Robustness Analysis of Two-Dimensional Networks

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Abstract: After geodetic networks are established, relevant measurements are made and point coordinates are estimated by the least-squares method. However, the least-squares method does not give any information about the robustness of networks. To measure robustness of a network, the degree of deformation of individual points of the network is measured by strain. Furthermore, threshold values are needed to evaluate networks. These threshold values will enable us to evaluate the robustness of the network. If the displacements of individual points of the network are worse than the threshold values, we must redesign the network by changing the configuration or improving the measurements until we obtain a network of acceptable robustness. This paper describes how to obtain the displacements at individual points of a network, employs the specifications of the Geodetic Survey Division, and shows the power of the technique on different geodetic networks.

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Introduction

Reliability of geodetic control networks (the detection of outliers/gross errors/blunders among the observations) can be measured using a technique pioneered by the geodesist Baarda and reported in Baarda (1968). Subsequent studies about Baarda's approach can be found in Proszynski (1994, 1997). In Baarda's method, a statistical test (data snooping) is used to detect outliers. What happens if one or more observations are burdened with an error? It is clear that these outliers will affect the observations and produce incorrect estimates of the parameters. If the outliers are detected by a statistical test then those contaminated observations are removed, the network is readjusted, and we obtain the final results.

In the approach described here, traditional reliability analysis (Baarda's approach) has been augmented with geometrical strength analysis using strain to create a technique called robustness analysis. In statistical literature, robustness is a measure of the insensitivity to outliers in the data. Robustness analysis is a natural merger of reliability and strain and is defined as the ability to resist deformations induced by the largest undetectable outliers as determined from internal reliability analysis.

This paper addresses the consequences of outliers not being detected by Baarda's test. This failure may happen for two reasons: (i) the observation is not sufficiently checked by other independent observations or (ii) the test does not recognize the gross error. By how much do these undetected errors influence the network? If the influence of the undetected errors is small, the network is called robust; if it is not, it is called a weak network.

The maximum undetectable errors $\mathbf{\Delta l}$ among the observations that would not be detected by a statistical test are given by Baarda (1968) as

$$\Delta l_i = \sqrt{\lambda_0}\frac{\sigma_{l_i}}{\sqrt{r_i}} \tag{1}$$

where $\sqrt{\lambda_0}$=value of the shift (noncentrality parameter) of the postulated distribution in the alternative hypothesis. Calculation of $\sqrt{\lambda_0}$ is given in Vaníček et al. (2001). σ_{l_i}=a priori value of standard deviation of the ith observation; and r_i=redundancy number of the ith observation.

The estimate for the displacements $\mathbf{\Delta x}$ caused by the maximum undetectable errors $\mathbf{\Delta l}$ in the observations is given by

$$\mathbf{\Delta x} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}\Delta l \tag{2}$$

where $\mathbf{A}$=design matrix; and $\mathbf{P}$=weight matrix.

Nonetheless, the problem with the displacements is that their estimates are datum dependent. This means that these estimates depend not only on the geometry of the network and the accuracy of the observations but also on the selection of constraints for the adjustment; this has nothing to do with the network deformation. Robustness of a network should depend only on the network geometry and accuracy of the observations. Therefore, we use the strain technique as it is independent of adjustment constraints and reflects only the network geometry and accuracy of the observations. Traditional reliability analysis has been augmented with, strain technique termed robustness analysis; this is outlined in the following section.

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Robustness Analysis of Two-Dimensional Networks

Robustness analysis of two-dimensional (2D) networks is outlined in Vaníček et al. (2001) as follows. Let us denote the displacement of a point P_i by

$$\Delta\mathbf{x}_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \tag{3}$$

where u=displacement in the x direction; and v=displacement in the y direction. Then, the tensor gradient with respect to position is

$$\mathbf{E}_i = \begin{bmatrix} \dfrac{\partial u_i}{\partial x} & \dfrac{\partial u_i}{\partial y} \\ \dfrac{\partial v_i}{\partial x} & \dfrac{\partial v_i}{\partial y} \end{bmatrix} \tag{4}$$

For $\forall j=0,1,\ldots,t$ [t=number of connection(s)] the displacements u and v can be calculated as follows:

$$a_i + \left(\frac{\partial u_i}{\partial x}\right)(X_j - X_i) + \left(\frac{\partial u_i}{\partial y}\right)(Y_j - Y_i) = u_j$$

$$b_i + \left(\frac{\partial v_i}{\partial x}\right)(X_j - X_i) + \left(\frac{\partial v_i}{\partial y}\right)(Y_j - Y_i) = v_j \tag{5}$$

where all the partial derivatives as well as the absolute terms a_i, b_i, and the coordinates X_i, Y_i refer to points P_i, and P_j is connected (by an observation) to the point of interest, point P_i. In matrix form

$$\forall i \text{ in the network} \quad \mathbf{K}_i \begin{bmatrix} a_i \\ \dfrac{\partial u_i}{\partial x} \\ \dfrac{\partial u_i}{\partial y} \end{bmatrix} = \mathbf{u}_i \tag{6}$$

$$\forall i \text{ in the network} \quad \mathbf{K}_i \begin{bmatrix} b_i \\ \dfrac{\partial v_i}{\partial x} \\ \dfrac{\partial v_i}{\partial y} \end{bmatrix} = \mathbf{v}_i \tag{7}$$

where $\mathbf{K}_i = \lfloor 1\ (X_j - X_i)\ (Y_j - Y_i) \rfloor$. If these equations are solved using the least-squares method, we get

$$\forall i \text{ in the network} \quad \begin{bmatrix} a_i \\ \dfrac{\partial u_i}{\partial x} \\ \dfrac{\partial u_i}{\partial y} \end{bmatrix} = (\mathbf{K}_i^T \mathbf{K}_i)^{-1} \mathbf{K}_i^T \mathbf{u}_i = \mathbf{Q}_i \mathbf{u}_i \tag{8}$$

$$\forall i \text{ in the network} \quad \begin{bmatrix} b_i \\ \dfrac{\partial v_i}{\partial x} \\ \dfrac{\partial v_i}{\partial y} \end{bmatrix} = (\mathbf{K}_i^T \mathbf{K}_i)^{-1} \mathbf{K}_i^T \mathbf{v}_i = \mathbf{Q}_i \mathbf{v}_i \tag{9}$$

Assembling Eqs. (8) and (9) into a hypermatrix, we get

Table 1. Values of Factor C for Different Order of Geodetic Networks

Order	Average length (km)	Factor C
First	20	2
Second	15	5
Third	10	12
Fourth	5	30

$$\forall i \text{ in the network} \quad \begin{bmatrix} a_i \\ \dfrac{\partial u_i}{\partial x} \\ \dfrac{\partial u_i}{\partial y} \\ b_i \\ \dfrac{\partial v_i}{\partial x} \\ \dfrac{\partial v_i}{\partial y} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_i & 0 \\ 0 & \mathbf{Q}_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \tag{10}$$

Since we are looking for the relation between the displacement vector and the strain matrix, absolute terms are of no interest to us. So, we can eliminate the first row of the $\mathbf{Q}_i$ matrix. If we show

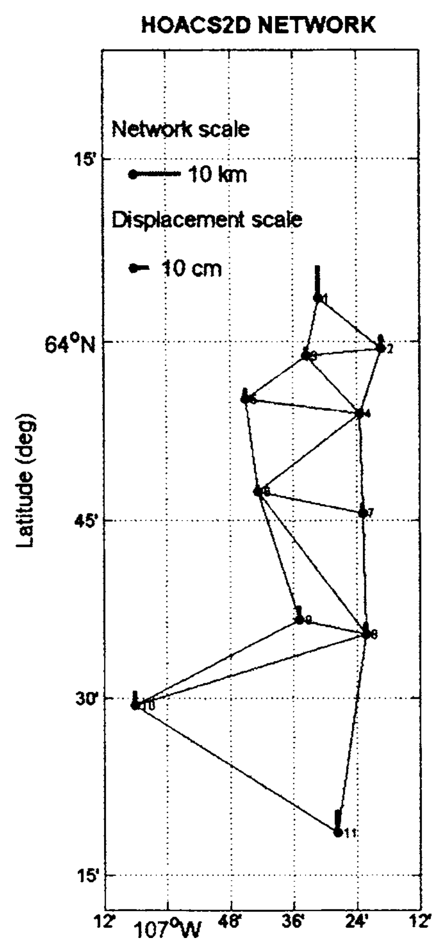

Fig. 1. Displacements in HOACS2D network

Table 2. Relative Displacements and Threshold Values for HOACS2D Network

Points	δr_{ij} (m)	r_{ij} (m)
1–2	0.40	1.59
1–3	0.35	2.82
2–3	0.07	1.43
2–4	0.13	1.34
3–4	0.08	1.51
3–5	0.10	1.45
4–5	0.09	0.88
4–6	0.06	0.99
4–7	0.12	0.79
5–6	0.04	1.82
6–7	0.18	0.84
6–8	0.20	0.57
6–9	0.22	0.42
7–8	0.03	0.97
8–9	0.02	1.33
8–10	0.08	0.76
8–11	0.13	0.62
9–10	0.08	0.59
10–11	0.10	0.76

the reduced matrix with **T** and substitute from Eqs. (3) and (4), we obtain

$$\forall i \text{ in the network} \quad \text{vec}(\mathbf{E}_i) = \mathbf{T}_i \Delta \mathbf{x}_i \tag{11}$$

Substituting Eq. (2) into Eq. (11)

$$\forall i \text{ in the network} \quad \text{vec}(\mathbf{E}_i) = \mathbf{T}_i (\mathbf{A}^T \mathbf{P} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{P} \Delta \mathbf{l} \tag{12}$$

To determine the displacements of point P_i, we introduce the "initial conditions" X_0 and Y_0. Initial conditions are the coordinates that are obtained minimizing the norm of the displacement vectors at all points in the network. This means that to calculate X_0 and Y_0, the displacements in the network points should be minimized. Since the formulas to do this are bulky they are not given here but can be seen in Berber (2006). Once X_0 and Y_0 have been determined $\bar{u}_i$ and $\bar{v}_i$ are determined from

$$\begin{bmatrix} \bar{u}_i \\ \bar{v}_i \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_i}{\partial x} & \dfrac{\partial u_i}{\partial y} \\ \dfrac{\partial v_i}{\partial x} & \dfrac{\partial v_i}{\partial y} \end{bmatrix} \begin{bmatrix} X_i - X_0 \\ Y_i - Y_0 \end{bmatrix} \tag{13}$$

After computing the displacements $\bar{u}$ and $\bar{v}$ for each point in the network, we can calculate the total displacement at each point from

$$\text{Disp}_i = \sqrt{\bar{u}_i^2 + \bar{v}_i^2} \tag{14}$$

Determination of Threshold Values

In this paper we use the specifications given by the Geodetic Survey Division (GSD) to compute threshold values. The GSD specifications are given in GSD (1978). These threshold values are going to enable us to evaluate the robustness of the network. The specifications are:

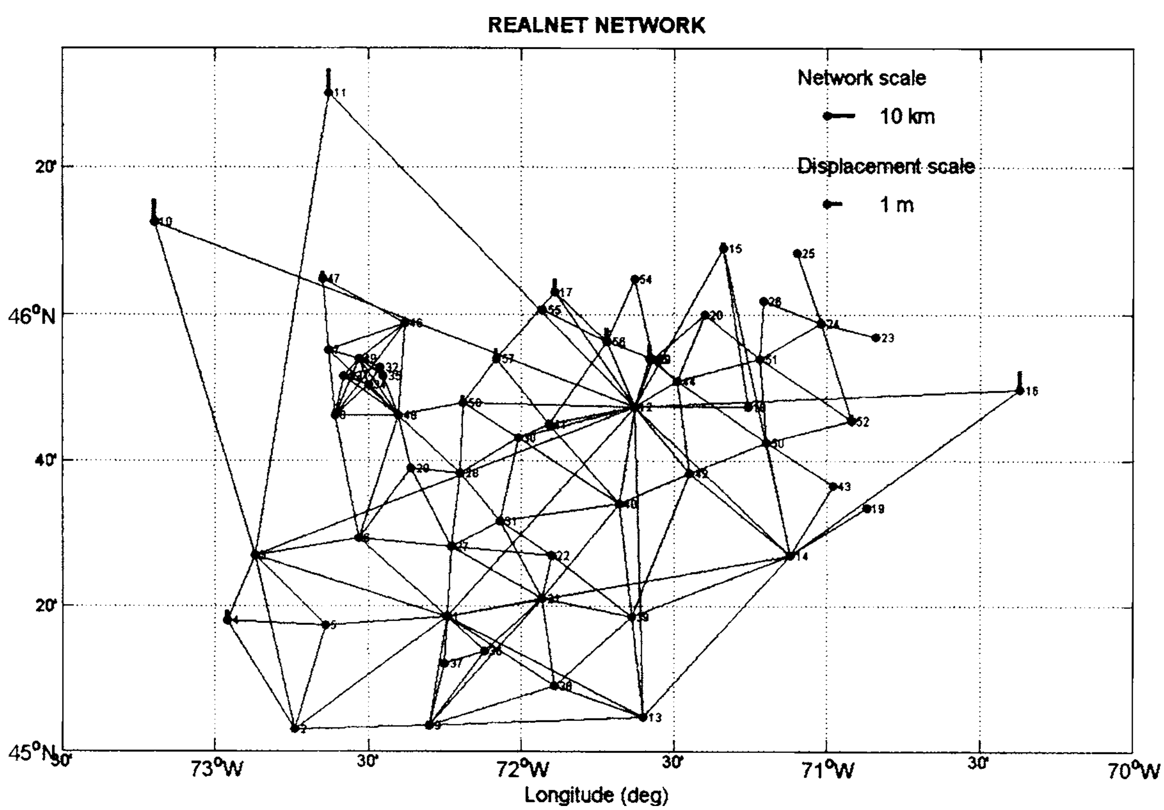

Fig. 2. Displacements in Realnet network

Table 3. Relative Displacements and Threshold Values for Realnet Network

Points	δr_{ij} (m)	r_{ij} (m)
1–27	0.20	0.88
1–12	0.34	1.43
1–14	0.42	1.79
1–21	0.33	0.50
1–13	0.11	1.13
1–38	0.23	0.66
1–36	0.33	1.60
1–37	0.34	1.49
1–9	0.30	0.57
1–2	0.45	0.98
1–5	0.31	0.63
1–3	0.21	1.03
1–6	0.18	0.60
*2–4	0.76	0.65
2–3	0.65	0.92
2–5	0.14	0.57
2–9	0.19	0.70
*3–10	2.24	1.78
*3–11	2.54	2.38
3–12	0.54	2.08
3–6	0.33	0.55
*3–5	0.52	0.51
*3–4	1.39	0.91
*4–5	0.88	0.50
*6–8	0.68	0.64
6–48	0.56	0.67
6–29	0.43	0.45
6–27	0.07	0.47
7–47	0.92	0.92
*7–46	0.80	0.42
7–49	0.04	2.59
7–48	0.79	0.48
7–8	0.50	0.84
8–33	0.69	3.04
8–49	0.51	0.82
*8–46	1.28	0.58
8–34	0.47	1.39
*8–48	1.23	0.81
9–37	0.63	0.81
*9–36	0.64	0.48
9–21	0.44	0.86
9–38	0.47	0.68
9–13	0.19	1.10
*10–12	2.75	2.60
*11–12	2.99	2.20
*12–17	1.38	0.71
*12–56	1.20	0.89
*12–45	1.72	1.56
12–53	0.44	1.61
12–15	0.47	0.92
12–44	0.27	1.61
12–16	1.77	1.97
12–18	0.21	0.59
12–14	0.11	1.11
12–42	0.27	0.45
12–13	0.23	1.58
12–40	0.40	0.50
12–28	0.59	0.95
12–30	0.31	0.61
12–41	0.35	0.44
*12–58	1.11	0.87
13–38	0.31	0.48
13–39	1.12	0.51
13–14	0.31	1.11
14–15	0.39	1.59
14–50	0.53	0.61
14–43	0.27	0.42
*14–16	1.66	1.45
14–39	0.42	0.88
14–42	0.37	0.67
15–18	0.66	0.80
*17–56	2.35	0.92
*17–57	0.66	0.45
20–51	0.39	0.93
20–44	0.28	0.91
20–53	0.22	0.88
21–22	0.19	1.50
21–40	0.36	0.64
21–14	0.51	1.29
21–39	0.38	0.46
21–38	0.20	0.44
21–27	0.20	0.55
21–31	0.22	0.47
22–31	0.09	0.81
22–39	0.19	0.52
22–27	0.14	0.53
24–26	0.48	0.76
24–51	0.69	0.92
*24–52	1.12	0.53
26–51	0.41	1.79
27–28	0.25	0.95
27–31	0.07	1.71
27–29	0.37	0.45
28–58	0.58	0.92
28–30	0.43	0.89
28–31	0.31	0.78
28–3	0.54	1.13
28–29	0.54	1.51
*28–48	0.53	0.45
29–48	0.28	1.78
*30–58	0.84	0.83
30–41	0.47	2.62
30–40	0.10	0.62
30–31	0.13	0.45
31–40	0.15	0.63
32–49	0.24	1.85
32–35	0.23	0.57
32–48	0.83	1.49
32–34	0.09	1.59

Table 3. (*Continued.*)

Points	δr_{ij} (m)	r_{ij} (m)
32–33	0.30	2.93
33–49	0.25	2.11
33–34	0.29	2.04
33–48	0.56	0.84
34–49	0.30	2.33
34–48	0.77	1.28
35–49	0.29	2.32
36–37	0.17	1.30
38–39	0.28	0.52
39–40	0.06	0.59
39–42	0.18	0.81
40–42	0.19	1.00
*40–41	0.57	0.55
*41–57	1.13	0.42
*41–56	1.02	0.51
42–44	0.06	0.48
42–50	0.19	0.41
43–50	0.40	0.43
44–45	1.60	2.61
44–53	0.38	2.56
44–51	0.14	0.43
44–50	0.17	0.55
46–47	0.21	0.48
46–49	0.81	1.68
46–48	0.19	0.46
48–58	0.31	0.84
48–49	0.81	0.88
50–15	0.69	0.99
50–51	0.04	0.42
*50–52	0.55	0.46
*51–52	0.56	0.56
53–54	0.37	0.41
53–56	1.19	1.50
*54–56	1.23	0.89
57–58	1.70	1.74

Note: *=pairs of points.

A survey point of a network is classified according to whether the semimajor axis of the 95% confidence region, with respect to other points of the network, is less than or equal to

$$r_{ij} = C(d_{ij} + 0.2) \qquad (15)$$

where r_{ij} is in centimeters, d_{ij}=distance in kilometers between points P_i and P_j; and C=factor assigned according to the order of survey as given in Table 1.

In robustness analysis the effect of the maximum undetectable errors which would not be detected by the statistical test on the network is searched. The GSD specifications are given for random errors, this means that by definition some certain amount of error is accepted in the GSD networks. Here, a tacit acceptance has been made of certain values as "acceptable," random or systematic. In this paper these specifications are used as an example but they might vary from country to country.

As the GSD specifications relate to pairs of points, in this study we implement the following formula:

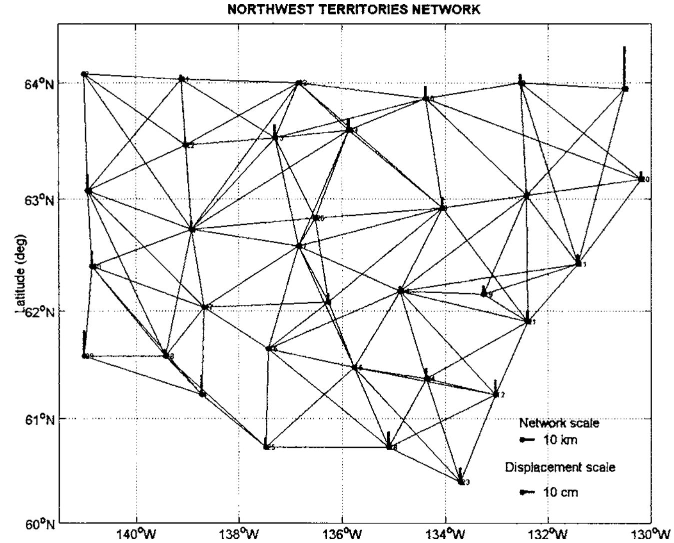

Fig. 3. Displacements in Northwest Territories network

$$\delta_{r_{ij}} = \sqrt{(u_j - u_i)^2 + (v_j - v_i)^2} \qquad (16)$$

where P_i and P_j=points in question; and $\delta_{r_{ij}}$=relative displacement between points P_i and P_j.

Here we calculate the relative displacements using Eq. (16) to be able to compare them with the specifications determined by GSD. We also compute the absolute displacements at network points using Eq. (14) for help in the interpretation and graphical display.

We calculate $\delta_{r_{ij}}$ and compare it with r_{ij}. If for $\forall ij$ in the network $\delta_{r_{ij}} < r_{ij}$, this means that the network is robust. If $\delta_{r_{ij}} > r_{ij}$, the network is weak. Examples are shown in the following section.

Examples

To be able to show the power of the technique we have examined three different networks. The first network is the HOACS2D network. It is a synthetic horizontal network, shown in Fig. 1. The network consists of 11 points, one of which (Point 1) is fixed, 38 directions, 19 distances, and 1 azimuth from Point 6 to Point 5. The distances were assigned a realistic standard deviation of 3 mm+2 ppm while the directions were assigned a standard deviation of 0.5″. The datum orientation was defined by the azimuth with a standard deviation of 1″.

In this network all directions and distances are measured. On the other hand, as can be seen from Fig. 1, the geometry of the network is not good as the controllability (number of connections) of the points at the edge of the network are low compared to the other points in the network. Hence we get bigger displacements at edge points.

From the detailed analysis of the original observations we find out that since the distances between Points 8–10, 8–11, 9–10, and 10–11 are longer than the other distances in the network their standard deviations are larger compared to the other distance observations in the network. Hence we get bigger displacements at these points.

We computed the displacements using Eq. (14) and plotted them in Fig. 1. We then calculated relative displacements using

Eq. (16) and compared them with the specifications; the comparisons are given in Table 2. In this network for $\forall ij$ in the network $\delta_{r_{ij}} < r_{ij}$ so it is a robust network.

The second network is called Realnet and is shown in Fig. 2. This an example of a real horizontal terrestrial network in southern Quebec consisting of 58 points, one of which (Point 1) is fixed, 307 directions, 125 distances, and 1 azimuth observation from Point 1 to Point 3. The ranges of the standard deviations for the direction observations are $0''.6-2''.0$ for the distance observations 1–34 cm and the standard deviation for the azimuth observation is $1''$. Note that the robustness of 4 points (Numbers 19, 23, 25, 55) is undefined. This is due to a singularity as these points are linked to the rest of the network by only one observation. Singularity cases are addressed in Vaníček et al. (2001).

As can be seen from Fig. 2, the geometry of the network is not good and there are some very low controlled points such as Points 10, 11, and 16. Moreover, most of the distances in the network are not measured. Therefore, the controllability of these points is rather low. Hence we obtain very big displacements at these points. Therefore, in this network for some pairs of points $\delta_{r_{ij}} > r_{ij}$; these pairs of points are identified in Table 3 by the * symbol.

We computed the displacements again using Eq. (14) and plotted them in Fig. 2 and then we again calculated relative displacements using Eq. (16) to be able to compare them with the specifications; the comparisons are given in Table 3.

As can be seen in Fig. 2, the identified pairs of points have a big displacement at least at one of the points. The reasons for these displacements are, first, the distances are not measured (discovered from detailed analysis of the original observations) and, second, the points are not well controlled.

However, if the controllability increases the displacements get smaller. For example, although the distances from Point 17 to the connected points are not measured, the controllability is higher at Point 17 compared to Points 10, 11, and 16. This causes the displacement at Point 17 to be smaller than the displacements at Points 10, 11, and 16, and similarly for Points 24, 56, and 57.

The third network is called Northwest Territories network; it is shown in Fig. 3. It is an example of a real GPS network. It consists of 33 points, one of which (Point 1) is fixed, and 402 coordinate differences. The range of the baseline component standard deviations are 8–774 mm since it is a rather old GPS network. Note that although GPS networks are intrinsically three-dimensional (3D) only the horizontal two-dimensional (2D) component of the network is analyzed here.

As can be seen from Fig. 3, generally the displacements are bigger at the edge of the network since the controllability of these points is rather small compared to the other points in the network. However, as soon as the controllability increases the displacements get smaller. For example, at Point 9 there are three connections, whereas at Point 20 there are 4 connections and the displacement is smaller at Point 20 than at Point 9. Nevertheless, Points 8, 10, 13, and 33 have some observations that have large standard deviations (see Table 4). Therefore, we get centimeter level displacements at these points. However, at Point 2, standard deviations of the observations are smaller compared to the maximum standard deviations at Points 8, 10, 13, and 33. Similar situations also occur at Points 31 and 32.

We computed the displacements using Eq. (14) and plotted them in Fig. 3 and then we calculated relative displacements using Eq. (16), allowing us to compare them with the specifications; the comparisons are given in Table 5. In this network for $\forall ij$ in the network $\delta_{r_{ij}} < r_{ij}$ so it is a robust network.

Table 4. Standard Deviations of Some of the Observations in Northwest Territories Network

Points	$\sigma_{\Delta X}$ (mm)	$\sigma_{\Delta Y}$ (mm)
2–5	82.66	63.42
2–3	59.56	41.84
2–31	50.52	18.48
2–22	33.61	26.67
8–20	27.91	17.75
8–11	151.91	71.76
8–10	67.70	63.86
8–9	82.84	53.74
8–7	24.22	11.93
13–31	38.42	25.22
13–26	78.63	73.35
13–33	92.78	52.93
13–32	73.42	53.04
13–22	49.55	44.06
13–17	53.58	67.49
13–10	110.63	77.52
13–5	121.46	66.60
33–32	147.18	100.06
33–13	94.66	46.38
33–6	125.07	91.07
33–10	35.95	40.62
33–26	71.62	68.52
33–17	59.99	76.54
33–13	92.78	52.93
10–32	109.54	97.13
10–33	35.95	40.62
10–6	87.68	103.26
10–7	122.63	99.28
10–8	67.70	63.86
10–20	104.06	93.71

Conclusion

In order to be able to calculate the displacements in 2D networks, the initial conditions must be computed. Furthermore the threshold values are needed to evaluate the networks. These threshold values enable us to assess the robustness of networks. In this study, the specifications given by the Geodetic Survey Division are used to compute the threshold values. The numerical results prove that this approach works well.

Robustness analysis is a very powerful technique capable of providing a picture of the analyzed network. If a network has some defects, the robustness analysis technique reveals them and portrays them.

When the controllability of the network points are lower we obtain bigger displacements. If a network has some deficiency (such as if the distances are not measured) we can determine the weakness of the network for these points. This lack of measurement also lowers the controllability of the network for these points. However, if the controllability increases the displacements get lower. The observations with large standard deviations cause bigger displacements at the connected points. This technique can also be used for GPS networks, although only the horizontal component of the GPS networks is analyzed here.

It seems that the robustness of a planned network, a robustness preanalysis, may prove to be more important than a postanalysis

Table 5. Relative Displacements and Threshold Values for Northwest Territories Network

Points	δr_{ij} (m)	r_{ij} (m)
1–15	0.05	1.55
1–16	0.06	1.44
1–17	0.07	1.28
1–26	0.08	1.68
1–27	0.07	2.52
l–6	0.09	2.95
2–22	0.03	2.37
2–3	0.12	2.28
2–31	0.03	1.84
2–5	0.06	3.67
3–31	0.14	2.84
3–5	0.07	2.20
3–22	0.14	2.13
3–27	0.13	3.28
3–28	0.09	3.64
3–30	0.04	1.48
4–25	0.24	1.71
4–27	0.16	1.84
4–28	0.11	1.15
4–29	0.06	2.59
4–30	0.05	3.49
5–13	0.05	2.43
5–17	0.07	2.14
5–26	0.07	2.45
5–33	0.04	3.64
5–22	0.08	1.67
5–31	0.08	2.93
5–32	0.07	3.51
5–27	0.06	1.58
5–28	0.05	2.61
5–30	0.07	2.13
6–14	0.05	1.85
6–7	0.04	1.68
6–10	0.08	2.14
6–21	0.05	2.86
6–17	0.09	2.96
6–26	0.09	2.53
6–33	0.02	2.39
6–32	0.08	3.69
7–11	0.06	1.72
7–14	0.03	3.16
7–19	0.10	2.14
7–20	0.02	2.27
7–8	0.05	2.14
7–9	0.34	2.79
7–21	0.07	2.55
7–10	0.09	2.70
8–11	0.01	3.69
8–20	0.05	2.97
8–10	0.15	1.85
8–9	0.29	1.97
9–11	0.29	3.56
10–32	0.08	2.44
10–33	0.11	1.58
10–20	0.11	4.46
10–13	0.13	2.97
11–14	0.08	3.62
11–19	0.14	1.99
11–21	0.13	1.56
11–20	0.05	2.10
12–23	0.03	1.94
12–24	0.04	1.48
12–18	0.13	2.49
12–16	0.14	3.01
12–14	0.12	2.93
12–21	0.09	1.67
13–26	0.12	1.76
13–33	0.02	1.43
13–31	0.13	2.12
13–32	0.11	1.13
13–17	0.11	2.16
13–22	0.13	1.74
14–15	0.05	2.94
14–16	0.04	1.83
14–17	0.04	2.25
14–19	0.08	1.69
14–21	0.05	2.66
14–24	0.08	1.91
15–17	0.01	2.17
15–16	0.01	1.80
15–18	0.14	3.25
15–25	0.10	2.06
15–27	0.02	1.56
16–23	0.12	3.30
16–24	0.09	1.54
16–17	0.01	2.71
16–18	0.13	1.84
16–25	0.09	2.49
17–26	0.01	0.63
17–27	0.01	2.27
17–33	0.10	2.48
18–23	0.15	1.70
18–24	0.10	1.62
18–25	0.07	2.61
19–21	0.10	1.06
21–24	0.08	2.39
22–31	0.01	1.27
22–32	0.05	2.46
23–24	0.05	2.26
25–28	0.14	2.84
26–33	0.10	1.86
27–28	0.06	1.28
27–30	0.13	2.42
28–29	0.17	1.65
28–30	0.07	2.35
29–30	0.11	1.81
31–32	0.06	2.22
32–33	0.09	1.30

of an already established network. In this case we must redesign the network by changing the configuration or improving the measurements until we obtain a network of acceptable robustness.

It is planned to further develop the technique to analyze all three dimensions of 3D networks.

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Notation

The following symbols are used in this paper:

$\mathbf{A}$ = design matrix;
$\mathbf{E}$ = strain matrix;
$\mathbf{P}$ = weight matrix;
r = redundancy number;
r_{ij} = semimajor axis of the confidence region;
u = displacement in the x direction;
v = displacement in the y direction;
$\Delta\mathbf{l}$ = maximum undetectable error;
$\Delta\mathbf{x}$ = displacement;
$\delta_{r_{ij}}$ = relative displacement;
$\sqrt{\lambda_0}$ = noncentrality parameter; and
σ = standard deviation.

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