BOLLETTINO DI GEOFISICA TEORICA ED APPLICATA VOL. XXVII, N. 108 - DICEMBRE 1985

P. VANÍČEK* and A. KLEUSBERG**

WHAT AN EXTERNAL GRAVITATIONAL POTENTIAL CAN REALLY TELL US ABOUT MASS DISTRIBUTION

Summary. A three-dimensional orthogonal series for density distribution within the Earth is studied. This series is chosen so that it allows a direct comparison of its coefficients with those of the harmonic development of the gravitational potential. Through this comparison it is possible to show that some specific density series coefficients are easily determinable from potential coefficients while the rest is not determinate. Thus a density series where the determinable coefficients are all put equal to zero, represents the most general Schiaparelli's "body of vanishing outer potential". The roles of the isotropic and anisotropic parts of the density distribution are discussed. As a by-product an expression for harmonic density compatible with a known external gravitational potential is also derived.

Received September 25, 1983; revised version July 25, 1984

1. Introduction

When one starts looking into the classical inverse problem of gravimetry, i.e., the determination of the (Earth) mass distribution from the external potential field, one quickly runs into confusion. On the one hand there are assurances like: "No potential due to spreads in regular regions and on regular surfaces, finite in number, with continuous densities and moments, can be due to any other spreads of the same character" (Kellogg, 1929, Chap. VIII, Theorem IX). On the other hand, one learns that it is not possible to obtain the mass distribution from external potential (Krarup, 1978; Marussi, 1980).

In statistics, it is known that given all the moments of a bounded cumulative probability distribution function, the function can be reconstructed in a unique way (Wilks, 1962). A completely analogous situation must, of course, exist in mechanics where one may take moments of inertia of all orders and look into the determination of mass density distribution (see, e.g., Shohat and Tamarkin, 1943). The question then reduces to whether it is possible to uniquely obtain all the moments of inertia from external gravitational potential. Pizzetti (1910) tells us that only $2n + 1$ out of $(n + 1)(n + 2)/2$ independent elements of the symmetrical tensor of rank n can be derived from potential coefficients of order n, leaving $n(n - 1)/2$ elements indeterminate. But an enquiry into the "moment problem" shows that only certain linear combinations of moments are needed to reconstruct the density. What is then the real story? That is what we had set out to investigate and the results of our investigation are presented here below.

* University of Toronto, Erindale College, Mississauga, Ontario L5L 1C6, Canada
** Geodätisches Institut, Universität Stuttgart, 7000 Stuttgart 1, FRG

Present address of the authors:
Dept. of Surveying Engineering, University of New Brunswick, Fredericton, N.B., Canada E3B 5A3

In a longhand form, eqn. (13) reads

$$c_{ijk} = (c^c_{ijk}, c^s_{ijk}) = \int_S (\widetilde{Y^c}_{jk}(\theta, \lambda), \widetilde{Y^s}_{jk}(\theta, \lambda)) \left[\int_0^1 \rho(r) \widetilde{G}_i(3, 3, r)\, dr\right] dS$$

$$= \int_S (\widetilde{Y^c}_{jk}(\theta, \lambda), \widetilde{Y^s}_{jk}(\theta, \lambda)) \left[\int_0^1 r^2 \rho(r) \widetilde{G}_i(3, 3, r)\, dr\right] d\upsilon, \tag{14}$$

where S is the spherical surface of radius r and $d\upsilon$ is a solid angle differential:

$$dS = r^2\, d\upsilon = r^2\, d\theta \cos\theta\, d\lambda. \tag{15}$$

It will be shown later that these coefficients are very closely linked with potential coefficients (eqn. (5)). Before this link is shown, however, we wish to clarify the role played by the isotropic (dependent on radius only) part of the mass distribution.

4. Isotropic and anisotropic parts of the mass distribution

It is very interesting to see what happens if we divide the density ρ into isotropic, ρ_I (i.e., depending only on r), and anisotropic, $\delta\rho$, parts as follows:

$$\rho = \rho_I + \delta\rho. \tag{16}$$

Clearly, for this equation to be satisfied, $\delta\rho$ must be such that

$$\forall\, r \in \langle 0, 1\rangle : \int_S \delta\rho(r)\, dS = 0. \tag{17}$$

Splitting the density according to eqn. (16), eqn. (13) can be rewritten as

$$\begin{aligned} c_{ijk} &= \int_B \rho_I(r)\, \phi_{ijk}(r)\, dB + \int_B \delta\rho(r)\, \phi_{ijk}(r)\, dB \\ &= c^{(I)}{}_{ijk} + c^{(A)}{}_{ijk}. \end{aligned} \tag{18}$$

Realizing that

$$\rho_I(\mathbf{r}) = \rho_I(r),$$

the isotropic part can then be rewritten as

$$\begin{aligned} c^{(I)}{}_{ijk} &= \int_S \widetilde{Y}_{jk}(\theta, \lambda) \left[\int_0^1 r^2 \rho_I(r) \widetilde{G}_i(3, 3, r)\, dr\right] d\upsilon \\ &= \int_0^1 r^2 \rho_I(r) \widetilde{G}_i(3, 3, r)\, dr \int_S \widetilde{Y}_{jk}(\theta, \lambda)\, d\upsilon. \end{aligned} \tag{19}$$

Here the second integral equals to zero for $j \neq 0 \vee k \neq 0$ because of the orthogonality of (surface) spherical harmonic functions.

Realizing that $\widetilde{Y}_{oo}(\theta, \lambda) = (4\pi)^{1/2}$ we can conclude that, except for

$$\begin{aligned} c^{(I)}{}_{ioo} &= (4\pi)^{1/2} \int_B \rho(\mathbf{r}) \widetilde{G}_i(3, 3, r)\, dB = (4\pi)^{1/2} \\ &\int_0^1 r^2 \rho_I(r) \widetilde{G}_i(3, 3, r)\, dr,\ i = 0, 1, \ldots, \end{aligned} \tag{20}$$

the isotropic part of density distribution does not affect any other coefficients. Arbitrary isotropic distribution can thus be added to the actual distribution without any changes ensuing to higher order coefficients. Also, we note that the isotropic part of density distribution is uniquely represented by c_{ioo}, $i = 0, 1, \ldots$.

5. Relation between density coefficients and potential coefficients

Let us now get back to the expression for the density coefficients. Equation (13) can be rewritten as

$$c_{ijk} = \int_B \rho(r)\, \widetilde{G}_i(3, 3, r)\, \widetilde{Y}_{jk}(\theta, \lambda)\, dB = \int_B \rho(r) \sum_{l=0}^{i} b_l^{(i)} r^l\, \widetilde{Y}_{jk}(\theta, \lambda)\, dB. \tag{21}$$

This equation can also be written as

$$c_{ijk} = \sum_{l=0}^{i} b_l^{(i)} \int_B \rho(r)\, r^l\, \widetilde{Y}_{jk}(\theta, \lambda)\, dB. \tag{22}$$

On the other hand, for a unit ball B,

$$\int_B \rho(r)\, r^j (Y^c_{jk}(\theta,\lambda), Y^s_{jk}(\theta,\lambda))\, dB = \frac{p_k\,(j+k)!}{4G\,(j-k)!}(A_{jk}, B_{jk}) \tag{23}$$

(cf. Vaníček and Krakiwsky, 1982), where G is the gravitational constant.
Thus:

$$\int_B \rho(r)\, r^j (\widetilde{Y}^c_{jk}, \widetilde{Y}^s_{jk})\, dB = \frac{1}{4G}\left(\frac{p_k\,(j+k)!\,(2j+1)}{\pi\,(j-k)!}\right)^{1/2} (A_{jk}, B_{jk}), \tag{24}$$

which provides the sought relation between the density coefficients and potential coefficients. We note that the relation is defined only when the power of r coincides with the first subscript of the harmonic functions.

Let us now denote

$$\overset{*}{c}_{ljk} = (c^{c*}_{ljk}, c^{s*}_{ljk}) = \int_B \rho(r)\, r^j (Y^c_{jk}, Y^s_{jk})\, dB. \tag{25}$$

Then clearly,

$$c_{ijk} = \sum_{l=0}^{i} b_l^{(i)}\, \overset{*}{c}_{ljk} = \sum_{l=0}^{i} \widetilde{c^{(i)}}_{ljk} \tag{26}$$

and eqn. (12) reads

$$\rho(r) = \sum_{i=0}^{\infty} \sum_{l=0}^{i} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \widetilde{c^{(i)}}_{ljk}\, \phi_{ijk}(r). \tag{27}$$

The coefficients $c^{(i)}_{ljk}$ can be uniquely determined from $\overset{*}{c}_{ljk}$, out of which only $\overset{*}{c}_{jjk}$ are (linear) functions of potential coefficients. The remaining $\overset{*}{c}_{ljk}$ (i.e., those for which $l \neq j$) *cannot be determined from the external potential.*

It is interesting to note that a hypothetical mass density given by

$$\rho^* (r) = \sum_{i=0}^{\infty} \sum_{\substack{l=0 \\ l \neq j}}^{i} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \widetilde{c^{(i)}}_{ljk} \, \phi_{ijk} (r), \tag{28}$$

with arbitrarily selected coefficients $c^{(i)}_{ljk}$, may be added to the actual mass density of $\rho (r)$ without changing its external potential. Because of (12) and (27), expression (28) thus evidently represents the most general description of a body of vanishing outer potential studied by Schiaparelli (1875-76) and others.

6. Case of harmonic density

It seems rather obvious that, if we are willing to regard density coefficients as three-dimensional quantities (because of its 3 subscripts) then only two dimensions are recoverable from the external potential, (with the exception of an occasional special combination of indeces). The third dimension, corresponding in an oblique manner to depth, has to be obtained from some other source.

It is easy to show that, in the unrealistic case of harmonic density, the depth variation of the coefficients is determinable. Let us assume a density distribution $H(r)$ harmonic in B. According to Heiskanen and Moritz (1966), we can write:

$$H (r) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} r^j C_{jk} \widetilde{Y}_{jk} (\theta, \lambda). \tag{29}$$

Substitution of H for ρ in eqn. (25) yields

$$\begin{aligned} \dot{c}_{lnm} (H) &= \int_B \sum_{j=0}^{\infty} \sum_{k=0}^{j} r^j C_{jk} \widetilde{Y}_{jk} (\theta, \lambda) \, r^l \, \widetilde{Y}_{nm} (\theta, \lambda) \, dB \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} C_{jk} \int_B r^{j+l} \, \widetilde{Y}_{jk} (\theta, \lambda) \, \widetilde{Y}_{nm} (\theta, \lambda) \, dB \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} C_{jk} \int_0^1 r^{j+l} \, dr \int_S \widetilde{Y}_{jk} (\theta, \lambda) \, \widetilde{Y}_{nm} (\theta, \lambda) \, dS. \end{aligned} \tag{30}$$

In the last equation, due to the orthonormality of normalized spherical harmonic functions, we have:

$$\int_S Y_{jk} (\theta, \lambda) \, Y_{nm} (\theta, \lambda) \, dS = r^2 \, \delta^{jk}{}_{nm} \tag{31}$$

and eqn. (30) reduces to

$$\dot{c}_{lnm} (H) = C_{nm} \int_0^1 r^{n+l+2} \, dr. \tag{32}$$

Since

$$\int_0^1 r^{n+l+2} \, dr = (n + l + 3)^{-1}, \tag{33}$$

we finally get

$$\dot{c}_{lnm} (H) = \frac{C_{nm}}{3 + n + l}. \tag{34}$$

This equation links the three-dimensional density coefficients c_{lnm} with the two-dimensional (superficial) density coefficients C_{nm} determinable on S.

The question now arises naturally as to whether any realistic property of ρ could be used to obtain the third dimension in the density coefficients. As discussed by Tscherning and Sünkel (1980) - on a slightly different case of an ellipsoidal integration domain - it is not too difficult to come up with a variety of artificial "depth functions" that can replace harmonicity to supply the third dimension.

7. Conclusions

A three-dimensional series for Earth's density distribution has been constructed such that it permits direct comparison of its coefficients with external gravitational potential coefficients. The base functions of the series are orthogonal with weight equal to r^2 on a ball of unit radius that contains the Earth. By means of this series it is possible to show explicitly that:

(i) Any isotropic distribution, e.g., such as those obtained from seismic data, is compatible with any external gravitational potential provided its first coefficient c_{ooo} and the potential coefficient A_{oo} are compatible. The compatibility of these two is ensured by

$$\begin{aligned} 2\sqrt{\pi}\,A_{oo} = GM &= G \int_B \sum_{i=0}^{\infty} c_{ioo}\, \widetilde{G}_i(3, 3, r)\, dB \\ &= 4\pi G \sum_{i=0}^{\infty} c_{ioo} \int_0^1 \widetilde{G}_i(3, 3, r)\, r^2\, dr \\ &= 4\pi G\, c_{ooo} / \sqrt{3} \end{aligned} \tag{35}$$

where M is the mass of the Earth.

(ii) Even in the anisotropic density series, there are only very few coefficients sensitive to, and thus determinable from external gravitational potential. The complete expression for the density distribution reads:

$$\begin{aligned} \rho(\mathbf{r}) = {} & \frac{A_{oo}}{2\sqrt{\pi}\,G} \sum_{i=o}^{\infty} b^{(i)}{}_o\, \widetilde{G}_i(3, 3, r) + \sum_{i=1}^{\infty} \sum_{l=1}^{i} \widetilde{c^{(i)}}{}_{loo}\, \widetilde{G}_i(3, 3, r) + \\ & + \frac{1}{4\sqrt{\pi}\,G} \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} b_j^{(i)} \sum_{k=0}^{j} \left(\frac{p_k\,(j+k)!\,(2j+1)}{(j-k)!} \right)^{1/2} (A_{jk}, B_{jk})\, \phi_{ijk}(\mathbf{r}) \\ & + \sum_{i=0}^{\infty} \sum_{\substack{l=0 \\ l \neq i}}^{i} \sum_{j=1}^{\infty} \sum_{k=0}^{j} \widetilde{c^{(i)}}{}_{ljk}\, \phi_{ijk}(\mathbf{r}). \end{aligned} \tag{36}$$

Here,for any distribution the first two terms describe the isotropic part of the distribution, the last two terms refer to the anisotropic part. While the first and third terms are determinable from a known external potential, the second and fourth are not; these represent a body of vanishing external potential (cf. 28). They give the most general expression for Schiaparelli's body (of vanishing external potential). For a density distribution constant in the ball, only the first term in the first series is different from zero, and for any distribution $c_{ooo} = \sqrt{3}\, M/(4\pi)$.

Clearly, there are infinitely many density distributions compatible with a known external gravitational potential. They can be obtained from each other by the addition of a Schiaparelli's body. If one desires to obtain a realistic distribution of density within the Earth, one has to use not only the known gravitational potential but to combine it with some other physical concepts. One source of information that must be taken into account is the actual shape of the Earth submerged in the ball of a unit radius. Another source should be seismology; as we have already seen, the isotropic part of distribution obtainable from seismic data complements the gravity information. Yet another source may be the knowledge (or a hypothesis) of material strengths neded to support the stresses induced by a density distribution. This possibility appears to be worth pursuing.

As a by-product, an expression for a density distribution compatible with a known external gravitational potential and harmonic within the unit ball has been derived. This solution has no practical use in geophysics. It is however of some theoretical interest.

Acknowledgements. We should like to acknowledge that most of the research described herein was carried out while the senior author was visiting professor at the University of Stuttgart. Sponsorship of the Natural Sciences and Engineering Research Council of Canada (grant no. IC-0084) is also gratefully acknowledged. Discussions with Prof. Dr. E.W. Grafarend have helped us in clarifying several important points.

References

Abramowitz M. and Stegun I.A.; 1964: *Handbook of mathematical functions.* Dover reprint 1965.

Aubin J.P.; 1979: *Applied functional analysis.* Wiley.

Heiskanen W. and Moritz H.; 1966: *Physical geodesy.* Freeman.

Hobson E.W.; 1931: *The theory of spherical and ellipsoidal harmonics.* Cambridge University Press.

Kellogg D.D.; 1929: *Foundations of potential theory.* Springer.

Krarup T.; 1978: *Some remarks about collocation, in: approximation methods in geodesy.* H. Wichmann, Karlsruhe.

Marussi A.; 1980: *On the density distribution in bodies of assigned outer Newtonian attraction.* Boll. Geof. Theor. App. XXII (86), no pagination.

Pizzetti P.; 1910: *Intorno alle possibili distribuzioni della massa all'interno della Terra.* Annali Matem., Pura Appl. XVII, Milano.

Schiaparelli G.V.; 1875-76: *Lezioni di Meccanica Celeste (Appunti).* Univ. di Pavia.

Shohat J.A. and Tamarkin J.D.; 1943: *The problem of Moments.* American Math. Soc.

Tscherning C.C. and Sünkel H.; 1980: *A method for the construction of spheroidal mass distribution consistent with the harmonic part of the earth's gravity potential.* Proceedings 4th International Symposium "Geodesy and Physics of the Earth", Zentralinstitut für Physik der Erde, Report 63/II, pp. 481-500.

Vaníček P. and Krakiwsky E.J.; 1982: *Geodesy: the Concepts.* North Holland.

Wilks S.S.; 1962: *Mathematical Statistics.* Wiley.