{\ell_0^2}{2R}}\ , \quad (34)$$

where

$$L(R,\psi,R+H(\Omega'))=$$
$$=\sqrt{(R+H(\Omega'))^2+R^2-2R(R+H(\Omega'))\cos\psi}=$$

$$= \sqrt{\ell_0^2 + \frac{H(\Omega')}{R}\ell_0^2 + H^2(\Omega')} \; , \qquad (35)$$

and ℓ_0 is the spatial distance between points (R,Ω) and (R,Ω'), i.e.,

$$\ell_0 \equiv L(R,\psi,R) = 2R\sin\frac{\psi}{2} \; . \qquad (36)$$

When the dummy point of integration comes close to the computation point ($\psi \to 0$ or $\ell_0 \to 0$), the first term on the right-hand side of eqn.(34) becomes equal to $(4R + H(\Omega'))H(\Omega')/2$; the second term goes to zero and the last term grows to infinity. This term behaves like $\ln \ell_0$ (we assume that $H(\Omega') > 0$). Because the following limit is valid

$$\lim_{\ell_0 \to 0} \frac{\ln \ell_0}{\frac{1}{\ell_0}} = 0 \; , \qquad (37)$$

the magnitude of the whole kernel $\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R}^{R+H(\Omega')}$ grows to infinity more slowly than the reciprocal distance $1/\ell_0$, when ψ approaches zero. Therefore, the numerical procedure of computing the Newton integral (32), based on eqn.(22) is very stable even near the computation point ($\psi = 0$).

Other approximations

The general formulae (31)-(33) are accurate to a spherical approximation and use the mean radial topographical density. They can be further simplified by accepting certain approximations permissible from the accuracy point of view.

Constant density of topographical masses

The first approximation is based on the fact that the density of topographical masses varies by about 10 - 20% around the mean crustal density $\varrho_0 = 2.67 \mathrm{g/cm^3}$. Martinec (1993) showed that these variations contribute at most 1 m to the geoidal heights even in the highest mountains. In the terrain with heights up to 1000 m, the contributions to geoidal heights by lateral variations of topographical density are below 1 cm and may be neglected. For such a case, the density of topographical masses is modelled by the mean crustal density ϱ_0.

Taking this approximation into account, i.e., putting $\overline{\varrho}(\Omega) \approx \varrho_0$, the general formulae (31)-(33) for the indirect topographical effect on potential become

$$\delta V(R,\Omega) = -2\pi G\varrho_0 H^2(\Omega)\left(1 + \frac{2}{3}\frac{H(\Omega)}{R}\right) +$$

$$+G\varrho_0 \int_{\Omega'} \left[\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H(\Omega)}^{R+H(\Omega')} - \right.$$

$$\left. -R^2 \frac{\tau(\Omega') - \tau(\Omega)}{\ell_0} \right] d\Omega' \; , \qquad (38)$$

where we have used ℓ_0 for $L(R,\psi,R)$ - cf., eqn.(36).

Restricted integration

Martinec (1993) also showed that the kernel

$$\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H(\Omega)}^{R+H(\Omega')} \qquad (39)$$

approaches the kernel

$$R^2\left[\tau(\Omega') - \tau(\Omega)\right] L^{-1}(R,\psi,R) \; , \qquad (40)$$

when the integration point moves away from the computation point (and $\tau(\Omega)$ is chosen according to eqn.(29)). For instance, the difference between kernels (39) and (40) falls 7 orders in magnitude when the integration point moves from $\psi = 0$ to $\psi = 1^o$. This confirms the well-known fact that the gravitational potential of topographical masses of a finite thickness behaves like the potential of a thin layer when it is observed from a larger distance. The integration over angular coordinates Ω' may be thus limited to a small area (of radius ψ_0) surrounding the computation point. To get a 1 cm accuracy of the residual potential $\delta V(R,\Omega)$, it is sufficient to integrate up to a distance $\psi_0 = 2^o$ (ibid.). The formula (38) for the indirect topographical effect on potential then becomes

$$\delta V(R,\Omega) \doteq -2\pi G\varrho_0 H^2(\Omega)\left(1 + \frac{2}{3}\frac{H(\Omega)}{R}\right) +$$

$$+G\varrho_0 \int_{\Omega'_1} \left[\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H(\Omega)}^{R+H(\Omega')} - \right.$$

$$\left. -R^2 \frac{\tau(\Omega') - \tau(\Omega)}{\ell_0} \right] d\Omega' \; , \qquad (41)$$

where Ω'_1 is a spherical cap of radius ψ_0 surrounding the computation point.

As we will see later, the integration kernels in integral (41) will be expressed by the distance ℓ_0, cf. eqn.(36), and heights $H(\Omega)$ and $H(\Omega')$. The restricted integration over Ω'_1 will enable us to neglect terms of the order of $\ell_0^2/2R^2$ with respect to 1, because these terms never exceed the value of 2×10^{-3}. This neglection is acceptable because it produces an error of the same order as the relative error of 3×10^{-3} of the spherical approximation used throughout the paper.

Planar approximation of distances

The planar approximation of distances (not to be confused with a planar approximation of the geoid) is based on the fact that the ratio H/R never exceeds the value of 1.4×10^{-3}. The planar approximation of distances is acceptable because it again produces an error of the same order of magnitude as the error in the spherical approximation of the geoid. Employing this approximation, quantities of the order of H/R are neglected with respect to 1. For example, the planar approximation of the spatial distance $L(R,\psi,H(\Omega'))$ between points (R,Ω) and $(R+H(\Omega'),\Omega')$ is simple to derive using eqn.(35):

$$L(R,\psi,R+H(\Omega')) \approx \sqrt{\ell_0^2+H^2(\Omega')}\ . \qquad (42)$$

Approximating the function $\tau(\Omega)$ by $H(\Omega)$, $\tau(\Omega')$ by $H(\Omega')$, the Bouguer term $-2\pi G\varrho_0 H^2(\Omega)\left(1+\frac{2}{3}\frac{H(\Omega)}{R}\right)$ becomes $-2\pi G\varrho_0 H^2(\Omega)$. Then the formula (41) reads

$$\delta V(R,\Omega) \doteq -2\pi G\varrho_0 H^2(\Omega) + +G\varrho_0\int_{\Omega'_1}\left[\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H(\Omega)}^{R+H(\Omega')} - -R^2\frac{H(\Omega')-H(\Omega)}{\ell_0}\right]d\Omega'\ . \qquad (43)$$

Let us find now the planar approximation of the integration kernel $\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H(\Omega)}^{R+H(\Omega')}$. Using eqn.(22), we have

$$\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H(\Omega)}^{R+H(\Omega')} = = \frac{R}{2}\left(1+\frac{H(\Omega')}{R}+3\cos\psi\right)L(R,\psi,R+H(\Omega')) - -\frac{R}{2}\left(1+\frac{H(\Omega)}{R}+3\cos\psi\right)L(R,\psi,R+H(\Omega)) + +\frac{R^2}{2}(3\cos^2\psi-1)\times \times\ln\frac{R(1-\cos\psi)+H(\Omega')+L(R,\psi,R+H(\Omega'))}{R(1-\cos\psi)+H(\Omega)+L(R,\psi,R+H(\Omega))}\ . \qquad (44)$$

Substituting for distances $L(R,\psi,R+H(\Omega'))$ and $L(R,\psi,R+H(\Omega))$ from eqn.(42), neglecting $H(\Omega)/R$ and $H(\Omega')/R$ with respect to 1 in the first two terms, and expressing the function $1-\cos\psi$ by means of the distance ℓ_0, we get

$$\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H(\Omega)}^{R+H(\Omega')} \doteq \doteq \frac{R}{2}(1+3\cos\psi)\left(\sqrt{\ell_0^2+H^2(\Omega')}-\sqrt{\ell_0^2+H^2(\Omega)}\right) + + \frac{R^2}{2}(3\cos^2\psi-1)\ln\frac{\frac{\ell_0^2}{2R}+H(\Omega')+\sqrt{\ell_0^2+H^2(\Omega')}}{\frac{\ell_0^2}{2R}+H(\Omega)+\sqrt{\ell_0^2+H^2(\Omega)}}\ . \qquad (45)$$

From eqn.(36), the function $1+3\cos\psi$ in terms of the distance ℓ_0 reads

$$1+3\cos\psi = 4-\frac{\ell_0^2}{2R^2}\ . \qquad (46)$$

Because of restricted integration over Ω' in eqn.(43), we may neglect the term $\ell_0^2/2R^2$ with respect to 4 getting

$$1+3\cos\psi = 4 - O\left(5\times10^{-4}\right)\ . \qquad (47)$$

Within the same accuracy, we may further write

$$3\cos^2\psi - 1 \doteq 2\ . \qquad (48)$$

Using approximations (47) and (48), we get the planar approximation of the integration kernel $\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H(\Omega)}^{R+H(\Omega')}$ as

$$\widetilde{L^{-1}}(R,\psi,r')\Big|_{r'=R+H(\Omega)}^{R+H(\Omega')} \doteq \doteq 2R\left(\sqrt{\ell_0^2+H^2(\Omega')}-\sqrt{\ell_0^2+H^2(\Omega)}\right) + + R^2\ln\frac{\frac{\ell_0^2}{2R}+H(\Omega')+\sqrt{\ell_0^2+H^2(\Omega')}}{\frac{\ell_0^2}{2R}+H(\Omega)+\sqrt{\ell_0^2+H^2(\Omega)}}\ . \qquad (49)$$

Finally, substituting the last formula into eqn.(43), the indirect topographical effect on potential may be approximated as

$$\delta V(R,\Omega) \doteq -2\pi G\varrho_0 H^2(\Omega) + +GR^2\varrho_0\int_{\Omega'_1}\left[2\frac{\sqrt{\ell_0^2+H^2(\Omega')}-\sqrt{\ell_0^2+H^2(\Omega)}}{R} + +\ln\frac{\frac{\ell_0^2}{2R}+H(\Omega')+\sqrt{\ell_0^2+H^2(\Omega')}}{\frac{\ell_0^2}{2R}+H(\Omega)+\sqrt{\ell_0^2+H^2(\Omega)}} - \frac{H(\Omega')-H(\Omega)}{\ell_0}\right]d\Omega' \qquad (50)$$

Discussion

Using the planar approximation of the geoid, Heck (1993) derived that the change of the gravitational potential due to the condensation (for $H\geq 0$) is equal to (ibid., eqn.(9))

$$\delta V_p(R,\Omega) = -\pi G\varrho_0 H^2(\Omega) + +GR^2\varrho_0\int_{\Omega'_1}\left[\ln\frac{H(\Omega')+\sqrt{\ell_0^2+H^2(\Omega')}}{H(\Omega)+\sqrt{\ell_0^2+H^2(\Omega)}} - -\frac{H(\Omega')-H(\Omega)}{\ell_0}\right]d\Omega'\ . \qquad (51)$$

Comparing this equation with eqn.(50), we can see that $\delta V_p(R,\Omega)$ differs from $\delta V(R,\Omega)$. Let us discuss the differences. The Bouguer term $-2\pi G\varrho_0 H^2(\Omega)$ in eqn.(50) derived from a spherical approximation of the geoid has a magnitude twice as large as the equivalent term in Heck's formula (51). The difference, $-\pi G\varrho_0 H^2(\Omega)$, is always negative and may reach up to -0.5 m in high mountains. Therefore, eqn. (51) derived from the planar approximation of the geoid is biased.

Equation (50) also differs from eqn.(50) in the terrain roughness term. We cannot neglect the term $\ell_0^2/2R$ with respect to $H(\Omega)$ in the logarithmic function because for distances over 1^o the magnitude of the term $\ell_0^2/2R$ may be comparable with $H(\Omega)$ or $H(\Omega')$. Nevertheless, this difference is perhaps not crucial because the terrain roughness term affects usually the geoid in the range of one or two decimetres at most.

Conclusion

All existing theories of topographical effects in Helmert's second condensation technique are based on the concept of planar approximation of the geoid. The geoid is considered as an infinite plane and topographical masses are condensed onto this plane. This approximation describes the actual situation only very roughly.

This theoretical study was motivated by the above inconsistency in the description of the problem. Modelling the geoid by a sphere removes infinite potentials and describes the actual situation much more accurately. Moreover, the density of topographical masses is here considered laterally varying which enables to model it better (cf., eqn.(32) and(33)) than by a constant value of 2.67 g/cm^3 used up to now.

We have derived the spherical formula for the indirect topographical effect on potential, see eqn.(31)-(33). Then we employed additional simplifications in order to compare our results with those recently derived by Heck (1993). We have shown that the planar formula is biased in a term corresponding to the indirect topographical effect of the Bouguer shell and that this bias may reach -0.5 m in high mountains. We can conclude that the expressions based on the planar model of the geoid cannot be used for evaluation of the topographical effects if one-centimetre accuracy in geoid determination is desired.

Acknowledgments

The authors would like to acknowledge that the research for this paper was done at the Geodetic Institute of the University of Stuttgart where the first author was stationed as a Fellow and the second author as a Distinguished Senior Scientist of the Alexander von Humboldt Foundation. Discussions with Prof. E.W. Grafarend and his staff had been of a significant value in this research. The authors are grateful also to Prof. A.Kleusberg for his constructive criticism of the manuscript. Comments to the submitted version of this manuscript by Prof. B. Heck and two anonymous reviewers have helped to clarify some points and improve the presentation.

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manuscripta geodaetica (1994) 19:220–230

manuscripta geodaetica

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Inversion of dynamic geodetic data with finite element method

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Received 16 March 1993; Accepted 2 December 1993

Abstract. On the basis of the inversion theory of dynamic geodetic data six kinds of inverse problems of Solid Mechanics are proposed. Numerical solutions for the nonlinear inverse problems are discussed and a parameter optimization method for inversion of geodetic data is presented. The feasibility of the method is demonstrated by numerical tests. Finally, by using the proposed method and geodetic data, a general method for the determination of regional boundary forces and rock medium parameters by inversion of geodetic data is presented, and the nonuniform rupture pattern of the Tangshan earthquake fault is analyzed. It is shown that, the parameter optimization method for inversion of geodetic data will improve the present methods used in this field.

1. Introduction

Since 1970's the research for inversion of dynamic geodetic data has been greatly developed and many achievements have been made (Matsu'ura 1977a; 1977b; Matsu'ura et al. 1986; Segall and Harris 1986; 1987; Ward and Barrientos 1986; Okada 1985). These results are very useful for studies of aseismic, preseismic, coseismic and postseismic fault movements. However, most of studies are based on the dislocation theory. Because there are some unreasonable assumptions in this theory, e.g., assuming that the dislocation components on each point of the fault are equal, called usually the uniform dislocation assumption, the research of the seismic nonuniform rupture can not be strictly proceeded with the dislocation theory. Although some authors obtained nonuniform slip pattern on the earthquake fault by inverting geodetic data (Ward and Barrientos 1986; Barrientos 1988; Yabuki and Matsu'ura 1992), these results are still based on the elastic dislocation theory, and further research in this field will be confined by the elastic dislocation theory. Therefore, it is necessary to develop a general method for inversion of dynamic geodetic data based on mechanical models.

Theory for inversion of dynamic geodetic data has been developed by Zhao (1991; 1992). In this paper, six kinds of the inverse problems of solid mechanics are proposed, and their solutions and properties are investigated. With the finite element method numerical solutions of the inverse problems are investigated and a parameter optimization method for inversion of geodetic data is presented. To demonstrate applications of the method, some examples in geodynamics and seismology are discussed.

The proposed method in this paper can be used to estimate regional boundary forces and part rock medium