Combining surface deformation parameters referred to different terrestrial coordinate systems

M. Berber*[1], H. S. Kutoglu[2], P. Dare[3] and P. Vaníček[3]

Transformation of deformation parameters from one coordinate system to another is investigated. For a robust deformation analysis, in addition to transformation parameters, their covariance matrix ought to be transformed to the desired coordinate system. This provides a check on the transformed parameters. For this purpose, a simulated network is employed. Results show that the outlined approach works well.

Keywords: Geodetic networks, Coordinate systems, Deformation, Strain

Introduction

The principal aim of establishing geodetic networks is to set up terrestrial coordinate systems. In addition, geodetic networks are used widely to monitor surface deformations on the surface of the Earth. To monitor surface deformations, a geodetic network is surveyed to gain multiple epochs of data. By comparing the coordinates which are acquired at different epochs, the deformations (displacements) at network points are obtained. As such the deformation parameters in the area where the geodetic network lies can be calculated. The outcome of this process yields deformation parameters in the coordinate system in which the geodetic network is defined.

In most cases coordinate systems, which geodetic networks are referred to, are dictated by the surveying techniques used for the network observations. Nowadays the most popular surveying technique uses the global positioning system (GPS). By using GPS, the coordinates on the surface of the Earth are determined in the satellite coordinate system. The satellite coordinates, on the other hand, are defined in the International Terrestrial Reference Frame (ITRF). As a consequence, if GPS is utilised, the coordinates of the network points are determined in ITRF. The satellite coordinates can be obtained from two different sources: broadcast ephemerides or precise ephemerides. The broadcast ephemerides (broadcast via the satellites and acquired at the observation epoch) are tabulated on the ITRF96 solution. The precise ephemerides (which are downloadable from IGS website) are always referred to the most up to date ITRF solution, which is currently ITRF05. As a result, deformation parameters which are computed on a network that has been surveyed using GPS for a long period of time might refer to different ITRF solutions.

Before GNSS positioning was introduced to geodesy, geodetic networks used to be surveyed utilising terrestrial measurement techniques only – terrestrial measurement techniques may still be used today. At that time, for deformation monitoring networks, the widespread approach was the constitution of a local topocentric coordinate system that may be defined with respect to one of the points in the network. Hence the deformation parameters which are acquired by the solution of consecutive periods were referring to the stated local topocentric systems. Lately, these types of networks have been surveyed using GPS as well. Thus current solutions are given in ITRF rather than in a local topocentric system.

In order to be able to describe the deformation characteristics of the land on which the geodetic network lies and to do the following risk analysis, the deformation parameters ought to be analysed both temporally and spatially over a long period of time. To be able to perform this analysis the deformation parameters must be determined in a common coordinate system. On the other hand, as it is indicated above, the coordinate systems in which geodetic networks are defined may differ in time with regard to the technology utilised. If this is the case, when these networks are surveyed again, sometimes it is difficult to get historic detailed information (i.e. approximate coordinates, observations and their weights, datum points, etc.) about these networks. Most of the time only the deformation parameters, which are the final products of the deformation monitoring networks, might be retrieved. Under these circumstances, for long period temporal and spatial analysis of deformation parameters, the practical and perhaps the most logical approach would be a coordinate transformation of deformation parameters from one system to another. Knowing the precise transformation parameters among different ITRF solutions and having the geographic

[1]Geomatics Engineering, College of Engineering and Computer Science, Florida Atlantic University, Port St Lucie, FL 34986, USA
[2]Geodesy & Photogrammetry Department, Faculty of Engineering, Zonguldak Karaelmas University, Zonguldak 67100, Turkey
[3]Department of Geodesy and Geomatics Engineering, University of New Brunswick, PO box 4400, Fredericton, NB E3B 5A3, Canada

*Corresponding author, email mberber@fau.edu

Received 5 October 2011; accepted 13 October 2011
DOI 10.1179/1752270611Y.0000000005

coordinates of the origin of the local topocentric system make transformations straightforward. Therefore, in this study, the transformation relations of the deformation parameters which are determined in different coordinate systems are investigated.

Relationship between terrestrial coordinate systems

From the past to the present, different coordinate systems have been defined and used along with advancing surveying technologies for different applications. Nowadays, The International Terrestrial Reference System (ITRS) is recommended not only for general use but also for monitoring purposes since today's most popular surveying technique, GPS, refers to it.

ITRS is the most up to date Conventional Terrestrial (CT) Coordinate System which is the ideal coordinate system for positioning purposes on the Earth. Its difference from the former description of CT systems is that ITRS is an Earth-crust fixed CT system and realised by coordinates and velocities of global tracking stations in which tidal effects are removed. The realisation of ITRS is called International Terrestrial Reference Frame (ITRF) [**5,2,4**]. Owing to technological advancements (i.e. computational techniques, reevaluated geodynamical models, improvements in data quality and an increasing number of the tracking stations) new solutions for ITRF are produced by The International Earth Rotation and Reference Systems Service.

In theory, coordinate transformation between CT systems is accomplished by six transformation parameters (three translations t_x, t_y, t_z and three rotations ε_x, ε_y, ε_z) as follows

$$\mathbf{r}_{CT,2}=\mathbf{t}+\mathbf{R}\,\mathbf{r}_{CT,1} \tag{1}$$

where $\mathbf{r}$ is the position vector, $\mathbf{t}=\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$ and $\mathbf{R}=\begin{bmatrix} 1 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{bmatrix}$. It should be noted that ε_x, ε_y, ε_z are small rotations angles.

Note that the equation includes no scale parameter because the scale difference is induced by the measurements; yet, coordinate systems are defined regardless of existence of measurements [**7**]. In this sense, reference frames such as ITRF cannot exist without measurements; thus, a coordinate transformation between reference frames needs the scale parameter (k) included in the rotation matrix as follows

$$\mathbf{R}=\begin{bmatrix} 1+k & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 1+k & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1+k \end{bmatrix} \tag{2}$$

Before the advent of satellite techniques, it was difficult to tie geodetic networks to a CT system using terrestrial (classical) surveying techniques. Hence, two different kinds of terrestrial coordinate systems were preferred for surveying purposes: (i) local geodetic (ellipsoidal) coordinate systems, defined by an ellipsoidal earth model positioned with respect to the CT system, and, (ii) for routine surveying practices, local astronomical coordinate systems, positioned and oriented with respect to an arbitrary point of geodetic network, mostly for deformation monitoring. Coordinates in the first group of coordinate systems are also transformed to the CT system or another ellipsoidal coordinate system via equation (1). In the case of local astronomical coordinate systems, the translation and rotation matrices in equation (1) are replaced by the following to transform a local astronomical system to the CT system

$$\mathbf{t}=\begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} \tag{3}$$

and

$$\mathbf{R}=\begin{bmatrix} \cos(180-\Lambda_0)\cos(90-\varphi_o) & \sin(180-\Lambda_0) & -\cos(180-\Lambda_0)\sin(90-\varphi_o) \\ -\sin(180-\Lambda_0)\cos(90-\varphi_o) & \cos(180-\Lambda_0) & \sin(180-\Lambda_0)\sin(90-\varphi_o) \\ \sin(90-\varphi_o) & 0 & \cos(90-\varphi_o) \end{bmatrix} \tag{4}$$

where (φ_o) and (Λ_0) are the astronomical latitude and longitude of the origin point of the local system, defined in the CT system, while X_o, Y_o, Z_o) are its Cartesian coordinates. The unit for the components of the rotation matrix is arc-degree.

Transformation of deformation (strain) parameters

Principally, geodetic networks have been established to realise terrestrial coordinate systems whose origins and three axes (x, y, z) are defined with respect to the Earth's centre of mass and axis of rotation. In addition, geodetic networks are used widely to monitor the Earth's surface deformations caused by landslides, crustal movements, etc. A deformation monitoring network is commonly composed of two groups of points: reference and object points. Reference points are monumented on stable ground and used to tie the network to a proper coordinate system. Whereas object points are for measuring deformation; therefore, they are established in the area where the deformation has occurred.

If the deformation network is surveyed at two different epochs (1 and 2) and if (x_i^1,y_i^1,z_i^1) and (x_i^2,y_i^2,z_i^2) are the coordinates of these two epochs, the displacement vector is obtained by

$$\mathbf{d}_i=\begin{bmatrix} x_i^2-x_i^1 \\ y_i^2-y_i^1 \\ z_i^2-z_i^1 \end{bmatrix}=\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \tag{5}$$

Then the strain matrix is

$$\mathbf{E}_i=\begin{bmatrix} \dfrac{\partial u_i}{\partial x} & \dfrac{\partial u_i}{\partial y} & \dfrac{\partial u_i}{\partial z} \\ \dfrac{\partial v_i}{\partial x} & \dfrac{\partial v_i}{\partial y} & \dfrac{\partial v_i}{\partial z} \\ \dfrac{\partial w_i}{\partial x} & \dfrac{\partial w_i}{\partial y} & \dfrac{\partial w_i}{\partial z} \end{bmatrix} \tag{6}$$

It is of interest to note that the following relation holds [**6**]

$$\mathbf{d}_i=\mathbf{E}_i\mathbf{r}_i+\mathbf{c}_0 \tag{7}$$

where $\mathbf{r}_i$ is the position vector and $\mathbf{c}_0$ is an arbitrary shift vector, constant for the network. The strain matrix can be decomposed into two parts as follows

$$\mathbf{E}=\frac{1}{2}(\mathbf{E}+\mathbf{E}^{\mathrm{T}})+\frac{1}{2}(\mathbf{E}-\mathbf{E}^{\mathrm{T}}) \tag{8}$$

$$\mathbf{E}=\mathbf{S}+\mathbf{A} \tag{9}$$

where the matrix **S** describes the expansion and contraction as well as the shearing deformation at a point and the matrix **A** describes the twisting deformation at a point – it is a rigid body motion. For each point **S** and **A** are calculated as follows

$$\mathbf{S}_{\mathrm{i}}=\begin{bmatrix}\frac{\partial u_{\mathrm{i}}}{\partial x} & \frac{1}{2}\left(\frac{\partial u_{\mathrm{i}}}{\partial y}+\frac{\partial v_{\mathrm{i}}}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_{\mathrm{i}}}{\partial z}+\frac{\partial w_{\mathrm{i}}}{\partial x}\right)\\ \frac{1}{2}\left(\frac{\partial v_{\mathrm{i}}}{\partial x}+\frac{\partial u_{\mathrm{i}}}{\partial y}\right) & \frac{\partial v_{\mathrm{i}}}{\partial y} & \frac{1}{2}\left(\frac{\partial v_{\mathrm{i}}}{\partial z}+\frac{\partial w_{\mathrm{i}}}{\partial y}\right)\\ \frac{1}{2}\left(\frac{\partial w_{\mathrm{i}}}{\partial x}+\frac{\partial u_{\mathrm{i}}}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w_{\mathrm{i}}}{\partial y}+\frac{\partial v_{\mathrm{i}}}{\partial z}\right) & \frac{\partial w_{\mathrm{i}}}{\partial z}\end{bmatrix} \tag{10}$$

and

$$\mathbf{A}_{\mathrm{i}}=\begin{bmatrix}0 & \frac{1}{2}\left(\frac{\partial u_{\mathrm{i}}}{\partial y}-\frac{\partial v_{\mathrm{i}}}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_{\mathrm{i}}}{\partial z}-\frac{\partial w_{\mathrm{i}}}{\partial x}\right)\\ \frac{1}{2}\left(\frac{\partial v_{\mathrm{i}}}{\partial x}-\frac{\partial u_{\mathrm{i}}}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_{\mathrm{i}}}{\partial z}-\frac{\partial w_{\mathrm{i}}}{\partial y}\right)\\ \frac{1}{2}\left(\frac{\partial w_{\mathrm{i}}}{\partial x}-\frac{\partial u_{\mathrm{i}}}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w_{\mathrm{i}}}{\partial y}-\frac{\partial v_{\mathrm{i}}}{\partial z}\right) & 0\end{bmatrix} \tag{11}$$

The above process yields deformation parameters in the coordinate system in which the geodetic network is defined since it is based on the coordinates of the network points.

Let the coordinates of the point i at the two epochs be denoted by $(x_{\mathrm{i}}^{1},y_{\mathrm{i}}^{1},z_{\mathrm{i}}^{1})$ and $(x_{\mathrm{i}}^{2},y_{\mathrm{i}}^{2},z_{\mathrm{i}}^{2})$ respectively. Let us now consider a second coordinate system in which the coordinates of the point i are expressed by $(x_{\mathrm{i}}^{1*},y_{\mathrm{i}}^{1*},z_{\mathrm{i}}^{1*})$ and $(x_{\mathrm{i}}^{2*},y_{\mathrm{i}}^{2*},z_{\mathrm{i}}^{2*})$. Based on equation (1), the relationship between the coordinates in each epoch is established as follows

$$\mathbf{r}_{\mathrm{i}}^{1}=\mathbf{t}+\mathbf{R}\mathbf{r}_{\mathrm{i}}^{1*} \tag{12}$$

$$\mathbf{r}_{\mathrm{i}}^{2}=\mathbf{t}+\mathbf{R}\mathbf{r}_{\mathrm{i}}^{2*} \tag{13}$$

where

$$\mathbf{r}_{\mathrm{i}}^{1}=\begin{bmatrix}x_{\mathrm{i}}^{1} & y_{\mathrm{i}}^{1} & z_{\mathrm{i}}^{1}\end{bmatrix}^{\mathrm{T}},\ \mathbf{r}_{\mathrm{i}}^{1*}=\begin{bmatrix}x_{\mathrm{i}}^{1*} & y_{\mathrm{i}}^{1*} & z_{\mathrm{i}}^{1*}\end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{r}_{\mathrm{i}}^{2}=\begin{bmatrix}x_{\mathrm{i}}^{2} & y_{\mathrm{i}}^{2} & z_{\mathrm{i}}^{2}\end{bmatrix}^{\mathrm{T}},\ \mathbf{r}_{\mathrm{i}}^{2*}=\begin{bmatrix}x_{\mathrm{i}}^{2*} & y_{\mathrm{i}}^{2*} & z_{\mathrm{i}}^{2*}\end{bmatrix}^{\mathrm{T}}.$$

Using equations (12) and (13), one can write the following for the displacement vector between the epoch 1 and 2

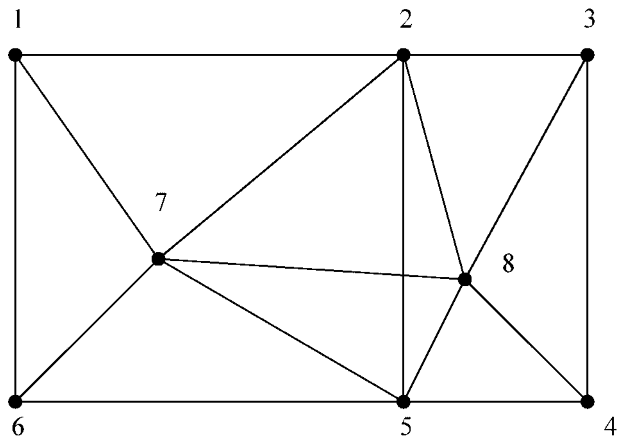
$$\mathbf{d}_{\mathrm{i}}=\mathbf{R}\mathbf{d}_{\mathrm{i}}^{*} \tag{14}$$

where $\mathbf{d}_{\mathrm{i}}=\mathbf{E}\mathbf{r}_{\mathrm{i}}^{1}+\mathbf{c}_{0}$ and $\mathbf{d}_{\mathrm{i}}^{*}=\mathbf{E}^{*}\mathbf{r}_{\mathrm{i}}^{1*}+\mathbf{c}_{0}^{*}$. Now let us write the displacement equation for a second point j under the effect of the same deformation parameters

$$\mathbf{d}_{\mathrm{j}}=\mathbf{R}\mathbf{d}_{\mathrm{j}}^{*} \tag{15}$$

where $\mathbf{d}_{\mathrm{j}}=\mathbf{E}\mathbf{r}_{\mathrm{j}}^{1}+\mathbf{c}_{0}$ while $\mathbf{d}_{\mathrm{j}}^{*}=\mathbf{E}^{*}\mathbf{r}_{\mathrm{j}}^{1*}+\mathbf{c}_{0}^{*}$. The difference of equations (14) and (15) reads

$$\mathbf{d}_{\mathrm{ij}}=\mathbf{d}_{\mathrm{j}}-\mathbf{d}_{\mathrm{i}}=\mathbf{E}\mathbf{r}_{\mathrm{j}}^{1}+\mathbf{c}_{0}-(\mathbf{E}\mathbf{r}_{\mathrm{i}}^{1}+\mathbf{c}_{0})=\mathbf{E}\Delta\mathbf{r}_{\mathrm{ij}} \tag{16}$$



1 RyeNet network

$$\mathbf{d}_{\mathrm{ij}}^{*}=\mathbf{d}_{\mathrm{j}}^{*}-\mathbf{d}_{\mathrm{i}}^{*}=\mathbf{E}^{*}\mathbf{r}_{\mathrm{j}}^{1*}+\mathbf{c}_{0}^{*}-(\mathbf{E}^{*}\mathbf{r}_{\mathrm{i}}^{1*}+\mathbf{c}_{0}^{*})=\mathbf{E}^{*}\Delta\mathbf{r}_{\mathrm{ij}}^{*} \tag{17}$$

Using equations (16) and (17), one can write the following for relative displacement vector between the points i and j as

$$\mathbf{E}\Delta\mathbf{r}_{\mathrm{ij}}=\mathbf{R}\mathbf{E}^{*}\Delta\mathbf{r}_{\mathrm{ij}}^{*} \tag{18}$$

Based on equation (1), the relationship between the relative coordinates is given by

$$\Delta\mathbf{r}_{\mathrm{ij}}=\mathbf{R}\Delta\mathbf{r}_{\mathrm{ij}}^{*} \tag{19}$$

Substituting this equation into equation (18) leads to

$$\mathbf{E}\mathbf{R}\Delta\mathbf{r}_{\mathrm{ij}}^{*}=\mathbf{R}\mathbf{E}^{*}\Delta\mathbf{r}_{\mathrm{ij}}^{*} \tag{20}$$

and finally, the resulting transformation equation for the deformation parameters is obtained as follows

$$\mathbf{E}=\mathbf{R}\mathbf{E}^{*}\mathbf{R}^{\mathrm{T}} \tag{21}$$

Once the strain matrix is transformed from one system to another the other deformation parameters, i.e. dilation, shear, differential rotation can be calculated by means of them (see (**1**)). Substituting equation (9) into equation (21), we get

$$\mathbf{E}=\mathbf{R}\ (\mathbf{S}^{*}+\mathbf{A}^{*})\ \mathbf{R}^{\mathrm{T}} \tag{22}$$

$$\mathbf{E}=(\mathbf{R}\ \mathbf{S}^{*}+\mathbf{R}\ \mathbf{A}^{*})\ \mathbf{R}^{\mathrm{T}} \tag{23}$$

$$\mathbf{E}=\mathbf{R}\ \mathbf{S}^{*}\ \mathbf{R}^{\mathrm{T}}+\mathbf{R}\ \mathbf{A}^{*}\ \mathbf{R}^{\mathrm{T}} \tag{24}$$

This equation proves that the transformation from first coordinate system to second coordinate system is commutative (see also (**1**)).

Table 1 Coordinates of points in RyeNet

	X/m	*Y*/m	*Z*/m
1	851699·0660	−4542606·4564	4380848·9293
2	851875·8826	−4542531·8097	4380890·2156
3	851988·5184	−4542486·9244	4380916·1366
4	852060·8208	−4542600·1418	4380781·2129
5	851946·8440	−4542650·0483	4380750·3791
6	851777·5913	−4542716·4263	4380717·5969
7	851823·9032	−4542624·1969	4380802·2016
8	851945·3241	−4542592·1050	4380811·8028

Table 2 The observations and their standard deviations in CTRS

From	To	ΔX/m	$\sigma_{\Delta X}$/mm	ΔY/m	$\sigma_{\Delta Y}$/mm	ΔZ/m	$\sigma_{\Delta Z}$/mm
1	2	176·8165	0·15	74·6470	0·24	41·2865	0·24
2	3	112·6355	0·15	44·8853	0·24	25·9212	0·24
3	4	72·3024	0·15	−113·2174	0·24	−134·9234	0·23
4	5	−113·9768	0·15	−49·9067	0·24	−30·8336	0·23
5	6	−169·2529	0·15	−66·3779	0·24	−32·7819	0·23
6	1	−78·5251	0·16	109·9696	0·24	131·3321	0·24
2	5	70·9616	0·15	−118·2384	0·24	−139·8365	0·23
1	7	124·8372	0·15	−17·7401	0·24	−46·7277	0·24
2	7	−51·9793	0·15	−92·3874	0·24	−88·0144	0·23
5	7	−122·9408	0·15	25·8516	0·24	51·8226	0·23
6	7	46·3119	0·15	92·2297	0·24	84·6045	0·23
2	8	69·4416	0·15	−60·2957	0·24	−78·4125	0·23
3	8	−43·1944	0·15	−105·1809	0·24	−104·3340	0·23
4	8	−115·4964	0·15	8·0369	0·24	30·5900	0·23
5	8	−1·5199	0·15	57·9432	0·24	61·4238	0·23
7	8	121·4209	0·15	32·0921	0·24	9·6010	0·24

Transformation of covariance matrix between coordinate systems

For a robust deformation analysis, in addition to transformation parameters, their covariance matrix ought to be transformed to the desired coordinate system. This provides a check on the transformed parameters.

After deformation, to determine the deformation parameters covariance matrix, the error propagation law needs to be applied to equation (21). In order to be able to apply the error propagation law easily to equation (21), let us rearrange this equation by multiplying both sides by $\mathbf{R}$, hence

$$\mathbf{ER}=\mathbf{RE}^{*}\mathbf{R}^{\mathrm{T}}\mathbf{R} \tag{25}$$

Since the rotation matrix $\mathbf{R}$ is orthogonal ($\mathbf{R}^{\mathrm{T}}\mathbf{R}=\mathbf{RR}^{\mathrm{T}}=I$), the following equation is obtained

$$\mathbf{ER}=\mathbf{RE}^{*} \tag{26}$$

Now that the above equation is in a suitable form for error propagation, the equation is differentiated with respect to strain matrix parameters

$$\mathrm{d}\mathbf{ER}=\mathbf{R}\mathrm{d}\mathbf{E}^{*} \tag{27}$$

If we square this equation, we reach

$$(\mathrm{d}\mathbf{ER})(\mathrm{d}\mathbf{ER})^{\mathrm{T}}=(\mathbf{R}\mathrm{d}\mathbf{E}^{*})(\mathbf{R}\mathrm{d}\mathbf{E}^{*})^{\mathrm{T}} \tag{28a}$$

$$\mathrm{d}\mathbf{ERR}^{\mathrm{T}}\mathrm{d}\mathbf{E}^{\mathrm{T}}=\mathbf{R}\mathrm{d}\mathbf{E}^{*}\mathrm{d}\mathbf{E}^{*}\mathbf{TR}^{\mathrm{T}} \tag{28b}$$

Then

$$\mathrm{d}\mathbf{E}\mathrm{d}\mathbf{E}^{\mathrm{T}}=\mathbf{R}\mathrm{d}\mathbf{E}^{*}\mathrm{d}\mathbf{E}^{*}\mathbf{TR}^{\mathrm{T}} \tag{29}$$

Table 3 Coordinates of points in LG

	x/m	y/m	z/m
1	30·3893	97·4469	158·3932
2	−12·8862	−62·5843	263·5504
3	−38·9189	−165·0188	328·3801
4	−68·9112	−256·9479	164·3687
5	−42·8552	−154·1207	92·3982
6	0·0000	0·0000	0·0000
7	−7·2623	−28·5215	130·1645
8	−37·5391	−141·9483	175·8019

According to the error propagation law the squares of differentiated strain matrix components are equal to the covariances of the related matrices [3]. As a result, we obtain

$$\mathbf{C}_{\mathbf{EE}}=\mathbf{RC}_{\mathbf{E}^{*}\mathbf{E}^{*}}\mathbf{R}^{\mathrm{T}} \tag{30}$$

Numerical example

In order to be able to verify the derivations, a simulated network, shown in Fig. 1, is examined. The coordinates of the network points are given in Table 1.

To be able to consider this network as a GPS network, first, the coordinate differences among the points are calculated. Then, to take these coordinate differences as observations, normally distributed random errors are generated using MATLAB. The standard deviations for GPS instruments were horizontal: 3 mm $\pm$ 0·5 ppm and vertical: 6 mm $\pm$ 0·5 ppm. Note that since this is a GPS network, standard deviations are transformed into Cartesian coordinate components from the coordinate components of horizontal and vertical using the following equation

$$\mathbf{K}_{\Delta X \Delta X}=\begin{bmatrix} -\sin\phi\cos\lambda & -\sin\lambda & \cos\phi\cos\lambda \\ -\sin\phi\sin\lambda & \cos\lambda & \cos\phi\sin\lambda \\ \cos\phi & 0 & \sin\phi \end{bmatrix} \begin{bmatrix} \sigma^2_{\Delta n\Delta n} & & \\ & \sigma^2_{\Delta e\Delta e} & \\ & & \sigma^2_{\Delta h\Delta h} \end{bmatrix} \begin{bmatrix} -\sin\phi\cos\lambda & -\sin\lambda & \cos\phi\cos\lambda \\ -\sin\phi\sin\lambda & \cos\lambda & \cos\phi\sin\lambda \\ \cos\phi & 0 & \sin\phi \end{bmatrix}^{\mathrm{T}} \tag{31}$$

Finally, the coordinates and standard deviations of observations are obtained in Conventional Terrestrial Reference System (CTRS), these are given in Table 2.

Next, in order to compute their counterparts in LG system, the coordinates are transformed from CTRS to LG using (point 6 is arbitrarily chosen as the reference point)

Table 4 Observations and their standard deviation in LG

From	To	α/deg	σ_α/''	z/deg	σz/''	s/m	σ_s/mm
1	2	254·867954	0·15	57·612191	0·15	196·3182	0·12
2	3	255·740724	0·15	58·475473	0·15	123·9897	0·11
3	4	251·930800	0·15	149·477245	0·15	190·3951	0·12
4	5	75·780799	0·15	124·155880	0·15	128·1876	0·11
5	6	74·460762	0·15	120·010917	0·15	184·7355	0·12
6	1	72·679722	0·15	32·799529	0·15	188·4352	0·12
2	5	251·871588	0·15	150·630994	0·15	196·3927	0·12
1	7	253·358716	0·15	102·117861	0·15	134·4713	0·11
2	7	80·624838	0·15	165·488720	0·15	137·7816	0·11
5	7	74·178036	0·15	73·864988	0·15	135·8981	0·11
6	7	255·714598	0·15	12·740871	0·15	133·4505	0·11
2	8	252·743662	0·15	136·556881	0·15	120·8561	0·11
3	8	86·577332	0·15	171·386627	0·15	154·3186	0·12
4	8	74·740880	0·15	84·521272	0·15	119·7489	0·11
5	8	66·407520	0·15	9·048828	0·15	84·4548	0·11
7	8	255·054568	0·15	68·756824	0·15	125·9566	0·11

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{LG}} = \begin{bmatrix} -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \begin{bmatrix} X-X_6 \\ Y-Y_6 \\ Z-Z_6 \end{bmatrix}_{\mathrm{CTRS}} \tag{32}$$

The obtained coordinates in LG system are given in Table 3.

Using these coordinates, the coordinate differences in Local Geodetic (LG) are calculated. Next, using these coordinate differences, the observations in LG are calculated using the following formulae

$$\text{Azimuth}: \ \alpha = \arctan\frac{\Delta y}{\Delta x} \tag{33}$$

$$\text{Zenith angle}: \ z = \arccos\frac{\Delta z}{s} \tag{34}$$

$$\text{Slope distance}: \ s = \left(\Delta x^2 + \Delta y^2 + \Delta z^2\right)^{1/2} \tag{35}$$

Again, using MATLAB, random errors are generated. Specifications for a typical instrument (for distances 2 mm$\pm$2 ppm and for angles 3″) are used. In LG system the observations and their standard deviations are given in Table 4.

In order to be able to create the coordinates for the second time period, the deformation matrix of no rotation (i.e. the matrix **A** in equation (9) is set to zero) is assumed as follows

$$\mathbf{E} = \mathbf{S} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} 0.000200 & 0.000556 & 0.000556 \\ 0.000556 & 0.000200 & 0.000556 \\ 0.000556 & 0.000556 & 0.000200 \end{bmatrix} \tag{36}$$

Then the same approach described above is followed for the second time period. The observations and their standard deviations in CTRS are given in Table 5 and the observations and their standard deviations in LG are given in Table 6.

The simulated data of the network referred to CTRS and LG systems are adjusted separately. After the adjustments the relative coordinates with respect to point 6 and displacements are determined; these are given in Table 7.

Using the adjusted coordinates, the deformation matrix is computed as

Table 5 The observations and their standard deviations in CTRS

From	To	ΔX/m	$\sigma_{\Delta X}$/mm	ΔY/m	$\sigma_{\Delta Y}$/mm	ΔZ/m	$\sigma_{\Delta Z}$/mm
1	2	176·9165	0·15	74·7832	0·24	41·4343	0·24
2	3	112·6977	0·15	44·9713	0·24	26·0135	0·24
3	4	72·1789	0·15	−113·2751	0·24	−134·9732	0·23
4	5	−114·0445	0·15	−49·9969	0·24	−30·9310	0·23
5	6	−169·3418	0·15	−66·5038	0·24	−32·9199	0·23
6	1	−78·4073	0·16	110·0209	0·24	131·3765	0·24
2	5	70·8324	0·15	−118·3006	0·24	−139·8907	0·23
1	7	124·8263	0·15	−17·7004	0·24	−46·6772	0·24
2	7	−52·0900	0·15	−92·4834	0·24	−88·1115	0·23
5	7	−122·9222	0·15	25·8172	0·24	51·7788	0·23
6	7	46·4194	0·15	92·3203	0·24	84·6983	0·23
2	8	69·3782	0·15	−60·3125	0·24	−78·4234	0·23
3	8	−43·3195	0·15	−105·2837	0·24	−104·4372	0·23
4	8	−115·4984	0·15	7·9909	0·24	30·5361	0·23
5	8	−1·4541	0·15	57·9879	0·24	61·4674	0·23
7	8	121·4682	0·15	32·1713	0·24	9·6888	0·24

Table 6 The observations and their standard deviations in LG

From	To	α/deg	σ_α/''	z/deg	σz/''	s/m	σ_s/mm
1	2	254·8753030	0·15	57·5708582	0·15	196·4978	0·1195
2	3	255·7472418	0·15	58·4334434	0·15	124·1077	0·1125
3	4	251·9211067	0·15	149·5200422	0·15	190·4394	0·119
4	5	75·7876241	0·15	124·1956300	0·15	128·3317	0·113
5	6	74·4679352	0·15	120·0540575	0·15	184·9027	0·1185
6	1	72·6716129	0·15	32·7584663	0·15	188·4511	0·119
2	5	251·8610887	0·15	150·6745013	0·15	196·4438	0·1195
1	7	253·3608995	0·15	102·0939961	0·15	134·4563	0·1135
2	7	80·6352977	0·15	165·4677899	0·15	137·9875	0·114
5	7	74·1796407	0·15	73·8829453	0·15	135·8705	0·1135
6	7	255·7425840	0·15	12·7641192	0·15	133·6170	0·1135
2	8	252·7394604	0·15	136·5865056	0·15	120·8654	0·112
3	8	86·5652857	0·15	171·3573125	0·15	154·5299	0·1155
4	8	74·7436170	0·15	84·5539970	0·15	119·7499	0·112
5	8	66·3380428	0·15	9·0044291	0·15	84·5241	0·1085
7	8	255·0602393	0·15	68·7097493	0·15	126·0419	0·1125

$$\mathbf{E}=(\mathbf{A}^{\mathrm{T}}\mathbf{Q}_{\mathrm{dd}}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{Q}_{\mathrm{dd}}^{-1}\mathbf{d} \tag{37}$$

$$\text{where } \mathbf{A}_{\mathrm{i}}=\begin{bmatrix} \Delta X_{\mathrm{i6}} & 0 & 0 & \Delta Y_{\mathrm{i6}} & \Delta Z_{\mathrm{i6}} & 0 \\ 0 & \Delta Y_{\mathrm{i6}} & 0 & \Delta X_{\mathrm{i6}} & 0 & \Delta Z_{\mathrm{i6}} \\ 0 & 0 & \Delta Z_{\mathrm{i6}} & 0 & \Delta X_{\mathrm{i6}} & \Delta Y_{\mathrm{i6}} \end{bmatrix} \tag{38}$$

and

$$\mathbf{Q}_{\mathrm{dd}}=\mathbf{Q}_{\mathrm{xx}}^{1}+\mathbf{Q}_{\mathrm{xx}}^{2} \tag{39}$$

where $\mathbf{Q}_{\mathrm{xx}}^{1}$ is the cofactor matrix of unknowns for the first period and $\mathbf{Q}_{\mathrm{xx}}^{2}$ is the cofactor matrix of unknowns for the second period. The deformation matrix components in LG and CTRS and their standard deviations are given in Table 8. Using equation (21) the deformation matrix is transformed from LG to CTRS and this is also shown in Table 8.

While transforming the parameters, the parameters are transformed from LG to CTRS. The reason for that is that presently GPS is the most utilised measurement technique. Nowadays deformation analysis with terrestrial techniques is rarely used. Therefore, the combination ought to be done in the new system. As can be seen the results in the third column in this table match closely with the assumed values given in equation (36). This proves that the outlined derivations are correct.

Conclusions

Geodetic networks are often surveyed to produce multiple epochs of data. After adjustment, by comparing the coordinates at networks points the deformations are obtained. This process yields deformation parameters in the coordinate system in which the geodetic network is defined. For long period temporal and spatial analysis of deformation parameters, the practical approach would be coordinate transformation of deformation parameters from one system to another. In this study, it is shown that coordinate transformation of deformation parameters from one system to another is possible. In addition, the transformation of the deformation parameters covariance matrix is formulated. Numerical results prove that the outlined approach works well.

Table 7 Relative coordinates with respect to point 6 and displacements

	ΔX		*d*	
	LG	CTRS	LG	CTRS
1	30·389	−78·525	−0·018	0·118
2	97·447	109·970	−0·107	0·052
3	158·393	131·332	0·082	0·044
4	−12·886	98·291	−0·016	0·218
5	−62·584	184·617	−0·180	0·187
6	263·550	172·618	0·295	0·192
7	−38·919	210·927	−0·015	0·280
8	−165·019	229·502	−0·225	0·273
9	328·380	198·540	0·428	0·285
10	−68·911	283·229	0·003	0·156
11	−256·948	116·285	−0·114	0·216
12	164·369	63·616	0·336	0·235
13	−42·855	169·253	0·003	0·089
14	−154·121	66·378	−0·064	0·126
15	92·398	32·782	0·196	0·138
16	−7·262	46·312	−0·008	0·107
17	−28·522	92·230	−0·089	0·091
18	130·165	84·604	0·144	0·094
19	−37·539	167·733	−0·004	0·155
20	−141·948	124·321	−0·121	0·171
21	175·802	94·206	0·267	0·181

Table 8 Deformation matrix components and their standard deviations

	LG	CTRS	From LG to CTRS
ε_{XX}	$-3{\cdot}14\times10^{-4}\pm0{\cdot}40\times10^{-4}$	$2{\cdot}01\times10^{-4}\pm0{\cdot}02\times10^{-4}$	$2{\cdot}02\times10^{-4}\pm0{\cdot}06\times10^{-4}$
ε_{yy}	$0{\cdot}03\times10^{-4}\pm0{\cdot}03\times10^{-4}$	$2{\cdot}19\times10^{-4}\pm0{\cdot}47\times10^{-4}$	$2{\cdot}21\times10^{-4}\pm0{\cdot}15\times10^{-4}$
ε_{zz}	$9{\cdot}53\times10^{-4}\pm0{\cdot}07\times10^{-5}$	$2{\cdot}18\times10^{-4}\pm0{\cdot}42\times10^{-4}$	$2{\cdot}19\times10^{-4}\pm0{\cdot}22\times10^{-4}$
ε_{xy}	$0{\cdot}25\times10^{-4}\pm0{\cdot}11\times10^{-4}$	$5{\cdot}51\times10^{-4}\pm0{\cdot}09\times10^{-4}$	$5{\cdot}51\times10^{-4}\pm0{\cdot}09\times10^{-4}$
ε_{xz}	$-0{\cdot}71\times10^{-4}\pm0{\cdot}01\times10^{-4}$	$5{\cdot}60\times10^{-4}\pm0{\cdot}09\times10^{-4}$	$5{\cdot}60\times10^{-4}\pm0{\cdot}12\times10^{-4}$
ε_{yz}	$-6{\cdot}81\times10^{-4}\pm0{\cdot}01\times10^{-4}$	$-5{\cdot}37\times10^{-4}\pm0{\cdot}45\times10^{-4}$	$-5{\cdot}36\times10^{-4}\pm0{\cdot}17\times10^{-4}$

References

1. Berber, M., 2006. Robustness Analysis of Geodetic Networks. Technical Report No 242, Department of Geodesy and Geomatics Engineering, University of New Brunswick, Fredericton, NB, Canada.
2. Bock, Y., 1998. Reference Systems. GPS for Geodesy, Springer, Berlin.
3. Koch, K. R., 1997. Parameter Estimation and Hypothesis Testing in Linear Models, 2nd edition. Springer, Berlin.
4. Leick, A., 2004. GPS Satellite Surveying. John Wiley & Sons, Hoboken, NJ.
5. McCarthy, D. D., 1996. International Earth Rotations Service Conventions 1996. IERS Technical Note 21, Observatoire de Paris.
6. Vaníček, P., Craymer, M. R. and Krakiwsky, E. J., 2001. Robustness Analysis of Geodetic Horizontal Networks. *Journal of Geodesy*, 75: 199–209.
7. Vaníček, P., Novák, P., Craymer, M. R. and Pagiatakis, S., 2002. On the Correct Determination of Transformation Parameters of a Horizontal Geodetic Datum. *Geomatica*, 56: 329–340.