

Geoid undulations, interpretations

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Synonyms

- Geoidal height, a.k.a. geoid undulation;
- Moho surface, a.k.a. compensation level for the Airy-Heiskanen isostatic hypothesis;
- Depth of Moho surface is the same as the thickness of the lithosphere;
- Schiaparelli's body, a.k.a. body of nil external potential;
- Stokes's integral truncated to the spherical distance of ψ_0 , a.k.a. inner zone contribution to Stokes's integral;
- spectral transfer function, a.k.a. admittance;
- sea surface topography (SST), a.k.a. dynamic topography;

Definitions

- Direct gravimetric problem is defined as follows: "Given the distribution of density within the Earth, derive the (external) gravity field the density generates".
- The inverse gravimetric problem (GIP) reads: "Given the external gravity field of the Earth, derive the distribution of density implied by the field."

Introduction

The main goal of geophysics is to learn about the physical processes as well as properties of the material within the Earth. These are studied by means of measurements (data) collected on the surface of the Earth and above it and, to a very limited degree, also in very shallow probes inside the Earth. The discussion of the data to be collected and the techniques to be used to study the physical processes and material properties within the earth are what this book is all about.

The focus of this section is on gravity field, the geoid in particular, to the exclusion of all other, such as seismological, magnetic, electrical, heat flow data. The only attribute of the Earth that can be studied by means of gravity, are the mass density variations as gravity is the acceleration caused by mass

attraction. Three aspects of gravity field can be used for these studies: gravity acceleration, the deflection of the vertical and gravity potential. Gravity acceleration, in the guise of gravity anomalies, are discussed in section #15, the deflections of the vertical are introduced in section #7 and for an in depth view of the definition of geoid the reader is advised to consult section #9.

One of the general properties of the gravity field is that deep-seeded density anomalies have long wavelength signature in the gravity field while shallow density anomalies have short wavelength signature, cf., section #25. Consequently, if one wishes to study shallow density features, one looks at the short wavelength part of gravity field and if one wishes to study deep-seeded features, one looks at the longer wavelength part. This is, of course true only as a general principle as there exist also shallow features of large lateral extent that have long wavelength signatures.

When it comes to the interpretation of gravity field, the data of choice by exploration geophysicists, i.e., people looking for ore, hydrocarbon and other deposits, are gravity anomalies. The reason for this is that these data are bountiful on land, relatively cheap and they contain all the short wavelength information. Observed deflections of the vertical as supplied by national geodetic offices are less popular as a source of information as they are not very accurate, and they are few and far apart. But we may live to see this situation change with the advent of Digital Zenith Cameras (Hirt et al., 2010).

Gravity potential data, such as the values of disturbing potential - divided by the value of gravity to convert them to the geoid - are much smoother than the first two kinds. This means that the short wavelength content is missing. We know, cf., section #19, that all three descriptors of gravity field we discuss here are functionally related: gravity acceleration at a point is a total gradient of the potential at that point, the deflection of the vertical at a point is a horizontal gradient of the equipotential surface passing through that point. Thus, there is nothing to stop us from using the potential data, the geoid, in such a way as to evaluate either the gravity anomalies or the deflections of the vertical and use these for the interpretation, is there?

That can be done of course, but this procedure cannot reconstitute the high frequency content as it has not been there to begin with. Thus the conversion would not help and it is better to use the geoid as is; this is the

thesis of the book (Vaníček and Christou, 1993). The direct use of the geoid has caught an interest of interpreters as the marine geoid data from satellite altimetry, see section #9, became quite reliable and freely available in the past 25 years.

Direct problem

Suppose we know the density distribution $\rho(\mathbf{r})$ within the Earth. Then we can determine the potential $V(\mathbf{r})$ outside the earth in two different ways. We can use the well known Newton's integral to get the gravitational potential in a spatial form at any point external to the Earth as (Vaníček and Krakiwsky, 1986, Eq.(20.48))

$$V(r, \Omega) = G \int_{\Omega_0} \int_{r=0}^{r_t(\Omega)} \frac{\rho(r', \Omega')}{|\mathbf{r} - \mathbf{r}'|} (r')^2 dr' d\Omega', \quad (1)$$

where G is Newton's gravitational constant, r is the radial distance from the centre of the Earth, Ω is the geocentric direction composed of co-latitude θ and longitude λ , Ω_0 is the full geocentric solid angle, r_t is the distance from the centre of the Earth to the topographical surface of the Earth.

Alternatively, the reciprocal distance $1/|\mathbf{r} - \mathbf{r}'|$ can be developed into a series of fully normalised spherical harmonic complex functions Y_{jm} to yield

$$V(r, \Omega) = \frac{GM}{a} \sum_{j=0}^{\infty} \left(\frac{a}{r}\right)^{j+1} \sum_{m=-j}^j A_{jm} Y_{jm}(\Omega), \quad (2)$$

where M is the mass of the Earth, a is the major semi-axis of the best fitting Earth ellipsoid and A_{jm} are the gravity potential coefficients. The potential coefficients can be evaluated from the known density $\rho(\mathbf{r}) = \rho(r, \Omega) = \rho(r, \theta, \lambda)$ within the Earth as

$$A_{jm} = \int_{\Omega_0} \int_{r=0}^{r_t(\Omega)} \rho(r, \Omega) G_{jm}(r, \Omega) dr d\Omega, \quad (3)$$

where the integration kernel G_{jm} is given by Martinec in (Vaníček and Christou, 1993, Ch. 7, Eqs.(13) and (14))

$$G_{jm}(r, \Omega) = \frac{4\pi}{Ma^j(2j+1)} r^{j+2} Y_{jm}^*(\Omega). \quad (4)$$

Here * denotes a complex conjugate. This is the spherical harmonic form of potential V , a dual form to the spatial form (Eq.1 above) – these two forms can be transformed into each other, back and forth any time. The relation between potential V and the geoid undulation N is explained in section # 7.

The direct approach can be used for solving some geophysical problems. Notably, it can be used to determine the depth of Mohorovicic's surface (cf., section #1, the surface that divides the asthenosphere (density about 3.2 g cm^{-3}) from the lithosphere (density about 2.67 g cm^{-3}). The Moho surface is also the compensation level for the Airy-Heiskanen isostatic hypothesis. The depth of the Moho surface, or the lithospheric (crustal) thickness derived this way, assuming the most realistic asthenospheric density and described by a spherical harmonic series of a limited degree and order is shown on a web page <http://geology.com/articles/mohorovicic-discontinuity.shtml>. The displayed map was compiled by US Geological Survey in 1999.

Inverse problem

The determination of density $\rho(\mathbf{r})$ as a function of position \mathbf{r} from any of the gravity field attributes, i.e., gravity anomaly $\Delta g(\mathbf{r})$, the components of the deflection of the vertical ($\xi(\mathbf{r}), \eta(\mathbf{r})$), or the geoidal height $N(\mathbf{r})$, is known as the inverse gravity problem (GIP). The non-uniqueness of GIP was known already to Schiaparelli, who in (1875-76) was speaking of “gravitationally transparent bodies”, specific distributions of density that do not generate any gravitational attraction outside the Earth. A complete mathematical description of a Schiaparelli's body was formulated by Vaníček and Kleusberg (1985) as follows: for arbitrary values of coefficients $c_{\ell j k}^{(i)}$ the following “body”:

$$\theta \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle, \lambda \in \langle 0, 2\pi \rangle, r \in \langle 0, 1 \rangle: \tilde{\rho}(\mathbf{r}) = \sum_{i=0}^{\infty} \sum_{\substack{\ell=0 \\ \ell \neq j}}^i \sum_{j=0}^{\infty} \sum_{k=0}^j c_{\ell j k}^{(i)} \phi_{ijk}(\mathbf{r}), \quad (5)$$

where

$$\forall i, j = 0, 1, \dots, k \leq j: \phi_{ijk}(\mathbf{r}) = J_i(3, 3, r)[Y_{jk}^c(\Omega), Y_{jk}^s(\Omega)], \quad (6)$$

J_i are normalized Jacobi polynomials (Abramowitz and Stegun, 1964) and Y_{jk}^c, Y_{jk}^s are the cosine and sine parts of spherical harmonic functions, has a nil external potential outside the ball of radius 1. The consequence of this behaviour is that if one tries to interpret the external gravity potential in terms of internal density distribution, any distribution that conforms to Eq.(5) can be randomly and legitimately added to the solution.

If the GIP does not have a unique solution, does it make any sense to even think about using gravity in studying the density distribution? True, it is not possible to take the gravity field outside the Earth and invert it uniquely into the density distribution within the Earth. But the external gravity field, particularly the geoid in our context here, can be used together with other data to get useful information on density.

Truncated geoid

There is a variety of techniques used in geophysical exploration to interpret gravity information obtained on the surface of the Earth in terms of anomalous density. One interesting approach uses the truncated geoid to determine the depth of a spherical density anomaly, where by “truncated geoid” we understand the following integral

$$N^{\psi_0}(P) = \frac{R}{4\pi\gamma} \int_{C_{\psi_0}} \Delta g(Q) S(P, Q) d\sigma, \quad (7)$$

that represents the contribution of gravity to the geoid within the spherical cap C of a radius ψ_0 . It can be viewed as the Stokes integral over the whole Earth, truncated to the spherical distance ψ_0 , where P is the point of interest, Q is the dummy point in the integration, $S(P, Q) = S(\psi)$ is the Stokes spherical kernel, cf., section #9, and $d\sigma$ is the surface element of the integration sphere of radius R .

It is the change of the truncated geoid with respect to the truncation parameter ψ_0 ,

$$\frac{dN^{\psi_0}}{d\psi_0} = \lim_{\Delta\psi_0 \rightarrow 0} \frac{N^{\psi_0 + \Delta\psi_0} - N^{\psi_0}}{\Delta\psi_0} \quad (8)$$

that is of real interest. At the location of the density anomaly, the sequence of surfaces or profiles $dN^{\psi_0}/d\psi_0$ displays a dimple for a specific value of ψ_0 – see Fig.1 (from Vajda and Vaníček, 1999). This specific value of ψ_0 is

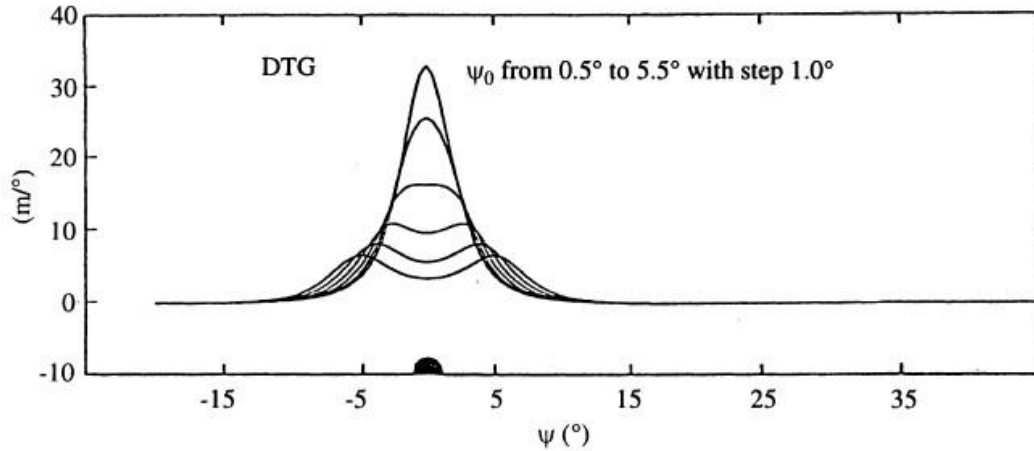


Figure 1. Sequence of profiles of the change of the truncated geoid with respect to the truncation parameter ψ_0 that shows the onset of the dimple for a value of $\psi_0 = 2.5^\circ$.

almost linearly related to the depth of the anomaly.

In Fig. 2, geoid from an area in Southern Slovakia is shown. Just north-east of the geographical center of the area, there is a local geoid low, known as “Kolarovo anomaly”, that has been intriguing geophysicists for some time. It has been postulated by several authors that the centre of mass of the density anomaly responsible for the shown pattern is about 9.5 km below the Earth surface. The application of the truncated geoid technique shows an estimate of 8.7 km (Vajda et al., 2002).

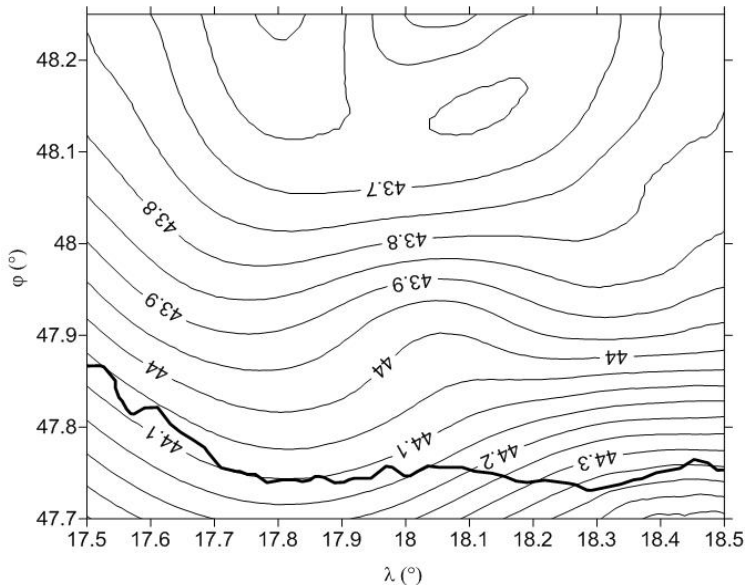


Figure 2. Geoid in the area of "Kolarovo anomaly" in Southern Slovakia (courtesy of J. Janak, Slovak University of Technology)

Some other applications

In addition to the above discussed applications, geoid can be used in many other applications; one of those applications is studying tectonic forces. Interested reader may read about the involved concepts in section #10. In another application, studies by Bowin (1991) show that the 4 to 10 degree geoid undulations are consistent with the present zones of subducted tectonic slabs. Bowin also speculated (1983) that the deepest Earth's mass anomalies cause geoid undulations only of degrees 2 to 10.

Casenave in (Vaníček and Christou, 1993, Ch. 13) discusses the use of the geoid in studying different isostatic compensation models using the spectral transfer function, a.k.a. admittance in the "theory of systems" parlance. She studies the mechanical behaviour of oceanic tectonic plates (flexure under seamount loading) and thermal evolution of oceanic lithosphere. She also argues that the signatures of oceanic hotspots in the geoid are in the medium wavelength of $n \in (5, 20)$ and in 1 to 10 metres in range.

It may be worth mentioning that meteorite craters are often clearly discernable in detailed geoid maps. The Manicouagan crater in Eastern Canada (location: N 51° 23' and W 68° 42') makes a very clear imprint on the geoidal map.

The last application of the geoid we want to show here concerns physical oceanography. A very good overview of this topic by Nerem and Koblinsky is in (Vaníček and Christou, 1993, Ch. 16). Geoid, being an equipotential (level) surface, should be represented by the mean sea level if sea water were homogeneous, which it is not. As a result, the mean sea level is not in equilibrium and currents develop. Some time after the initiation of the motion, these currents reach a steady state, called geostrophic, and swirl around the globe in a perpetual motion attenuated only by friction.

The equation that describes the geostrophic motion is ascribed to Navier and Stokes, and it reads as follows

$$\dot{\mathbf{V}} + (\mathbf{V} \cdot \nabla) \mathbf{V} + 2\boldsymbol{\Omega} \times \mathbf{V} = \frac{-\nabla p}{\rho} + \mathbf{g} + \mathbf{F}, \quad (9)$$

where \mathbf{V} is the flow velocity vector, $\boldsymbol{\Omega}$ is the rotation vector of the Earth, p is the hydrostatic pressure, ρ is the sea-water density, \mathbf{g} is the gravitation vector and \mathbf{F} is the vector representing frictional forces (*ibid.*Eq.(1)). This equation, together with measurements taken at sea, is then used to derive the height of the sea surface above the geoid, which is called the “dynamic topography”.

As discussed above, the instantaneous sea surface has been monitored by a technique called satellite altimetry (cf., section #9) for years. This surface reflects, of course, the sum of the geoid and the dynamic topography. Subtracting the geoid from the measured sea surface yields the dynamic topography that can be easily converted into information about currents. Conversely, subtracting the dynamic topography from the measured surface, yield the geoid. Interested reader can refer to following URL <http://www.aviso.oceanobs.com/en/data/products/auxiliary-products/index.html> for more detailed information.

Summary

The geoidal height (geoid undulation) is a quantity that can be quite readily interpreted in terms of density distribution within the Earth. The main application is in studying the processes that take place deep within the Earth, but even shallower density anomalies have an impact on a geoid map.

Nowadays, when sea surface height measured by satellite altimetry is freely available, marine geoid has become particularly handy tool to use.

Bibliography

Abramowitz, M. and I.A.Stegun, 1964. *Handbook of Mathematical Functions*, Dover reprint.

Bowin, C., 1983. Depth of principal mass anomalies contributing to the Earth's geoidal undulations and gravity anomalies, *Marine Geod.*, 7, 61.

Bowin, C., 1991. Speculations on a Mechanism for Maintenance of Absolute Plate Motion, AGU abstract, *EOS, Trans. A.G.U.* 72 (44), 244.

Hirt, C.,B. Burki, A. Somieski and G. Seeber, 2010. Modern Determination of Vertical Deflections Using Digital Zenith Cameras, *J. of Surv. Engg.* 136 (1), pp. 1-12.

Schiaparelli, G.V., 1875-76. *Lezioni di Meccanica Celeste (Appunti)*, University of Pavia, Italy.

Vajda, P. and P. Vaníček, 1999. Truncated geoid and gravity inversion for one point-mass anomaly, *J. of Geodesy*, 73, pp. 58-66.

Vajda, P., M. Bielik and V. Pohanka, 2002. Testing the application of the Truncation Filtering Methodology in interpreting real gravity data: the Kolárovo gravity anomaly, *Contributions to Geophysics and Geodesy*, 32/1, pp. 57-66

Vaníček, P. and A. Kleusberg, 1985. What an external gravitational potential can really tell us about mass distribution. *Bollettino di Geofisica Teorica ed Applicata*, Vol. XXCII, No. 108, December, pp. 243-250.

Vaníček, P. and E.J. Krakiwsky, 1986. *Geodesy: the concepts*. Second edition, North Holland, Amsterdam.

Vaníček, P. and N.T. Christou, 1993. *Geoid and its geophysical interpretations*. CRC Press, Boca Raton, FL, USA.

Cross-references

In this section the following other sections are referred to: 1, 7, 9, 10, 15, 19, 25.