CORRECTION TO HELMERT'S ORTHOMETRIC HEIGHT DUE TO ACTUAL LATERAL VARIATION OF TOPOGRAPHICAL DENSITY

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ABSTRACT

Helmert (1890) used Poincaré-Prey's gravity gradient for the definition of the orthometric height. According to this approach the gravity value needed for the evaluation of the height is obtained from the observed gravity at the earth surface reduced to the mid-point between the earth surface and the geoid, considering that the gravity gradient is constant along the plumbline. Moreover, the mean topographical density $\rho_o = 2.67 \text{ g.cm}^{-3}$ is assumed to approximate the actual distribution of topographical density. The correction to Helmert's orthometric height due to the lateral variation of topographical density has been introduced by Vaníček et al. (1995). In this paper, some numerical aspects of this correction are investigated.

Keywords: Gravity gradient, orthometric height, topographical density.

1. HELMERT'S ORTHOMETRIC HEIGHT

The fundamental formula for a definition of the orthometric height $H^{\mathrm{O}}(\Omega)$ reads (e.g., Heiskanen and Moritz, 1967, eqn. 4-21)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}: \qquad H^{\mathrm{O}}(\Omega) = \frac{C[r_t(\Omega)]}{\bar{g}(\Omega)}, \tag{1}$$

where $C[r_t(\Omega)]$ is the geopotential number, and $\bar{g}(\Omega)$ is the mean value of the actual gravity along the plumbline between the physical surface of the earth and the geoid surface.

The geocentric position is given by the geocentric spherical coordinates ϕ and λ; $\Omega = (\phi, \lambda)$, and the geocentric radius r; $r \in \Re^{+}$ $(\Re^{+} \in \langle 0, +\infty))$. In eqn. (1), $r_t(\Omega)$ further denotes the geocentric radius of the earth surface, and Ω_{O} stands for the total solid angle $[\phi \in \langle -\pi/2, \pi/2\rangle,\ \lambda \in \langle 0, 2\pi\rangle]$.

The geopotential number $C[r_t(\Omega)]$ is given by the difference of the actual gravity potential W_{o} of the geoid and the actual gravity potential $W[r_t(\Omega)]$ referred to the physical surface of the earth, so that

$$\forall\ \Omega \in \Omega_{\mathrm{O}}: \qquad C[r_t(\Omega)] = \mathrm{W}_{\mathrm{o}} - W[r_t(\Omega)]. \tag{2}$$

Helmert (1890) defined the approximate value of the mean gravity $\bar{g}(\Omega)$ along the plumbline by using Poincaré-Prey's gravity gradient. It reads

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\bar{g}(\Omega) \cong g[r_t(\Omega)] - \frac{1}{2}\frac{\partial g(r,\Omega)}{\partial \mathrm{t}}\bigg|_{r=r_t(\Omega)} H^{\mathrm{O}}(\Omega)$$

$$\approx g[r_t(\Omega)] - \frac{1}{2}\left[\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\bigg|_{r=r_t(\Omega)} + 4\pi \mathrm{G} \rho_{\mathrm{o}}\right] H^{\mathrm{O}}(\Omega), \tag{3}$$

where $\partial g(r,\Omega)/\partial \mathrm{t}$ represents the actual gravity gradient, and $\partial \gamma(r,\phi)/\partial \mathrm{n}$ is the normal gravity gradient.

According to this theory the gravity gradient is considered to be constant along the plumbline within the topography. Thus, the mean value of the gravity $\bar{g}(\Omega)$ is evaluated directly for the mid-point of the plumbline $H^{\mathrm{O}}(\Omega)/2$.

From the Poisson equation (Heiskanen and Moritz, 1967, eqn. 1-14)

$$\forall\ x, y, z \in \Re\ (\Re \in (-\infty, +\infty)):$$

$$\Delta W(x,y,z) = \frac{\partial^2 W(x,y,z)}{\partial x^2} + \frac{\partial^2 W(x,y,z)}{\partial y^2} + \frac{\partial^2 W(x,y,z)}{\partial z^2}$$

$$= -4\pi \mathrm{G} \rho_{\mathrm{o}} + 2\omega^2, \tag{4}$$

and from the expression for the mean curvature of the equipotential surface $J(r,\Omega)$:

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$

$$J(r,\Omega) = -\frac{1}{2g(r,\Omega)}\left[\frac{\partial^2 W(x,y,z)}{\partial x^2} + \frac{\partial^2 W(x,y,z)}{\partial y^2}\right], \tag{5}$$

the Bruns formula for the actual gravity gradient can be found (Heiskanen and Moritz, 1967)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$

$$\frac{\partial g(r,\Omega)}{\partial \mathrm{t}}\bigg|_{r} \cong -2g(r,\Omega)\, J(r,\Omega) + 4\pi \mathrm{G} \rho_{\mathrm{o}} - 2\omega^2. \tag{6}$$

In the above equations, ω denotes the mean value of the angular velocity of the earth spin, G is Newton's gravitational constant, and $\partial^2 W(x,y,z)/\partial x^2$, $\partial^2 W(x,y,z)/\partial y^2$ and $\partial^2 W(x,y,z)/\partial z^2$ are the second partial derivatives of the gravity potential in the local astronomical coordinate system $\mathrm{x}, \mathrm{y}, \mathrm{z}$, where the z-axis coincides with the outer normal of the local equipotential surface.

The normal gravity gradient $\partial \gamma(r,\phi)/\partial \mathrm{n}$ is defined by

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$

$$\frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\bigg|_{r} = -2\gamma(r,\phi)\, J_o(\phi) - 2\omega^2. \tag{7}$$

Furthermore, the mean curvature of the ellipsoid surface $J_o(\phi)$ is given by (e.g., Bomford, 1971)

$$\forall\ \phi \in \langle -\pi/2, \pi/2\rangle: \quad J_o(\phi) = \frac{1}{2}\left(\frac{1}{M(\phi)} + \frac{1}{N(\phi)}\right), \tag{8}$$

where $M(\phi)$ and $N(\phi)$ are the principal radii of curvature of the ellipsoid in North-South and East-West directions.

Under the following assumption

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$

$$g(r,\Omega) J(r,\Omega) \cong \gamma(r,\phi) J_o(\phi), \tag{9}$$

Poincaré-Prey's gravity gradient can finally be found in the form (e.g., Vaníček and Krakiwsky, 1986)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$

$$\frac{\partial g(r,\Omega)}{\partial \mathrm{t}}\bigg|_{r} \cong \frac{\partial \gamma(r,\phi)}{\partial \mathrm{n}}\bigg|_{r} + 4\pi \mathrm{G} \rho_{\mathrm{o}}$$

$$= -2\gamma(r,\phi)\, J_o(\phi) - 2\omega^2 + 4\pi \mathrm{G} \rho_{\mathrm{o}}. \tag{10}$$

2. EFFECT OF LATERAL VARIATION OF TOPOGRAPHICAL DENSITY ON HELMERT'S ORTHOMETRIC HEIGHT

The correction $\delta H^{\mathrm{O}}[\Omega : \rho(\Omega)]$ to Helmert's orthometric height due to the laterally varying topographical density $\rho(\Omega)$ is given by the following approximate expression (Vaníček et al., 1995)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\delta H^{\mathrm{O}}[\Omega : \rho(\Omega)] \approx 2\pi \mathrm{G}\, [H^{\mathrm{O}}(\Omega)]^2\, \frac{\delta\rho(\Omega)}{\gamma_o(\phi)}, \tag{11}$$

where $\delta\rho(\Omega) = \rho(\Omega) - \rho_{\mathrm{o}}$ is the anomalous laterally varying topographical density, and $\gamma_o(\phi)$ is the normal gravity referred to the ellipsoid surface.

According to Martinec (1993), the laterally varying topographical density $\rho(\Omega)$ is given by

$\forall\ \Omega \in \Omega_o$:

$$\rho(\Omega) = \frac{1}{H^{O}(\Omega)} \int_{r=r_g(\Omega)}^{r_g(\Omega)+H^{O}(\Omega)} \rho(r,\Omega)\, r^2\ \mathrm{d}r\,, \qquad (12)$$

where $r_g(\Omega)$ denotes the geocentric radius of the geoid surface.

3. NUMERICAL INVESTIGATION

In most of the topography the actual lateral density varies from $1.0\,\mathrm{g.cm^{-3}}$ (water) to $2.98\,\mathrm{g.cm^{-3}}$ (gabbro). Thereby, disregarding existing water bodies, the variation of topographical density $\delta\rho(\Omega)$ is mostly within $\pm 0.3\,\mathrm{g.cm^{-3}}$ around the mean value ρ_o. However, larger topographical density variations up to 20-30% are encountered in some local and regional geological structures. Regarding eqn. (11), it can be estimated that the variation of topographical density can cause centimeters and decimeters errors in orthometric height. The relation between the change of orthometric height $\delta H^{O}[\Omega : \rho(\Omega)]$ and the anomalous lateral topographical density $\delta\rho(\Omega)$ is shown in Fig. 1. Since the correction to the orthometric height due to the lateral variation of topographical density increasing exponentially with the height, this correction can approximately reach up to ± 0.5 m for the height 6000 m, while for the heights up to 2000 m is less than ± 6 cm .

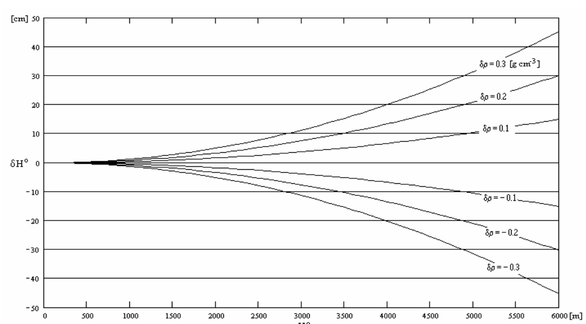

Figure. 1- Change of orthometric height $H^{O}(\Omega)$ due to the variation of lateral topographical density $\delta\rho(\Omega)$ [cm]

The lateral topographical density variation at the territory of Canada is shown in Fig. 2. The density model has been prepared by the Natural Resources of Canada. The correction $\delta H^{O}[\Omega : \rho(\Omega)]$ to Helmert's orthometric height due to the lateral variation of topographical density $\delta\rho(\Omega)$ is computed on GPS/leveling points over the territory of Canada. The result is shown in Fig. 3. As can be seen from the graphical interpretation the correction $\delta H^{O}[\Omega : \rho(\Omega)]$ ranges between –1.9 cm and +3.4 cm. Since the sufficient information about the accuracy of the density model is not available, the error estimation of the numerical result is not employed.

5. CONCLUSIONS

Based on the theoretical analysis in the previous paragraph (Fig. 1) it has been estimated that the effect of the lateral topographical density variation on the orthometric height can reach up to a several decimeters. As further follows from the result of the numerical investigation on GPS/leveling points this effect represents a change in orthometric height of only a few centimeters (see Fig. 3). It is because the leveling benchmarks are situated in regions of which the elevations are usually less than 2000 meters.

Helmert's orthometric heights are usually used for a definition of the geodetic vertical datum. When the proper density model is available, the accuracy of Helmert's orthometric heights can be improved. The correction to Helmert's orthometric height due to the lateral variation of topographical density can then be applied, especially when the high accuracy is required such as the determination of the orthometric heights on the leveling points.

6. REFERENCES

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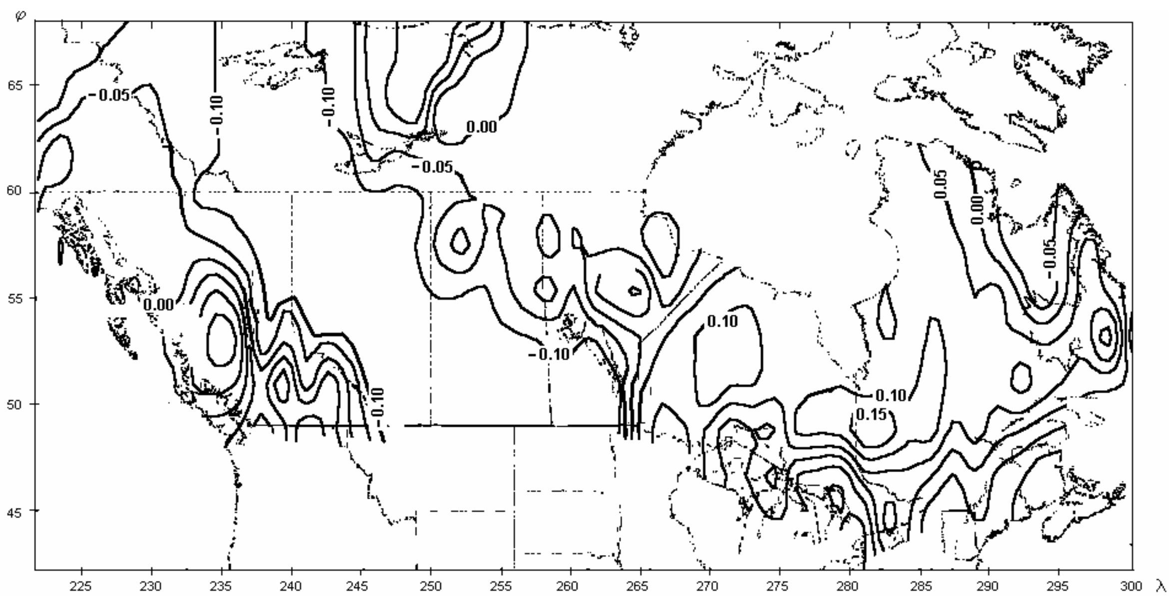

Figure. 2- Lateral variation of topographical density $\delta\rho(\Omega)$ at the territory of Canada [$g.cm^{-3}$]

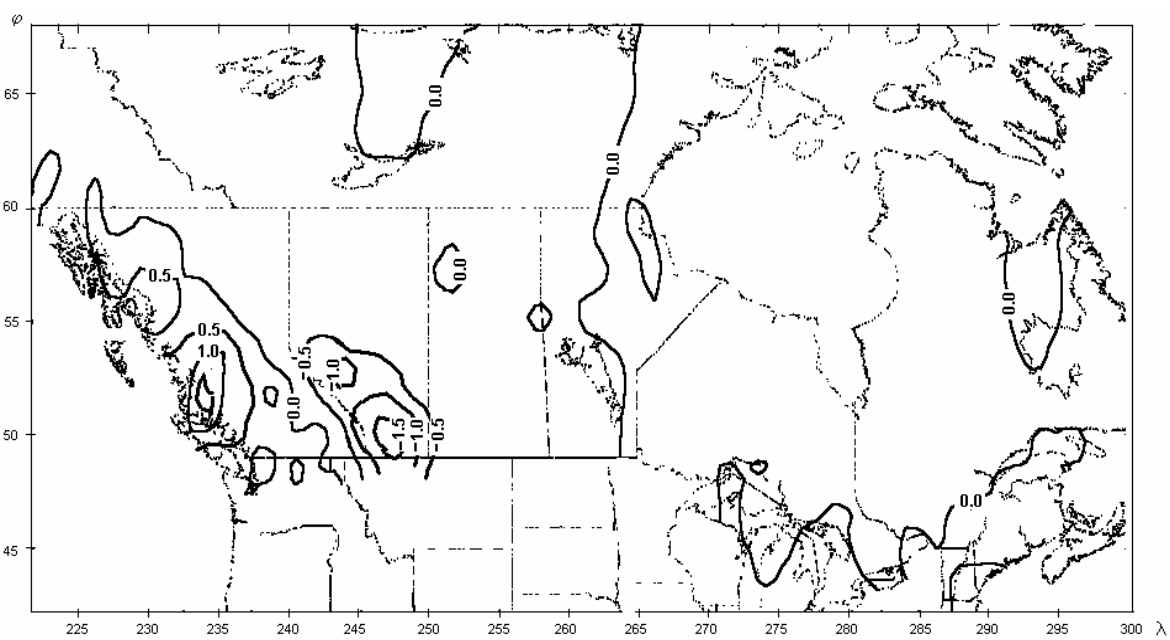

Figure. 3- Correction to Helmert's orthometric height $\delta H^{O}[\Omega : \rho(\Omega)]$ due to the actual lateral variation of topographical density [cm]