A Review of the UNB Approach for Precise Geoid Determination Based on the Stokes-Helmert Method

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ABSTRACT

Many methods for geoid determination exist and are in use around the world. One of the most advantageous methods is the Stokes-Helmert approach developed at the University of New Brunswick. The main theoretical developments of this method is attributed to Vaníček, along with the contribution of other authors, such as Martinec, Sjöberg, Kleusberg, Heck and Grafarend. The theoretical aspects of the UNB approach were published in more than fifty contributions (see References) and the general principles are summarized in Vaníček and Martinec (1994), Vaníček et al. (1999), Novák (2000), and Vaníček and Janák (2001). The main idea of this contribution is to offer to readers, in a more detailed form, the basic theoretical aspects of the Stokes-Helmert approach for geoid determination. Another purpose is to summarize publications related to this topic.

Key words: Atmospheric Effect – Bruns's Formula – Density of Topographical and Atmospheric Masses – Geodetic Boundary-Value Problem – Geoid – Gravitational Attraction – Gravitational Potential – Gravity – Gravity Anomaly – Helmert Space – Helmert's Second Condensation Method – Newton's Integral – Orthometric Height – Poisson's Integral – Stokes's Integral – Terrain – Topographical Effect

1 Introduction

In the classical sense of Gauss and Listing, the geoid is defined as an equipotential surface of the Earth's gravity field with the gravity potential value W_O. Gauss (1828) was the first to define this surface in the strict mathematical sense as a surface which is intersected everywhere by directions of gravity at right angle and which best approximates the mean sea level over the whole Earth. Later, Bessel (1837) stipulated this equipotential surface as a reference for all geodetic applications. Finally, Listing (1873) called this surface "geoid".

Stokes (1849) derived a theorem, which forms a theoretical foundation for estimation of the geoid based on gravity observations that refer to the geoid (assuming as harmonic the space above the geoid). The requirement of harmonicity was difficult to fulfill in practical applications of the Stokes theory since the distribution of actual topographical density between the geoid and the Earth's surface is not known with sufficient accuracy.

The first attempt to satisfy this requirement can be attributed to Helmert (1884). Helmert suggested that the Earth's topographical masses can be replaced by an infinitesimal condensation layer of a surface density that is equal to the product of the mean topographical density and height of the Earth's surface above the geoid. This layer could be located anywhere on or beneath the geoid without violating the required assumption of harmonicity. In the second condensation method that Helmert formulated, the condensation layer is placed right on the geoid (Lambert, 1930; Heck, 1992; Martinec et al., 1993).

According to Newton's theory of gravity, Martinec and Vaníček formulated principles for the description of the effect of topographical masses on the gravitational potential and attraction in the case of laterally varying topographical density distribution and for the spherical approximation of the geoid (Martinec, 1993; Martinec and Vaníček, 1994a, b). Sjöberg (1998 and 1999) and Novák (2000) studied the effect of the atmospheric masses in the Stokes-Helmert method of geoid determination.

Based on Molodensky's theory (Molodensky et al., 1960), Vaníček and Kleusberg (1987) introduced the idea of modification of the Stokes function to separate the reference and higher-degree gravity field. Theory of the reference gravity field and the spheroid, and the reformulation of Stokes's boundary-value problem for the higher-degree reference spheroid. were described by Vaníček and Kleusberg (1987), Vaníček and Sjöberg (1991), Vaníček et al. (1995), Vaníček and Featherstone (1998).

The solution of Dirichlet's boundary-value problem by applying the Poisson integral equation for the downward continuation of Helmert's gravity anomalies was investigated by Martinec (1996), Vaníček et al. (1996), Sun and Vaníček (1998) and Huang (2002).

The principle of the Stokes-Helmert scheme of geoid determination can be summarized in the following scheme (Vaníček et al., 1999; Vaníček and Janák, 2001):

- Formulation of the boundary-value problem of the third kind on the Earth's surface.
- Transformation of the boundary-value problem into a harmonic space, i.e., transformation of gravity anomalies from the real to Helmert space (according to the second condensation technique where the topographical and atmospheric masses are condensed directly onto the geoid).

- Solution of Dirichlet's boundary-value problem by applying the Poisson integral equation, i.e., the downward continuation of Helmert's gravity anomalies from the Earth's surface to the geoid.
- Reformulation of the geodetic boundary-value problem by decomposition of Helmert's gravity field into a low and high-frequency gravity field.
- Solution of the Stokes boundary-value problem for the high-frequency Helmert gravity field (by using the modified spheroidal Stokes kernel) and evaluation of Helmert's reference spheroid (from a satellite geopotential model).
- Transformation of the equipotential surface from the Helmert space back into the real space.

2. Geodetic boundary-value problem in the real space

Let us begin with the definition of the disturbing gravity potential $T(r_t(\Omega))$ which is reckoned at the Earth's surface, $\Omega \in \Omega_O : r_t(\Omega) = r_g(\Omega) + H^O(\Omega)$, as the difference of the Earth's gravity potential $W(r_t(\Omega))$ and the normal gravity potential $U(r_t(\Omega))$ generated by the reference geocentric ellipsoid of revolution (Somigliana, 1929; Pizzeti, 1894 and 1911)

$$\Omega \in \Omega_O : T(r_t(\Omega)) = W(r_t(\Omega)) - U(r_t(\Omega)), \tag{2.1}$$

where $r_t(\Omega)$ stands for the geocentric radius of the Earth's surface, $r_g(\Omega)$ is the geocentric radius of the geoid and $H^O(\Omega)$ is the orthometric height. A pair of the geocentric coordinates ϕ and λ represent the geocentric direction $\Omega = (\phi, \lambda)$ while Ω_O stands for the total solid angle $\left[\phi \in \langle -\pi/2, \pi/2 \rangle, \lambda \in \langle 0, 2\pi \rangle\right]$. Eqn. (2.1) is valid only if the normal gravity potential U_o on the reference ellipsoid equals to the gravity potential W_o on the geoid.

Approximating the geoid by the geocentric sphere of radius R, i.e., $\Omega \in \Omega_O : r_g(\Omega) \approx R$, the radial derivative of the disturbing gravity potential $T(r_t(\Omega))$ reads (Vaníček et al., 1999)

$$\forall\, \Omega \in \Omega_O :$$

$$\left.\frac{\partial\, T(r,\Omega)}{\partial\, r}\right|_{r=R+H^O(\Omega)} = \left.\frac{\partial\, W(r,\Omega)}{\partial\, r}\right|_{r=R+H^O(\Omega)} - \left.\frac{\partial\, U(r,\Omega)}{\partial\, r}\right|_{r=R+H^O(\Omega)} =$$

$$= \left|\,\mathbf{grad} W(r_t(\Omega))\right| \cos\left(\mathbf{grad} W(r_t(\Omega)), \mathbf{r^o}\right) - \left|\,\mathbf{grad} U(r_t(\Omega))\right| \cos\left(\mathbf{grad} U(r_t(\Omega)), \mathbf{r^o}\right), \tag{2.2}$$

where $\mathbf{r^o}$ is the unit vector in the radial direction. The vertical gradient of the gravity potential $W(r_t(\Omega))$ and the vertical gradient of the normal gravity potential define gravity $g(r_t(\Omega))$ and normal gravity $\gamma(r_t(\Omega))$:

$$\forall\, \Omega \in \Omega_O : \left| \mathbf{grad} W(r_t(\Omega)) \right| = -\left| \mathbf{g}(r_t(\Omega)) \right| = -g(r_t(\Omega)), \tag{2.3}$$

$$\forall\, \Omega \in \Omega_O : \left| \mathbf{grad} U(r_t(\Omega)) \right| = -\left| \boldsymbol{\gamma}(r_t(\Omega)) \right| = -\gamma(r_t(\Omega)). \tag{2.4}$$

The angle between the plumb line and the radial direction $\angle\left(-\mathbf{g}\,,\mathbf{r^o}\right)$, and the angle between the normal to the reference ellipsoid and the radial direction $\angle\left(-\boldsymbol{\gamma}\,,\mathbf{r^o}\right)$ can be written with sufficient accuracy as follows (Vaníček et al., 1999) :

$$\cos\left(-\mathbf{g}\,,\mathbf{r^o}\right) \cong 1 - \frac{\beta_g^2}{2}\,, \tag{2.5}$$

$$\cos\left(-\boldsymbol{\gamma}\,,\mathbf{r^o}\right) \cong 1 - \frac{\beta_\gamma^2}{2}\,. \tag{2.6}$$

Substituting Eqns. (2.3-2.6) back to Eqn. (2.2), the radial derivative of the disturbing gravity potential becomes (Vaníček et al., 1999)

$$\begin{aligned} &\forall\, \Omega \in \Omega_O : \\ &\left. \frac{\partial\, T(r,\Omega)}{\partial\, r} \right|_{r=R+H^O(\Omega)} = -g(r_t(\Omega)) + \gamma(r_t(\Omega)) + \frac{g(r_t(\Omega))}{2}\beta_g^2(r_t(\Omega)) - \frac{\gamma(r_t(\Omega))}{2}\beta_\gamma^2(r_t(\Omega)) = \\ &= -\delta g(r_t(\Omega)) + \varepsilon_{\delta g}(r_t(\Omega)), \end{aligned} \tag{2.7}$$

where the difference of gravity $g(r_t(\Omega))$ and normal gravity $\gamma(r_t(\Omega))$ defines the gravity disturbance, $\delta g(r_t(\Omega)) = g(r_t(\Omega)) - \gamma(r_t(\Omega))$, $\varepsilon_{\delta g}(r_t(\Omega))$ is the „*ellipsoidal correction to the gravity disturbance*“ (ibid)

$$\begin{aligned} &\forall\, \Omega \in \Omega_O : \\ &\varepsilon_{\delta g}(r_t(\Omega)) = \delta g(r_t(\Omega))\frac{\beta_\gamma^2(r_t(\Omega))}{2} + g(r_t(\Omega))\beta_\gamma(r_t(\Omega))\xi(r_t(\Omega)) + \frac{g(r_t(\Omega))}{2}\theta^2(r_t(\Omega)), \end{aligned} \tag{2.8}$$

and ξ, η stand for the components of the deflection of a vertical θ, $\theta = \sqrt{\xi^2 + \eta^2}$.
The angle β_γ is the difference of the geodetic latitude φ and geocentric latitude ϕ that is given by Bomford (1971)

$$\beta_{\gamma} \cong f \sin 2\varphi , \tag{2.9}$$

where $f = (a-b)/a$ is the first geometric flattening of the reference ellipsoid.

Considering only the second term on the right-hand side of Eqn. (2.8), the ellipsoidal correction to the gravity disturbance $\varepsilon_{\delta g}(r_t(\Omega))$ can be evaluated with sufficient accuracy by (Vaníček et al., 1999)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : \varepsilon_{\delta g}(r_t(\Omega)) \cong g(r_t(\Omega))\, f \sin 2\varphi\, \xi(r_t(\Omega)) \cong -\frac{f \sin 2\varphi}{r_t(\Omega)} \frac{\partial T(r,\Omega)}{\partial \varphi} . \tag{2.10}$$

Since the computation of normal gravity $\gamma(r_t(\Omega))$ on the Earth's surface requires the knowledge of the geodetic height $h(\Omega)$ above the reference ellipsoid, the gravity disturbance $\delta g(r_t(\Omega))$ is transformed into the gravity anomaly $\Delta g(r_t(\Omega))$. Gravity anomaly is given as a difference of gravity $g(r_t(\Omega))$ on the Earth's surface and normal gravity $\gamma(H^{\mathrm{N}}(\Omega))$ on the telluroid, i.e., $\forall\, \Omega \in \Omega_{\mathrm{O}} : r(\Omega) \cong r_o(\Omega) + H^{\mathrm{N}}(\Omega)$, see (Vaníček et al., 1999)

$$\begin{aligned}&\forall\, \Omega \in \Omega_{\mathrm{O}} : \\ &\Delta g(r_t(\Omega)) = g(r_t(\Omega)) - \gamma(H^{\mathrm{N}}(\Omega)) = \delta g(r_t(\Omega)) + \gamma(r_t(\Omega)) - \gamma(H^{\mathrm{N}}(\Omega)) + \varepsilon_{\delta g}(r_t(\Omega)),\end{aligned} \tag{2.11}$$

where $r_o(\Omega)$ is the geocentric radius of the reference ellipsoid and $H^{\mathrm{N}}(\Omega)$ stands for the normal height (Molodensky, 1945).

Considering Molodensky's approach (Molodensky et al., 1960), the difference of normal gravity $\gamma(r_t(\Omega))$ on the Earth's surface, $\Omega \in \Omega_{\mathrm{O}} : r_t(\Omega) = r_g(\Omega) + H^{\mathrm{O}}(\Omega) \cong r_o(\Omega) + h(\Omega)$, and normal gravity $\gamma(H^{\mathrm{N}}(\Omega))$ on the telluroid can be defined as

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : \gamma(r_t(\Omega)) - \gamma(H^{\mathrm{N}}(\Omega)) = \left| \mathbf{grad}\,\gamma(r_t(\Omega)) \right| \varsigma(\Omega) = \left. \frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}} \right|_{r = R + H^{\mathrm{O}}(\Omega)} \varsigma(\Omega), \tag{2.12}$$

where the derivative of normal gravity is taken with respect to the normal n to the reference ellipsoid and $\varsigma(\Omega)$ is the height anomaly (Molodensky et al., 1960). Using Bruns's spherical formula (Bruns, 1878), the expression on the right-hand side of Eqn. (2.12) can be rewritten as (Vaníček et al., 1999)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : \left.\frac{\partial \gamma(r,\Omega)}{\partial\ \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \varsigma(\Omega) = \left.\frac{\partial \gamma(r,\Omega)}{\partial\ \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \frac{T(r_t(\Omega))}{\gamma(H^{\mathrm{N}}(\Omega))}. \tag{2.13}$$

Substituting Eqn. (2.13) into Eqn. (2.11), the fundamental boundary condition takes the following form

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : \Delta g(r_t(\Omega)) = \delta g(r_t(\Omega)) + \varepsilon_{\delta g}(r_t(\Omega)) + \left.\frac{\partial \gamma(r,\Omega)}{\partial\ \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \frac{T(r_t(\Omega))}{\gamma(H^{\mathrm{N}}(\Omega))}. \tag{2.14}$$

Applying the following spherical approximation

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : \frac{1}{\gamma(H^{\mathrm{N}}(\Omega))} \left.\frac{\partial \gamma(r,\Omega)}{\partial\ \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} T(r_t(\Omega)) = -\frac{2}{r_t(\Omega)} T(r_t(\Omega)) - \varepsilon_{\mathrm{n}}(r_t(\Omega)), \tag{2.15}$$

the boundary condition in Eqn. (2.14) becomes (Vaníček et al., 1999)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :$$

$$\Delta g(r_t(\Omega)) = -\left.\frac{\partial\ T(r,\Omega)}{\partial\ r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \varepsilon_{\delta g}(r_t(\Omega)) - \frac{2}{r_t(\Omega)} T(r_t(\Omega)) - \varepsilon_{\mathrm{n}}(r_t(\Omega)). \tag{2.16}$$

The „*ellipsoidal correction for the spherical approximation*“ $\varepsilon_{\mathrm{n}}(r_t(\Omega))$ can be derived in the following form (Vaníček and Martinec, 1994)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : \varepsilon_{\mathrm{n}}(r_t(\Omega)) \cong 2\left[m + f\left(\cos 2\varphi - \frac{1}{3}\right)\right] \frac{T(r_t(\Omega))}{R}, \tag{2.17}$$

where $m = \omega^2 a^3 / GM$ stands for the Clairaut constant (Heiskanen and Moritz, 1967), GM is the geocentric gravitational constant, ω is the mean angular velocity of the Earth's rotation, and the mean radius of the Earth R can be evaluated by the following formula (Vaníček and Krakiwsky, 1986)

$$R = \sqrt[3]{a^2 b}\ . \tag{2.18}$$

3. Geodetic boundary-value problem in the Helmert space

To investigate the geodetic boundary-value problem in the Helmert space, „*Helmert's disturbing gravity potential*“ $T^{\mathrm{H}}(r_t(\Omega))$ referred to the Earth's surface is defined by Vaníček et al. (1999)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}\ :\ T^{\mathrm{H}}\left(r_t(\Omega)\right) = T\left(r_t(\Omega)\right) - \delta V^{t}\left(r_t(\Omega)\right) - \delta V^{a}\left(r_t(\Omega)\right), \tag{3.1}$$

where $\delta V^{t}\left(r_t(\Omega)\right)$ and $\delta V^{a}\left(r_t(\Omega)\right)$ are the so-called residual gravitational potentials of topographical and atmospheric masses.

Assuming the mean angular velocity ω of the Earth's rotation is equal to the mean angular velocity of rotation of the reference ellipsoid, the disturbing gravity potential $T^{\mathrm{H}}(r,\Omega)$ is harmonic everywhere above the geoid, i.e., $\forall\ \Omega \in \Omega_{\mathrm{O}},\ r > r_g(\Omega)$: $\Delta T^{\mathrm{H}}(r,\Omega) = 0$ in the Helmert space.

„*Helmert's gravity*" $g^{\mathrm{H}}\left(r_t(\Omega)\right)$ is related to actual gravity $g\left(r_t(\Omega)\right)$ as follows (Vaníček et al., 1999):

$$\forall\ \Omega \in \Omega_{\mathrm{O}}\ :\ g^{\mathrm{H}}\left(r_t(\Omega)\right) = g\left(r_t(\Omega)\right) + \left.\frac{\partial \delta V^{t}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \left.\frac{\partial \delta V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \tag{3.2}$$

„*Helmert's gravity disturbance*" $\delta g^{\mathrm{H}}\left(r_t(\Omega)\right)$ defined as the negative vertical gradient of the Helmert disturbing gravity potential can be described as a sum of the negative radial derivative of the Helmert disturbing gravity potential $T^{\mathrm{H}}\left(r_t(\Omega)\right)$ and the ellipsoidal correction $\varepsilon_{\delta g}\left(r_t(\Omega)\right)$ to the gravity disturbance

$$\begin{aligned}
&\forall\ \Omega \in \Omega_{\mathrm{O}}\ : \\
&\delta g^{\mathrm{H}}\left(r_t(\Omega)\right) = -\left.\frac{\partial T^{\mathrm{H}}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \varepsilon_{\delta g}\left(r_t(\Omega)\right) = \\
&= g\left(r_t(\Omega)\right) - \gamma\left(r_t(\Omega)\right) + \varepsilon_{\delta g}\left(r_t(\Omega)\right) + \left.\frac{\partial \delta V^{t}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \left.\frac{\partial \delta V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)}.
\end{aligned} \tag{3.3}$$

The relation between the gravity disturbance $\delta g^{\mathrm{H}}\left(r_t(\Omega)\right)$ and gravity anomaly $\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right)$ in the Helmert space can be obtained from the boundary condition (Heiskanen and Moritz, 1967)

$$\begin{aligned}
&\forall\ \Omega \in \Omega_{\mathrm{O}}\ : \\
&\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right) = -\left.\frac{\partial T^{\mathrm{H}}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \left.\frac{\partial \gamma(r,\Omega)}{\partial \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \frac{T^{\mathrm{H}}\left(r_t(\Omega)\right)}{\gamma\left(H^{\mathrm{N}}(\Omega)\right)} + \varepsilon_{\delta g}\left(r_t(\Omega)\right) =
\end{aligned}$$

$$= \delta g^{\mathrm{H}}\left(r_t(\Omega)\right) + \gamma\left(r_t(\Omega)\right) - \gamma_o(\phi) - \left.\frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\right|_{r=r_o(\Omega)} H^{\mathrm{N}}(\Omega) -$$

$$- \frac{1}{\gamma\left(H^{\mathrm{N}}(\Omega)\right)} \left.\frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \left[\delta V^{t}\left(r_t(\Omega)\right) + \delta V^{a}\left(r_t(\Omega)\right)\right] + \varepsilon_{\delta g}\left(r_t(\Omega)\right), \tag{3.4}$$

where $\gamma_o(\phi)$ is normal gravity on the reference ellipsoid (Somigliana, 1929).

If Helmert's orthometric height $H^{\mathrm{O}}(\Omega)$ is used (Helmert, 1890), the „*geoid-quasigeoid correction*" has to be applied to the boundary condition formulated in the Helmert space (Vaníček et al., 1999). The geoid-quasigeoid correction, i.e., the difference of the normal and orthometric heights, can be approximately described as a function of the simple Bouguer gravity anomaly $\Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right)$, see (Martinec, 1993),

$$\forall\, \Omega \in \Omega_{\mathrm{O}} :\; H^{\mathrm{N}}(\Omega) - H^{\mathrm{O}}(\Omega) \cong H^{\mathrm{O}}(\Omega) \frac{\Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right)}{\gamma_o(\phi)}. \tag{3.5}$$

The formula for the simple Bouguer gravity anomaly $\Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right)$ reads (Heiskanen and Moritz, 1967)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} :\; \Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right) = g\left(r_t(\Omega)\right) - \gamma\left(H^{\mathrm{O}}(\Omega)\right) - 2\pi\, G \rho_o H^{\mathrm{O}}(\Omega), \tag{3.6}$$

where G is the Newton (universal) gravitational constant. The third term on the right-hand side of Eqn. (3.6) stands for gravitational attraction generated by the infinite Bouguer plate (with the mean topographical density ρ_o and thickness equal to the orthometric height $H^{\mathrm{O}}(\Omega)$ at the computation point). Substituting Eqn. (3.5) into the boundary condition in Eqn. (3.4), Helmert's gravity anomaly $\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right)$ becomes (Vaníček et al., 1999)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} :$$

$$\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right) = \delta g^{\mathrm{H}}\left(r_t(\Omega)\right) + \gamma\left(r_t(\Omega)\right) - \gamma_o(\phi) - \left.\frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\right|_{r=r_o(\Omega)} H^{\mathrm{O}}(\Omega) \left[1 + \frac{\Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right)}{\gamma_o(\phi)}\right] -$$

$$- \frac{1}{\gamma\left(H^{\mathrm{N}}(\Omega)\right)} \left.\frac{\partial \gamma(r,\Omega)}{\partial\, \mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)} \left[\delta V^{t}\left(r_t(\Omega)\right) + \delta V^{a}\left(r_t(\Omega)\right)\right] + \varepsilon_{\delta g}\left(r_t(\Omega)\right) =$$

$$= g\left(r_t(\Omega)\right) - \gamma\left(r_t(\Omega)\right) + \left.\frac{\partial \delta V^{t}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \left.\frac{\partial \delta V^{a}\left(r_t(\Omega)\right)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} +$$

$$+\gamma\left(r_t(\Omega)\right)-\gamma_o(\phi)-\left.\frac{\partial\gamma(r,\Omega)}{\partial\,\mathrm{n}}\right|_{r=r_o(\Omega)} H^{\mathrm{O}}(\Omega)\left[1+\frac{\Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right)}{\gamma_o(\phi)}\right]-$$
$$-\frac{1}{\gamma\left(H^{\mathrm{N}}(\Omega)\right)}\left.\frac{\partial\gamma(r,\Omega)}{\partial\,\mathrm{n}}\right|_{r=R+H^{\mathrm{O}}(\Omega)}\left[\delta V^{t}\left(r_t(\Omega)\right)+\delta V^{a}\left(r_t(\Omega)\right)\right]+\varepsilon_{\delta g}\left(r_t(\Omega)\right). \tag{3.7}$$

Introducing the free-air gravity anomaly (Heiskanen and Moritz, 1967)

$$\forall\ \Omega\in\Omega_{\mathrm{O}}:$$
$$\Delta g^{\mathrm{FA}}\left(r_t(\Omega)\right)=g\left(r_t(\Omega)\right)-\gamma\left(H^{\mathrm{O}}(\Omega)\right)=g\left(r_t(\Omega)\right)-\gamma_o(\phi)-\left.\frac{\partial\gamma(r,\Omega)}{\partial\,\mathrm{n}}\right|_{r=r_o(\Omega)} H^{\mathrm{O}}(\Omega), \tag{3.8}$$

and applying the spherical approximation from Eqn. (2.15), the boundary condition in Eqn. (3.7) can subsequently be written in the form (Vaníček et al., 1999)

$$\forall\ \Omega\in\Omega_{\mathrm{O}}:$$
$$\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right)=\Delta g^{\mathrm{FA}}\left(r_t(\Omega)\right)+\left.\frac{\partial\delta V^{t}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)}+\left.\frac{\partial\delta V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)}+\varepsilon_{\delta g}\left(r_t(\Omega)\right)+$$
$$+\frac{2}{R}H^{\mathrm{O}}(\Omega)\,\Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right)+\frac{2}{r_t(\Omega)}\delta V^{t}\left(r_t(\Omega)\right)+\frac{2}{r_t(\Omega)}\delta V^{a}\left(r_t(\Omega)\right)-\varepsilon_{\mathrm{n}}\left(r_t(\Omega)\right). \tag{3.9}$$

The second and third term on the right-hand side of Eqn. (3.9) are the direct topographical and atmospheric effects on gravitational attraction. The fifth term stands for the „*geoid-quasigeoid correction to the boundary-value problem*“, and the sixth and seventh terms represent the secondary indirect topographical and atmospheric effects on gravitational attraction.

Helmert’s gravity anomaly can be also formulated as a function of the Bouguer gravity anomaly. The complete Bouguer gravity anomaly $\Delta g^{\mathrm{CB}}\left(r_t(\Omega)\right)$ is defined by the following formula (Heiskanen and Moritz, 1967)

$$\forall\ \Omega\in\Omega_{\mathrm{O}}:$$
$$\Delta g^{\mathrm{CB}}\left(r_t(\Omega)\right)=\Delta g^{\mathrm{SB}}\left(r_t(\Omega)\right)+\delta g^{tc}\left(r_t(\Omega)\right)=g\left(r_t(\Omega)\right)-\gamma\left(H^{\mathrm{O}}(\Omega)\right)-2\pi\,G\rho_o H^{\mathrm{O}}(\Omega)+$$
$$+\delta g^{tc}\left(r_t(\Omega)\right)=\Delta g^{\mathrm{FA}}\left(r_t(\Omega)\right)-2\pi\,G\rho_o H^{\mathrm{O}}(\Omega)+\delta g^{tc}\left(r_t(\Omega)\right), \tag{3.10}$$

where $\delta g^{tc}\left(r_t(\Omega)\right)$ is the „*gravimetric terrain correction*“ (Vaníček et al., 1999), i.e., the correction for gravitational attraction of topography taken relative to the height of the evaluation point (r,Ω). The curvature effect $-8\pi\, G\rho_o \left[H^{\mathrm{O}}(\Omega)\right]^2 / R$ is usually not considered in the definition of the complete Bouguer gravity anomaly (Vaníček and Krakiwsky, 1986).

Substituting Eqn. (3.10) into Eqn. (3.9), the relation between the Helmert gravity anomaly $\Delta g^{\mathrm{H}}(r_t(\Omega))$ and the complete Bouguer gravity anomaly $\Delta g^{\mathrm{CB}}(r_t(\Omega))$ is given by Vaníček et al. (1999)

$$\begin{aligned}
&\forall\, \Omega \in \Omega_{\mathrm{O}} : \\
&\Delta g^{\mathrm{H}}(r_t(\Omega)) = \Delta g^{\mathrm{CB}}(r_t(\Omega)) + 2\pi\, G\rho_o H^{\mathrm{O}}(\Omega) - \delta g^{tc}(r_t(\Omega)) + \varepsilon_{\delta g}(r_t(\Omega)) - \varepsilon_{\mathrm{n}}(r_t(\Omega)) + \\
&+ \left.\frac{\partial \delta V^t(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \left.\frac{\partial \delta V^a(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} + \\
&+ \frac{2}{R} H^{\mathrm{O}}(\Omega)\, \Delta g^{\mathrm{SB}}(r_t(\Omega)) + \frac{2}{r_t(\Omega)}\, \delta V^t(r_t(\Omega)) + \frac{2}{r_t(\Omega)}\, \delta V^a(r_t(\Omega)).
\end{aligned} \tag{3.11}$$

4. Effect of topographical masses on gravitational attraction

To evaluate the Helmert gravity anomaly $\Delta g^{\mathrm{H}}(r_t(\Omega))$ on the Earth's surface according to Eqn. (3.9), the topographical effect on gravitational attraction has to be computed. The topographical effect on gravitational attraction, which is reckoned on the Earth's surface, is represented by the direct and secondary indirect topographical effects (Martinec, 1993; Martinec and Vaníček, 1994a, b; Martinec et al., 1995 and 1996; Vaníček et al., 1995a and 1999; Novák et al., 2001; Huang et al., 2001).

4.1 Residual gravitational potential of topographical masses

The „*residual gravitational potential of topographical masses*" $\delta V^t(r_t(\Omega))$ is defined as a difference of the gravitational potential $V^t(r,\Omega)$ of topographical masses and gravitational potential $V^{ct}(r,\Omega)$ of topographical masses condensed according to the Helmert second condensation method directly onto the geoid (Martinec et al., 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r \in \Re^+ : \delta V^t(r,\Omega) = V^t(r,\Omega) - V^{ct}(r,\Omega). \tag{4.1}$$

The „*gravitational potential of topographical masses*" $V^t(r,\Omega)$ is given by the Newton volume integral (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$
$$V^{t}(r,\Omega) = G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=r_g(\Omega')}^{r_g(\Omega')+H^{\mathrm{O}}(\Omega')} \rho(r',\Omega')\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega', \tag{4.2}$$

where $\rho(r,\Omega)$ is the actual density of topographical masses (i.e., masses between the geoid and the Earth's surface). The spatial distance $l[r,\psi(\Omega,\Omega'),r']$ between two points with the geocentric positions (r,Ω) and (r',Ω') reads

$$\forall\ \Omega,\Omega' \in \Omega_{\mathrm{O}},\ r,r' \in \Re^{+}:\ l[r,\psi(\Omega,\Omega'),r'] = \sqrt{r^{2} + r'^{2} - 2r\ r' \cos\psi(\Omega,\Omega')}, \tag{4.3}$$

and the spherical distance $\psi(\Omega,\Omega')$, $\psi \in \langle\ 0,\pi\ \rangle$, is given by the law of cosines

$$\forall\ \Omega,\Omega' \in \Omega_{\mathrm{O}}:\ \cos\psi(\Omega,\Omega') = \sin\phi' \sin\phi + \cos\phi' \cos\phi \cos(\lambda' - \lambda). \tag{4.4}$$

The „*gravitational potential of condensed topographical masses*“ $V^{ct}(r,\Omega)$ can be computed by the Newton surface integral (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:\ V^{ct}(r,\Omega) = G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \sigma(\Omega')\, l^{-1}\left[r,\psi(\Omega,\Omega'), r_g(\Omega')\right] r_g^{2}(\Omega')\, \mathrm{d}\Omega', \tag{4.5}$$

where $\sigma(\Omega)$ is the surface density of topographical masses condensed onto the geoid.

Approximating the geoid by the geocentric sphere of radius R, i.e., $\forall\ \Omega \in \Omega_{\mathrm{O}}:\ r_g(\Omega) \approx R$, and the actual density $\rho(r,\Omega)$ of topographical masses by the laterally varying topographical density $\rho(\Omega)$, see (Martinec, 1993),

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:\ \rho(\Omega) = \frac{1}{H^{\mathrm{O}}(\Omega)} \int_{r=R}^{R+H^{\mathrm{O}}(\Omega)} \rho(r,\Omega)\, r^{2}\ \mathrm{d}r, \tag{4.6}$$

the gravitational potential $V^{t}(r,\Omega)$ of topographical masses in Eqn. (4.2) takes the following form (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:\ V^{t}(r,\Omega) = G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right] r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega'. \tag{4.7}$$

The surface density $\sigma(\Omega)$ of condensed topographical masses is according to the principle of mass-conservation of topographical masses (Wichiencharoen, 1982), i.e., the mass of the condensation layer is equal to the mass of actual topographical masses, in an integral representation (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : \iint_{\Omega} \rho(\Omega) \int_{r=R}^{R+H^{\mathrm{O}}(\Omega)} r^2 \ \mathrm{d}r \ \mathrm{d}\Omega = R^2 \iint_{\Omega} \sigma(\Omega)\, \mathrm{d}\Omega\,. \tag{4.8}$$

According to Eqn. (4.8) the surface density $\sigma(\Omega)$ becomes (Martinec and Vaníček, 1994a)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :$$
$$\sigma(\Omega) = \frac{\rho(\Omega)}{R^2} \int_{r=R}^{R+H^{\mathrm{O}}(\Omega)} r^2 \ \mathrm{d}r = \rho(\Omega) H^{\mathrm{O}}(\Omega) \left[1 + \frac{H^{\mathrm{O}}(\Omega)}{R} + \frac{\left[H^{\mathrm{O}}(\Omega)\right]^2}{3R^2} \right] = \rho(\Omega) \frac{r_t^3(\Omega) - R^3}{3R^2}\,. \tag{4.9}$$

The gravitational potential $V^{ct}(r,\Omega)$ of condensed topographical masses, see Eqn. (4.5), is then (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^+ : V^{ct}(r,\Omega) = G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3} l^{-1}\left[r, \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega'\,. \tag{4.10}$$

Considering the gravitational potential $V^t(r,\Omega)$ of topographical masses in Eqn. (4.7) and the gravitational potential $V^{ct}(r,\Omega)$ of condensed topographical masses in Eqn. (4.10), the residual gravitational potential $\delta V^t(r,\Omega)$ of topographical masses (in the spherical approximation of the geoid $\forall\ \Omega \in \Omega_{\mathrm{O}} : r_g(\Omega) \approx R$) becomes (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^+ :$$
$$\delta V^t(r,\Omega) = G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^2 \ \mathrm{d}r' \ \mathrm{d}\Omega' -$$
$$- G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3} l^{-1}\left[r, \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega'\,. \tag{4.11}$$

The radial integral of the reciprocal spatial distance $l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right]$ multiplied by r'^2 can be described by the analytical form (Gradshteyn and Ryzhik, 1980)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :$$
$$\int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^2 \ \mathrm{d}r' = \Bigg| \frac{1}{2}\left[r' + 3r\cos\psi(\Omega,\Omega')\right] l\left[r, \psi(\Omega,\Omega'), r'\right] +$$

$$+\frac{r^2}{2}\left(3\cos^2\psi(\Omega,\Omega')-1\right)\ln\left| r'-r\cos\psi(\Omega,\Omega')+l[r,\psi(\Omega,\Omega'),r']\right|\Bigg|_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} . \tag{4.12}$$

4.2 Direct topographical effect

The radial derivative of the residual gravitational potential $\delta V^t(r,\Omega)$ of topographical masses in Eqn. (4.1) referred to the Earth's surface defines the „*direct topographical effect on gravitational attraction*“ (Martinec, 1993; Martinec and Vaníček, 1994a)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}: \frac{\partial\, \delta V^t(r,\Omega)}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)} = \frac{\partial V^t(r,\Omega)}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)} - \frac{\partial V^{ct}(r,\Omega)}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)} . \tag{4.13}$$

The „*gravitational attraction of the topographical masses*“ is given by the radial derivative of the gravitational potential $V^t(r,\Omega)$ of topographical masses, see Eqn. (4.7), referred to the Earth's surface (Martinec and Vaníček, 1994a)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\frac{\partial V^t(r,\Omega)}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)} = G \iint_{\Omega'\in\Omega_{\mathrm{O}}} \rho(\Omega') \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} \frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)} r'^2\ \mathrm{d}r'\ \mathrm{d}\Omega' . \tag{4.14}$$

The radial derivative of the gravitational potential $V^{ct}(r,\Omega)$ of condensed topographical masses, see Eqn. (4.10), which is also reckoned on the Earth's surface, represents the „*gravitational attraction of condensed topographical masses*“ (Martinec and Vaníček, 1994a)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\frac{\partial V^{ct}(r,\Omega)}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)} = GR^2 \iint_{\Omega'\in\Omega_{\mathrm{O}}} \sigma(\Omega') \frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),R]}{\partial r}\Bigg|_{r=R+H^{\mathrm{O}}(\Omega)} \mathrm{d}\Omega' . \tag{4.15}$$

The radial integral of the radial derivative of the reciprocal spatial distance $\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r'] / \partial\, r$ multiplied by r'^2 can be expressed analytically as follows (Martinec, 1993):

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} \frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial r}\Bigg|_{r} r'^2\ \mathrm{d}r' = -\int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} \frac{r - r'\cos\psi(\Omega,\Omega')}{l^3[r,\psi(\Omega,\Omega'),r']} r'^2\ \mathrm{d}r' =$$

$$= \left| \frac{r'^2 \cos\psi(\Omega,\Omega') + 3\,r^2 \cos\psi(\Omega,\Omega') + r\,r' - 6\,r\,r' \cos^2\psi(\Omega,\Omega')}{l[r,\psi(\Omega,\Omega'),r']} + \right.$$

$$\left. + \, r\left(3\cos^2\psi(\Omega,\Omega') - 1\right) \ln\left| \, r' - r\cos\psi(\Omega,\Omega') + l[r,\psi(\Omega,\Omega'),r'] \, \right| \, \right|_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} . \tag{4.16}$$

To remove the weak singularity of the Newton integral (Kellogg, 1929) in the computation point, the gravitational attraction of the spherical Bouguer shell with the lateral topographical density $\rho(\Omega)$ and thickness $H^{\mathrm{O}}(\Omega)$ equal to the orthometric height of the computation point can be subtracted from and added to gravitational attraction of topographical masses (Martinec, 1993; Martinec et al., 1995). The gravitational potential $V^{shell}(r,\Omega)$ of the spherical Bouguer shell with the topographical density $\rho(\Omega)$ and thickness $H^{\mathrm{O}}(\Omega)$ is equal to (Wichiencharoen, 1982)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :$$

$$V^{shell}(r,\Omega) = G\rho(\Omega) \iint_{\Omega'\in\Omega_{\mathrm{O}}} \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega)} l^{-1}[r,\psi(\Omega,\Omega'),r']\, r'^2 \ \mathrm{d}r'\ \mathrm{d}\Omega' =$$

$$= \begin{cases} 4\pi G\rho(\Omega) \dfrac{R^2}{r} H^{\mathrm{O}}(\Omega) \left[1 + \dfrac{H^{\mathrm{O}}(\Omega)}{R} + \dfrac{[H^{\mathrm{O}}(\Omega)]^2}{3R^2} \right], & r \ge R + H^{\mathrm{O}}(\Omega), \\ 2\pi G\rho(\Omega) \left[R^2 + 2RH^{\mathrm{O}}(\Omega) + [H^{\mathrm{O}}(\Omega)]^2 - \dfrac{2}{3}\dfrac{R^3}{r} - \dfrac{1}{3} r^2 \right], & R \le r \le R + H^{\mathrm{O}}(\Omega), \\ 4\pi G\rho(\Omega) H^{\mathrm{O}}(\Omega) \left[R + \dfrac{1}{2} H^{\mathrm{O}}(\Omega) \right], & r \le R. \end{cases} \tag{4.17}$$

Moreover, gravitational attraction $\partial V^{shell}(r,\Omega)/\partial\, r$ of the spherical Bouguer shell is (Vaníček et al., 2001)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :$$

$$\left. \frac{\partial V^{shell}(r,\Omega)}{\partial\, r} \right|_r = G\rho(\Omega) \iint_{\Omega'\in\Omega_{\mathrm{O}}} \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega)} \left. \frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial\, r} \right|_r r'^2 \ \mathrm{d}r'\ \mathrm{d}\Omega' =$$

$$= \begin{cases} -4\pi G\rho(\Omega) \dfrac{R^2}{r^2} H^{\mathrm{O}}(\Omega) \left[1 + \dfrac{H^{\mathrm{O}}(\Omega)}{R} + \dfrac{[H^{\mathrm{O}}(\Omega)]^2}{3R^2} \right], & r \ge R + H^{\mathrm{O}}(\Omega), \\ -\dfrac{4}{3}\pi G\rho(\Omega) \dfrac{r^3 - R^3}{r^2}\,, & R \le r \le R + H^{\mathrm{O}}(\Omega), \\ 0, & r \le R. \end{cases} \tag{4.18}$$

Subtracting and adding gravitational attraction of the spherical Bouguer shell, gravitational attraction of topographical masses in Eqn. (4.14) becomes (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :$$

$$\left.\frac{\partial V^{t}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} = -4\pi\, G\rho(\Omega)\frac{R^2}{r_t^2(\Omega)} H^{\mathrm{O}}(\Omega)\left[1+\frac{H^{\mathrm{O}}(\Omega)}{R}+\frac{\left[H^{\mathrm{O}}(\Omega)\right]^2}{3R^2}\right]+$$

$$+G\iint_{\Omega'\in\Omega_{\mathrm{O}}}\left[\rho(\Omega')\int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')}\left.\frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} r'^2\ \mathrm{d}r' -\right.$$

$$\left.-\rho(\Omega)\int_{r'=R}^{R+H^{\mathrm{O}}(\Omega)}\left.\frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} r'^2\ \mathrm{d}r'\right]\mathrm{d}\Omega' . \tag{4.19}$$

A similar procedure can be applied to remove the weak singularity of the Newton surface integral from gravitational attraction of condensed topographical masses, see Eqn. (4.15). The gravitational potential $V^{layer}(r,\Omega)$ of the spherical condensation layer with the surface density $\sigma(\Omega)$ is (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :$$

$$V^{layer}(r,\Omega) = GR^2\sigma(\Omega)\iint_{\Omega'\in\Omega_{\mathrm{O}}} l^{-1}[r,\psi(\Omega,\Omega'),R]\,\mathrm{d}\Omega' = \begin{cases} 4\pi G\sigma(\Omega)\dfrac{R^2}{r}, & r > R, \\ 4\pi G\sigma(\Omega)R, & r \le R, \end{cases} \tag{4.20}$$

and gravitational attraction $\partial V^{layer}(r,\Omega)/\partial\, r$ of the spherical condensation layer is

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :$$

$$\left.\frac{\partial V^{layer}(r,\Omega)}{\partial r}\right|_{r} = GR^2\sigma(\Omega)\iint_{\Omega'\in\Omega_{\mathrm{O}}}\left.\frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),R]}{\partial r}\right|_{r}\mathrm{d}\Omega' = \begin{cases} -4\pi G\sigma(\Omega)\dfrac{R^2}{r^2}, & r > R, \\ 0, & r < R. \end{cases} \tag{4.21}$$

Subtracting and adding gravitational attraction of the spherical condensation layer to Eqn. (4.15), gravitational attraction of condensed topographical masses takes the following form (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} :$$

$$\left.\frac{\partial V^{ct}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} = -4\pi\, G\sigma(\Omega)\frac{R^2}{r_t^2(\Omega)}+$$

$$+ GR^2 \iint\limits_{\Omega' \in \Omega_O} \left[\sigma(\Omega') - \sigma(\Omega) \right] \frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), R \right]}{\partial\, r} \Bigg|_{r = R + H^O(\Omega)} \mathrm{d}\Omega' \,. \tag{4.22}$$

Comparing gravitational attraction of the spherical Bouguer shell referred to the Earth's surface, that is given by the first term on the right-hand side of Eqn. (4.19), with gravitational attraction of the spherical condensation layer referred to the Earth's surface, that is given by the first term on the right-hand side of Eqn. (4.22), they are equal. Substituting gravitational attraction of topographical masses in Eqn. (4.19) and gravitational attraction of condensed topographical masses in Eqn. (4.22) back into Eqn. (4.13), the direct topographical effect on gravitational attraction becomes (Martinec, 1993)

$$\forall\ \Omega \in \Omega_O :$$

$$\frac{\partial \delta V^t(r,\Omega)}{\partial\, r} \Bigg|_{r = R + H^O(\Omega)} = G \iint\limits_{\Omega' \in \Omega_O} \Bigg[\rho(\Omega') \int_{r'=R}^{R + H^O(\Omega')} \frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), r' \right]}{\partial\, r} \Bigg|_{r = R + H^O(\Omega)} r'^2 \,\mathrm{d}r' -$$

$$- \rho(\Omega) \int_{r'=R}^{R + H^O(\Omega)} \frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), r' \right]}{\partial\, r} \Bigg|_{r = R + H^O(\Omega)} r'^2 \,\mathrm{d}r' \Bigg] \mathrm{d}\Omega' -$$

$$- GR^2 \iint\limits_{\Omega' \in \Omega_O} \left[\sigma(\Omega') - \sigma(\Omega) \right] \frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), R \right]}{\partial\, r} \Bigg|_{r = R + H^O(\Omega)} \mathrm{d}\Omega' \,. \tag{4.23}$$

Separating the laterally varying topographical density $\rho(\Omega)$ into the mean value $\rho_o = 2.67\ \mathrm{g.cm}^{-3}$ and the laterally varying anomalous topographical density $\delta\rho(\Omega)$:

$$\forall\ \Omega \in \Omega_O : \rho(\Omega) = \rho_o + \delta\rho(\Omega), \tag{4.24}$$

and substituting them into Eqn. (4.9), the surface density $\sigma(\Omega)$ becomes (Martinec, 1993)

$$\forall\ \Omega \in \Omega_O : \sigma(\Omega) = \sigma_o + \delta\sigma(\Omega) = \left[\rho_o + \delta\rho(\Omega) \right] H^O(\Omega) \left[1 + \frac{H^O(\Omega)}{R} + \frac{\left[H^O(\Omega) \right]^2}{3R^2} \right]. \tag{4.25}$$

Applying the above mentioned decomposition of densities into gravitational attraction of topographical masses and of condensed topographical masses, the direct topographical effect on gravitational attraction finally takes the following form (Martinec, 1993)

$$\forall\ \Omega \in \Omega_O :$$

$$\frac{\partial \delta V^t(r,\Omega)}{\partial\, r} \Bigg|_{r = R + H^O(\Omega)} = G\rho_o \iint\limits_{\Omega' \in \Omega_O} \int_{r' = R + H^O(\Omega)}^{R + H^O(\Omega')} \frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), r' \right]}{\partial\, r} \Bigg|_{r = R + H^O(\Omega)} r'^2 \,\mathrm{d}r' \mathrm{d}\Omega' -$$

$$
-G\rho_o \iint\limits_{\Omega'\in\Omega_O} \frac{r_t^3(\Omega')-r_t^3(\Omega)}{3} \frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),R]}{\partial r}\bigg|_{r=R+H^O(\Omega)} \mathrm{d}\Omega' +
$$

$$
+G \iint\limits_{\Omega'\in\Omega_O} \delta\rho(\Omega') \int_{r'=R}^{R+H^O(\Omega')} \frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),r']}{\partial r}\bigg|_{r=R+H^O(\Omega)} r'^2 \,\mathrm{d}r'\,\mathrm{d}\Omega' -
$$

$$
-G \iint\limits_{\Omega'\in\Omega_O} \delta\rho(\Omega') \frac{r_t^3(\Omega')-R^3}{3} \frac{\partial\, l^{-1}[r,\psi(\Omega,\Omega'),R]}{\partial r}\bigg|_{r=R+H^O(\Omega)} \mathrm{d}\Omega' . \tag{4.26}
$$

The first term on the right-hand side of Eqn. (4.26) is the so-called „*spherical terrain correction*“, and the second term stands for the „*spherical condensed terrain correction*“ (Martinec and Vaníček, 1994a). The third and fourth terms represent together the contribution of the laterally varying topographical density to the direct topographical effect.

4.3 Secondary indirect topographical effect

The „*secondary indirect topographical effect on gravitational attraction*“, which refers to the Earth’s surface, is given by the following equation (Martinec and Vaníček, 1994b)

$$
\forall\, \Omega\in\Omega_O : \frac{2}{r_t(\Omega)}\delta V^t(r_t(\Omega)) = \frac{2}{r_t(\Omega)} V^t(r_t(\Omega)) - \frac{2}{r_t(\Omega)} V^{ct}(r_t(\Omega)). \tag{4.27}
$$

Dividing the laterally varying topographical density $\rho(\Omega)$ into the mean and laterally varying anomalous topographical density, see Eqn. (4.24), the gravitational potential $V^t(r_t(\Omega))$ of topographical masses given by Eqn. (4.7) takes the following form (Martinec, 1993)

$$
\forall\, \Omega\in\Omega_O :
$$

$$
V^t(r_t(\Omega)) = 4\pi G\rho_o \frac{R^2}{r_t(\Omega)} H^O(\Omega)\left[1+\frac{H^O(\Omega)}{R}+\frac{[H^O(\Omega)]^2}{3R^2}\right]+
$$

$$
+G\rho_o \iint\limits_{\Omega'\in\Omega_O} \int_{r'=R+H^O(\Omega)}^{R+H^O(\Omega')} l^{-1}[r_t(\Omega),\psi(\Omega,\Omega'),r']\, r'^2 \,\mathrm{d}r'\mathrm{d}\Omega' +
$$

$$
+G \iint\limits_{\Omega'\in\Omega_O} \delta\rho(\Omega') \int_{r'=R}^{R+H^O(\Omega')} l^{-1}[r_t(\Omega),\psi(\Omega,\Omega'),r']\, r'^2 \,\mathrm{d}r'\,\mathrm{d}\Omega' , \tag{4.28}
$$

where the first term stands for the gravitational potential of the spherical Bouguer shell referred to the Earth’s surface, see Eqn. (4.17). Similarly, the gravitational potential $V^{ct}(r_t(\Omega))$ of condensed topographical masses, see Eqn. (4.10), becomes (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$V^{ct}\left(r_t(\Omega)\right) = 4\pi\ G\sigma_o \frac{R^2}{r_t(\Omega)} + G\rho_o \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3}\, l^{-1}\left[r_t(\Omega), \psi(\Omega, \Omega'), R\right] \mathrm{d}\Omega' +$$
$$+ G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3}\, l^{-1}\left[r_t(\Omega), \psi(\Omega, \Omega'), R\right] \mathrm{d}\Omega' , \tag{4.29}$$

where the first term on the right-hand side represents the gravitational potential of the spherical condensation layer at the point above the geoid, see Eqn. (4.20).

Considering the gravitational potential $V^t\left(r_t(\Omega)\right)$ of topographical masses, see Eqn. (4.28), and the gravitational potential $V^{ct}(r, \Omega)$ of condensed topographical masses, see Eqn. (4.29), the secondary indirect topographical effect in Eqn. (4.27) can be written as follows (Martinec, 1993)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$
$$\frac{2}{r_t(\Omega)} \delta V^t\left(r_t(\Omega)\right) = \frac{2}{r_t(\Omega)} G\rho_o \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega)}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[r_t(\Omega), \psi(\Omega, \Omega'), r'\right] r'^2\ \mathrm{d}r' \mathrm{d}\Omega' -$$
$$- \frac{2}{r_t(\Omega)} G\rho_o \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3}\, l^{-1}\left[r_t(\Omega), \psi(\Omega, \Omega'), R\right] \mathrm{d}\Omega' +$$
$$+ \frac{2}{r_t(\Omega)} G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} \left. \frac{\partial\, l^{-1}\left[r, \psi(\Omega, \Omega'), r'\right]}{\partial r} \right|_{r=R+H^{\mathrm{O}}(\Omega)} r'^2\ \mathrm{d}r'\ \mathrm{d}\Omega' -$$
$$- \frac{2}{r_t(\Omega)} G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \delta\rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3}\, l^{-1}\left[r_t(\Omega), \psi(\Omega, \Omega'), R\right] \mathrm{d}\Omega' . \tag{4.30}$$

The gravitational potential of the spherical Bouguer shell and the gravitational potential of the spherical condensation layer are subtracted from Eqn. (4.30), because, if reckoned on the Earth's surface, they are equal.

5. Effect of atmospheric masses on gravitational attraction

Transforming the boundary-value problem, as formulated in the real space by Eqn. (2.16), into the Helmert space according to Eqn. (3.9), the effect of atmospheric masses on gravitational attraction is represented by the direct and secondary indirect atmospheric effects.

5.1 Residual gravitational potential of atmospheric masses

Similarly to the residual gravitational potential of topographical masses, the „*residual gravitational potential of atmospheric masses*“ $\delta V^{a}(r,\Omega)$ is given by the difference of the gravitational potential $V^{a}(r,\Omega)$ of atmospheric masses and the gravitational potential $V^{ca}(r,\Omega)$ of atmospheric masses condensed (according to the Helmert second condensation method) onto the geoid (Vaníček et al., 1999)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+} :\ \delta V^{a}(r,\Omega) = V^{a}(r,\Omega) - V^{ca}(r,\Omega)\,. \tag{5.1}$$

Under the spherical approximation of the geoid ($\forall\ \Omega \in \Omega_{\mathrm{O}} : r_g(\Omega) \approx R$), the „*gravitational potential of atmospheric masses*“ $V^{a}(r,\Omega)$ reads (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+} :$$
$$V^{a}(r,\Omega) = G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\mathrm{lim}}} \rho^{a}(r,\Omega')\, l^{-1}\left[r,\psi(\Omega,\Omega'),r'\right] r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega'\,, \tag{5.2}$$

where $\rho^{a}(r,\Omega)$ is the actual atmospheric density and r_{lim} is the upper limit of the atmosphere where the atmospheric density becomes negligible (approximately 50 km above the sea level).

The „*gravitational potential of condensed atmospheric masses*“ $V^{ca}(r,\Omega)$ is (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+} :\ V^{ca}(r,\Omega) = GR^{2} \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \sigma^{a}(\Omega')\, l^{-1}\left[r,\psi(\Omega,\Omega'),R\right] \mathrm{d}\Omega'\,, \tag{5.3}$$

where $\sigma^{a}(\Omega)$ is the surface density of condensed atmospheric masses.

Using the laterally homogenous atmospheric density distribution

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \left\langle R + H^{\mathrm{O}}(\Omega),\ r_{\mathrm{lim}} \right\rangle :\ \rho^{a}(r) = \frac{1}{\Omega} \iint\limits_{\Omega} \rho^{a}(r,\Omega)\,\mathrm{d}\Omega\,, \tag{5.4}$$

the gravitational potential $V^{a}(r,\Omega)$ of atmospheric masses in Eqn. (5.2) can be written in the following form (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:\ V^{a}(r,\Omega) = G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}} \rho^{a}(r')\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega' . \qquad (5.5)$$

According to the principle of the mass conservation, the atmospheric surface density $\sigma^{a}(\Omega)$ is defined by (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:\ \sigma^{a}(\Omega) = \frac{1}{R^{2}} \int_{r=R+H^{\mathrm{O}}(\Omega)}^{r_{\lim}} \rho^{a}(r)\, r^{2}\ \mathrm{d}r . \qquad (5.6)$$

Substituting Eqn. (5.6) for the atmospheric surface density $\sigma^{a}(\Omega)$ into Eqn. (5.3), the gravitational potential $V^{ca}(r,\Omega)$ of condensed atmospheric masses takes the following form (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:\ V^{ca}(r,\Omega) = G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}} \rho^{a}(r')\, r'^{2}\ \mathrm{d}r'\ l^{-1}\left[r, \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega' . \qquad (5.7)$$

Formally, the Earth's atmospheric masses can be split into the spherical shell bounded by the maximum geocentric radius of the topography and of the upper limit of the atmosphere, $\forall\ \Omega \in \Omega_{\mathrm{O}}: r = r_{\lim}$, and the roughness term bounded by the Earth's surface, $\forall\ \Omega \in \Omega_{\mathrm{O}}: r_t(\Omega) = R + H^{\mathrm{O}}(\Omega)$ and the maximum geocentric radius of the topography (Novák, 2000).

The gravitational potential $V^{a}(r,\Omega)$ of atmospheric masses can then be described by (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$
$$V^{a}(r,\Omega) = G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\lim}} \rho^{a}(r')\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega' +$$
$$+ G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^{a}(r')\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right] r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega' . \qquad (5.8)$$

Dividing also the integration domain of the atmospheric surface density $\sigma^{a}(\Omega)$ in Eqn. (5.6) as follows

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:\ \sigma^{a}(\Omega) = \frac{1}{R^{2}} \int_{r=R+H^{\mathrm{O}}(\Omega)}^{R+H_{\lim}} \rho^{a}(r)\, r^{2}\ \mathrm{d}r + \frac{1}{R^{2}} \int_{r(\Omega)=R+H_{\lim}}^{r_{\lim}} \rho^{a}(r)\, r^{2}\ \mathrm{d}r , \qquad (5.9)$$

the gravitational potential $V^{ca}(r,\Omega)$ of condensed atmospheric masses becomes (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r \in \Re^{+}:$$

$$V^{ca}(r,\Omega) = G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\lim}} \rho^{a}(r')\, r'^{2}\ \mathrm{d}r'\ l^{-1}\left[r, \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega' +$$

$$+ G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^{a}(r')\, r'^{2}\ \mathrm{d}r'\ l^{-1}\left[r, \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega' . \tag{5.10}$$

5.2 Direct atmospheric effect

The „*direct atmospheric effect on gravitational attraction*“ is defined as the radial derivative of the residual gravitational potential $\delta V^{a}(r_t(\Omega))$ of atmospheric masses referred to the Earth’s surface (Vaníček et al., 1999; Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}: \left.\frac{\partial\, \delta V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} = \left.\frac{\partial V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} - \left.\frac{\partial V^{ca}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} . \tag{5.11}$$

Since the gravitational attraction of the atmospheric spherical shell (bounded by the geocentric radii of the upper limit of topography and of the upper limit of the atmosphere) at the inner point $r < R + H_{\lim}$ is equal to zero (Mac Millan, 1930)

$$\forall\ r < R + H_{\lim}: G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^{a}(r') \left.\frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right]}{\partial r}\right|_{r} r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega' = 0 , \tag{5.12}$$

the „*gravitational attraction of atmospheric masses*“ is given as the radial derivative of the gravitational potential of the atmospheric roughness term. The roughness term, which represents gravitational attraction of the atmosphere between the topography ($\forall\ \Omega \in \Omega_{\mathrm{O}}: r_t(\Omega) = R + H^{\mathrm{O}}(\Omega)$) and the upper limit of topography ($\forall\ \Omega \in \Omega_{\mathrm{O}}:$ $r = R + H_{\lim}$), is given by (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\left.\frac{\partial V^{a}(r,\Omega)}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} = G \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\lim}} \rho^{a}(r') \left.\frac{\partial\, l^{-1}\left[r, \psi(\Omega,\Omega'), r'\right]}{\partial r}\right|_{r=R+H^{\mathrm{O}}(\Omega)} r'^{2}\ \mathrm{d}r'\ \mathrm{d}\Omega' . \tag{5.13}$$

Considering that gravitational attraction of the spherical condensation layer with the surface atmospheric density $\sigma^a(\Omega)$ at the outer point above the condensation layer $r > R$ is equal to a constant (Mac Millan, 1930)

$$\forall \;\; r > R:$$
$$G \iint_{\Omega' \in \Omega_O} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' \frac{\partial\, l^{-1}[r, \psi(\Omega,\Omega'), R]}{\partial\, r}\bigg|_r \mathrm{d}\Omega' = -4\pi G \frac{R^2}{r^2} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r', \qquad (5.14)$$

„*gravitational attraction of condensed atmospheric masses*“ becomes

$$\forall \; \Omega \in \Omega_O :$$
$$\frac{\partial V^{ca}(r,\Omega)}{\partial\, r}\bigg|_{r=R+H^O(\Omega)} = -4\,\pi\, G \frac{R^2}{r_t^2(\Omega)} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' +$$
$$+ G \iint_{\Omega' \in \Omega_O} \int_{r'=R+H^O(\Omega')}^{R+H_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' \, \frac{\partial\, l^{-1}[r, \psi(\Omega,\Omega'), R]}{\partial\, r}\bigg|_{r=R+H^O(\Omega)} \mathrm{d}\Omega'. \qquad (5.15)$$

Substituting gravitational attraction of atmospheric masses in Eqn. (5.13) and the gravitational attraction condensed atmospheric masses in Eqn. (5.15) back into Eqn. (5.11), the direct atmospheric effect on gravitational attraction takes the following form

$$\forall \; \Omega \in \Omega_O :$$
$$\frac{\partial\, \delta V^a(r,\Omega)}{\partial\, r}\bigg|_{r=R+H^O(\Omega)} = G \iint_{\Omega' \in \Omega_O} \int_{r'=R+H^O(\Omega')}^{R+H_{\lim}} \rho^a(r') \frac{\partial\, l^{-1}[r, \psi(\Omega,\Omega'), r']}{\partial\, r}\bigg|_{r=R+H^O(\Omega)} r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' +$$
$$+ 4\,\pi\, G \frac{R^2}{r_t^2(\Omega)} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' -$$
$$- G \iint_{\Omega' \in \Omega_O} \int_{r'=R+H^O(\Omega')}^{R+H_{\lim}} \rho^a(r')\, r'^2 \, \mathrm{d}r' \, \frac{\partial\, l^{-1}[r, \psi(\Omega,\Omega'), R]}{\partial\, r}\bigg|_{r=R+H^O(\Omega)} \mathrm{d}\Omega'. \qquad (5.16)$$

5.3 Secondary indirect atmospheric effect

The „*secondary indirect atmospheric effect on gravitational attraction*“, stipulated as being on the Earth's surface, can be described by the following expression (Novák, 2000)

$$\forall \; \Omega \in \Omega_O : \frac{2}{r_t(\Omega)} \delta V^a(r_t(\Omega)) = \frac{2}{r_t(\Omega)} V^a(r_t(\Omega)) - \frac{2}{r_t(\Omega)} V^{ca}(r_t(\Omega)). \qquad (5.17)$$

If Eqns. (5.8) and (5.10) are considered, the residual gravitational potential $\delta V^{a}\left(r_t(\Omega)\right)$ in Eqn. (5.17) takes the following form

$$\forall\ \Omega \in \Omega_{\mathrm{O}}:$$

$$\begin{aligned}
\delta V^{a}\left(r_t(\Omega)\right) &= 4\pi G \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^{a}(r')\, r'\, \mathrm{d}r' \ - 4\pi G \frac{R^2}{r_t(\Omega)} \int_{r'=R+H_{\lim}}^{r_{\lim}} \rho^{a}(r')\, r'^2\, \mathrm{d}r' \ + \\
&+ G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\lim}} \rho^{a}(r')\, l^{-1}\left[r_t(\Omega), \psi(\Omega,\Omega'), r'\right] r'^2\, \mathrm{d}r'\, \mathrm{d}\Omega' - \\
&- G \iint_{\Omega' \in \Omega_{\mathrm{O}}} \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{R+H_{\lim}} \rho^{a}(r')\, r'^2\, \mathrm{d}r'\, l^{-1}\left[r_t(\Omega), \psi(\Omega,\Omega'), R\right] \mathrm{d}\Omega' . \qquad (5.18)
\end{aligned}$$

6. Downward continuation of Helmert's gravity anomalies

To obtain gravity anomalies on the geoid, that are needed for solving the Stokes boundary-value problem, the downward continuation of gravity anomalies from the Earth's surface to the geoid in the Helmert space has to be evaluated. The downward continuation is evaluated by the Poisson integral equation, which is the inverse operation to Poisson's integral.

Since topographical and atmospheric masses are condensed onto the geoid, the Helmert space above the geoid (approximated by the geocentric sphere of radius R, i.e., $\forall\ \Omega \in \Omega_{\mathrm{O}} : R \approx r_g(\Omega)$) is harmonic. Helmert's gravity anomaly $\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right)$ multiplied by the geocentric radius of the Earth's surface $r_t(\Omega)$ then satisfies the Laplace differential equation in the space everywhere above the geoid $\forall\ \Omega \in \Omega_{\mathrm{O}},\ r_t(\Omega) > R : \Delta\left[r_t(\Omega)\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right)\right] = 0$ (Vaníček et al., 1996). „*Poisson's integral*" is given by the following formula (Kellogg, 1929)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r_t(\Omega) \geq R :$$

$$\Delta g^{\mathrm{H}}\left(r_t(\Omega)\right) = \frac{R}{4\pi\, r_t(\Omega)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \mathrm{K}\left[r_t(\Omega), \psi(\Omega,\Omega'), R\right] \Delta g^{\mathrm{H}}(R,\Omega')\, \mathrm{d}\Omega' , \qquad (6.1)$$

where $\mathrm{K}\left[r_t(\Omega), \psi(\Omega,\Omega'), R\right]$ is the „*spherical Poisson integral kernel*" (Sun and Vaníček, 1998)

$$\forall\ \Omega, \Omega' \in \Omega_{\mathrm{O}},\ r_t(\Omega) \geq R :$$

$$\mathrm{K}[r_t(\Omega), \psi(\Omega,\Omega'), R] = \sum_{n=2}^{\infty} (2n+1) \left[\frac{R}{r_t(\Omega)} \right]^{n+1} \mathrm{P_n}(\cos\psi(\Omega,\Omega')) =$$

$$= R \left[\frac{r_t^2(\Omega) - R^2}{l^3 [r_t(\Omega), \psi(\Omega,\Omega'), R]} - \frac{1}{r_t(\Omega)} - \frac{3R}{r_t^2(\Omega)} \cos\psi(\Omega,\Omega') \right]. \tag{6.2}$$

The discrete form of „*Poisson's integral equation*“ which generic form is the Fredholm integral equation of the first kind, can be expressed as (Martinec, 1996; Huang, 2002)

$$\mathbf{\Delta g^H}(r_t(\Omega)) = \mathbf{K}[r_t(\Omega), \psi(\Omega,\Omega'), R] \ \mathbf{\Delta g^H}(R,\Omega'), \tag{6.3}$$

where $\mathbf{\Delta g^H}(r_t(\Omega))$ is the vector of Helmert's gravity anomalies on the Earth's surface, $\mathbf{\Delta g^H}(R,\Omega')$ is the vector of Helmert's gravity anomalies on the co-geoid (approximated again by the reference sphere), and $\mathbf{K}[r_t(\Omega), \psi(\Omega,\Omega'), R]$ is the matrix of values of the Poisson integral kernel multiplied by the factor $R / r_t(\Omega)$ and constant $1/4\pi$.

According to Jacobi's iteration approach (Ralston, 1965) for solution of a system of linear algebraic equations, the matrix $\mathbf{K}[r_t(\Omega), \psi(\Omega,\Omega'), R]$ can be expressed in the form

$$\mathbf{K}[r_t(\Omega), \psi(\Omega,\Omega'), R] = \mathbf{E} - \mathbf{B}[r_t(\Omega), \psi(\Omega,\Omega'), R], \tag{6.4}$$

where $\mathbf{E}$ is the unit matrix. Substituting Eqn. (6.4) into Eqn. (6.3), the following system of algebraic equations is obtained (Martinec, 1996)

$$\mathbf{\Delta g^H}(R,\Omega') = \mathbf{\Delta g^H}(r_t(\Omega)) - \mathbf{B}[r_t(\Omega), \psi(\Omega,\Omega'), R] \ \mathbf{\Delta g^H}(R,\Omega'). \tag{6.5}$$

The system of Eqns. (6.5) may be solved iteratively starting with the vector $\mathbf{\Delta g^{FA}}(r_t(\Omega))$ of free-air gravity anomalies on the Earth's surface (because of free-air gravity anomalies on the Earth's surface are similar to Helmert's gravity anomalies on the geoid)

$$\forall\ \Omega \equiv \Omega' : \ \mathbf{\Delta g^H}(R,\Omega')\Big|_0 = \mathbf{\Delta g^{FA}}(r_t(\Omega)). \tag{6.6}$$

The k-th stage of iteration ($k > 0$) $\mathbf{\Delta g^H}(R,\Omega')\Big|_k$ is carried out according to equation (Martinec, 1996)

$$\mathbf{\Delta g^H}(R,\Omega')\Big|_k = \mathbf{B}[r_t(\Omega), \psi(\Omega,\Omega'), R] \ \mathbf{\Delta g^H}(R,\Omega')\Big|_{k-1}. \tag{6.7}$$

When the difference of results from two successive steps $\left| \mathbf{\Delta g^H}(R,\Omega')\big|_k - \mathbf{\Delta g^H}(R,\Omega')\big|_{k-1} \right|$ is smaller than some tolerance ε, the iterative process stops. The result of this operation yields the solution of Eqn. (6.3), see (Martinec, 1996),

$$\mathbf{\Delta g^H}(R,\Omega') = \mathbf{\Delta g^H}(r_t(\Omega)) + \sum_{k=1}^{\bar{k}} \mathbf{\Delta g^H}(R,\Omega')\big|_k \,, \tag{6.8}$$

where $\bar{k}$ is the final number of iteration steps.

7. Reference field and spheroid in the Helmert space

To solve the Stokes boundary-value problem, the gravity anomalies over the entire boundary surface are required. To reduce the truncation errors, i.e., the far-zone contribution in the Stokes integration, the low and high-frequency parts of Helmert's gravity field are defined (Vaníček and Sjöberg, 1991).

The reference gravity field of degree $\bar{n}$ can be expressed by the „*reference gravity potential*" $W_{\mathrm{ref}}(r,\Omega)$ as (Vaníček et al., 1995)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r \geq r_t(\Omega) : W_{\mathrm{ref}}(r,\Omega) = \frac{GM}{r} - \sum_{n=2}^{\bar{n}} \left(\frac{a_o}{r} \right)^{n+1} \sum_{m=-n}^{n} \mathrm{W}_{\mathrm{n,m}}\ \mathrm{Y}_{\mathrm{n,m}}(\Omega), \tag{7.1}$$

where $\mathrm{W}_{\mathrm{n,m}}$ are the geopotential coefficients of the harmonic expansion of the Earth's gravity field, $\mathrm{Y}_{\mathrm{n,m}}$ are the normalized spherical functions of degree n and order m, a_o is an arbitrary parameter of length (usually the major semi-axis of the reference ellipsoid), and $\bar{n}$ stands for the maximum degree of retained harmonics. In the Helmert space the reference gravity potential $W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega)$ reads

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r \in \Re^+ : W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega) = W_{\mathrm{ref}}(r,\Omega) - \delta V^{t}_{\mathrm{ref}}(r,\Omega) - \delta V^{a}_{\mathrm{ref}}(r,\Omega)\,, \tag{7.2}$$

where $\delta V^{t}_{\mathrm{ref}}(r,\Omega)$ and $\delta V^{a}_{\mathrm{ref}}(r,\Omega)$ are the reference residual gravitational potentials of the topographical and atmospheric masses.

7.1 Reference residual gravitational potential of topographical masses

According to Eqn. (4.1) the „*reference residual gravitational potential of topographical* *masses*“ $\delta V_{\mathrm{ref}}^{t}(r,\Omega)$ can be defined as the difference of the „*reference gravitational potential of topographical masses*“ $V_{\mathrm{ref}}^{t}(r,\Omega)$ (Vaníček et al., 1995)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r > R + H_{\mathrm{lim}} :$$

$$V_{\mathrm{ref}}^{t}(r,\Omega) \approx G\rho_o \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \int\limits_{H'=0}^{H^{\mathrm{O}}(\Omega')} \frac{1}{r} \sum_{n=0}^{\bar{n}} \left(\frac{R+H'}{r} \right)^{n} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right)(R+H')^2 \, \mathrm{d}H' \, \mathrm{d}\Omega' , \tag{7.3}$$

and the „*reference gravitational potential of condensed topographical masses*“ $V_{\mathrm{ref}}^{ct}(r,\Omega)$ (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}, r > R : V_{\mathrm{ref}}^{ct}(r,\Omega) \approx GR \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \sigma(\Omega') \sum_{n=0}^{\bar{n}} \left(\frac{R}{r} \right)^{n+1} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' . \tag{7.4}$$

For points $\forall\, \Omega \in \Omega_{\mathrm{O}} : r > R + H_{\mathrm{lim}}$ outside the Brillouin sphere (minimal geocentric sphere containing all the Earth’s mass), the reference gravitational potential $V_{\mathrm{ref}}^{t}(r,\Omega)$ of topographical masses in Eqn. (7.3) takes the following form (Vaníček et al., 1995)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r > R + H_{\mathrm{lim}} :$$

$$V_{\mathrm{ref}}^{t}(r,\Omega) = G\rho_o R^2 \sum_{n=0}^{\bar{n}} \left(\frac{R}{r} \right)^{n+1} \frac{1}{n+3} \sum_{k=1}^{n+3} \binom{n+3}{k} \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \left[\frac{H^{\mathrm{O}}(\Omega')}{R} \right]^{k} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' . \tag{7.5}$$

Differencing the reference gravitational potential $V_{\mathrm{ref}}^{t}(r,\Omega)$ of topographical masses, see Eqn. (7.5), and the reference gravitational potential $V_{\mathrm{ref}}^{ct}(r,\Omega)$ of condensed topographical masses, see Eqn. (7.4), the reference residual gravitational potential $\delta V_{\mathrm{ref}}^{t}(r,\Omega)$ of topographical masses becomes (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r > R + H_{\mathrm{lim}} :$$

$$\delta V_{\mathrm{ref}}^{t}(r,\Omega) \cong G\rho_o R^2 \sum_{n=0}^{\bar{n}} \left(\frac{R}{r} \right)^{n+1} \left\{ \frac{1}{n+3} \sum_{k=1}^{n+3} \binom{n+3}{k} \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \left[\frac{H^{\mathrm{O}}(\Omega')}{R} \right]^{k} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' - \right.$$

$$\left. - \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} \frac{H^{\mathrm{O}}(\Omega')}{R} \left[1 + \frac{H^{\mathrm{O}}(\Omega')}{R} + \frac{\left[H^{\mathrm{O}}(\Omega')\right]^2}{3R^2} \right] \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' \right\} . \tag{7.6}$$

Since for $H^{\mathrm{O}}(\Omega') << R$ the summation over k converges very quickly (Vaníček et al., 1995), Eqn. (7.6) can be rewritten into the following form (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r > R + H_{\mathrm{lim}}:$$
$$\delta V^{t}_{\mathrm{ref}}(r,\Omega) \cong G\rho_o R^2 \sum_{n=1}^{\bar{n}} \frac{n}{2}\left(\frac{R}{r}\right)^{n+1} \left\{ \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{\left[H^{\mathrm{O}}(\Omega')\right]^2}{R^2} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' + \right.$$
$$\left. + \frac{n+3}{3} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{\left[H^{\mathrm{O}}(\Omega')\right]^3}{R^3} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' \right\}. \tag{7.7}$$

Expressing the surface harmonics of the orthometric height as (Kellogg, 1929)

$$\iint_{\Omega' \in \Omega_{\mathrm{O}}} H^{\mathrm{O}}(\Omega')\, \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \mathrm{d}\Omega' = \frac{4\pi}{2n+1} \sum_{m=-n}^{n} \mathrm{H_{n,m}}(\Omega)\, \mathrm{Y_{n,m}}(\Omega), \tag{7.8}$$

the reference residual gravitational potential $\delta V^{t}_{\mathrm{ref}}(r,\Omega)$ of topographical masses becomes (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r > R + H_{\mathrm{lim}}(\Omega):$$
$$\delta V^{t}_{\mathrm{ref}}(\Omega) \cong 2\pi G\rho_o \sum_{n=1}^{\bar{n}} \frac{n}{2n+1}\left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{H^2_{n,m}}(\Omega)\, \mathrm{Y_{n,m}}(\Omega) +$$
$$+ \frac{2\pi}{3R} G\rho_o \sum_{n=1}^{\bar{n}} \frac{n(n+3)}{2n+1}\left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{H^3_{n,m}}(\Omega)\, \mathrm{Y_{n,m}}(\Omega). \tag{7.9}$$

7.2 Reference residual gravitational potential of atmospheric masses

The „*reference gravitational potential of atmospheric masses*“ $V^{a}_{\mathrm{ref}}(r,\Omega)$ can be described in the form (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}},\, r > r_{\mathrm{lim}}:$$
$$V^{a}_{\mathrm{ref}}(r,\Omega) = G \sum_{n=0}^{\bar{n}} \frac{1}{r^{n+1}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \int_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\mathrm{lim}}} \rho^a(r')\, r'^{\,n+2}\, \mathrm{d}r'\, \mathrm{d}\Omega'. \tag{7.10}$$

To define the reference residual gravitational potential $\delta V^{a}_{\mathrm{ref}}(r,\Omega)$ of the atmospheric masses, the atmospheric density $\rho^a(r)$ given by Eqn. (5.4) can be replaced by the laterally symmetrical density model (Sjöberg, 1998; Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}, r \in \left\langle R + H^{\mathrm{O}}(\Omega)\,, r_{\lim} \right\rangle, \nu > 2 \wedge \nu \in \Im^{+} : \rho^{a}(r) = \rho_{o}^{a}\left[\frac{R}{R + H^{\mathrm{O}}(\Omega)}\right]^{\nu}, \tag{7.11}$$

where ρ_o^a is the atmospheric density at the sea level, and the positive integer constant $\nu \in \Im^{+}$ ($\Im^{+} = 1, 2, \ldots$) determines the atmospheric density distribution model.

If the integration over the geocentric radius r from the Earth's surface $r_t(\Omega)$ to the upper limit $r_{\lim}$ of the atmosphere is evaluated by using the atmospheric model density from Eqn. (7.11)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}, \nu > 2 \wedge \nu \in \Im^{+}, n = 1, 2, \ldots, \bar{n} :$$

$$\int_{r=R+H^{\mathrm{O}}(\Omega)}^{r_{\lim}} \rho^{a}(r)\, r^{n+2}\, \mathrm{d}r = \rho_{o}^{a} \int_{r=R+H^{\mathrm{O}}(\Omega)}^{r_{\lim}} \left(\frac{R}{r}\right)^{\nu} r^{n+2}\, \mathrm{d}r\,, \tag{7.12}$$

the reference gravitational potential $V_{\mathrm{ref}}^{a}(r, \Omega)$ of the atmospheric masses can be written as (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}, r > r_{\lim}, \nu > 2 \wedge \nu \in \Im^{+} :$$

$$V_{\mathrm{ref}}^{a}(r, \Omega) \cong G R^{\nu} \rho_{o}^{a} \sum_{n=0}^{\bar{n}} \frac{1}{r^{n+1}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \left| \frac{r'^{\,n-\nu+3}}{n-\nu+3} \right|_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}} \mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega, \Omega')\right) \mathrm{d}\Omega' \cong$$

$$\cong G R^{\nu} \rho_{o}^{a} \sum_{n=0}^{\bar{n}} \frac{1}{r^{n+1}} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{R^{n-\nu+3}}{n-\nu+3} \sum_{k=1}^{n-\nu+3} \binom{n-\nu+3}{k} \frac{(r_{\lim} - R)^{k} - \left[H^{\mathrm{O}}(\Omega')\right]^{k}}{R^{k}}\, \mathrm{P}_{\mathrm{n}}\left(\cos\psi(\Omega, \Omega')\right) \mathrm{d}\Omega'\,. \tag{7.13}$$

Applying the binomial theorem to the evaluation of the surface atmospheric density $\sigma^{a}(\Omega)$, see (Novák, 2000),

$$\forall\, \Omega \in \Omega_{\mathrm{O}}, \nu > 2 \wedge \nu \in \Im^{+} :$$

$$\sigma^{a}(\Omega) = \frac{\rho_{o}^{a}}{R^{2}} \int_{r=R+H^{\mathrm{O}}(\Omega)}^{r_{\lim}} \left(\frac{R}{r}\right)^{\nu} r^{2}\, \mathrm{d}r =$$

$$= \rho_{o}^{a} \frac{R^{3-\nu}}{3-\nu} \sum_{k=1}^{3-\nu} \binom{3-\nu}{k} \frac{(r_{\lim} - R)^{k} - \left[H^{\mathrm{O}}(\Omega)\right]^{k}}{R^{k}}\,, \tag{7.14}$$

the „*reference gravitational potential of condensed atmospheric masses*" $V^{ca}_{\mathrm{ref}}(r,\Omega)$ takes the following form (Novák, 2000)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\, r > r_{\lim},\, \nu > 2 \wedge \nu \in \Im^{+}:$$

$$V^{ca}_{\mathrm{ref}}(r,\Omega) \cong \frac{G\rho^{a}_{o}}{R^{1-\nu}} \sum_{n=0}^{\bar{n}} \left(\frac{R}{r}\right)^{n+1} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \left| \frac{r'^{3-\nu}}{3-\nu} \right|_{r'=R+H^{\mathrm{O}}(\Omega')}^{r_{\lim}} \mathrm{P_n}(\cos\psi(\Omega,\Omega'))\, \mathrm{d}\Omega' =$$

$$= \frac{G\rho^{a}_{o}}{R^{1-\nu}} \sum_{n=0}^{\bar{n}} \left(\frac{R}{r}\right)^{n+1} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \frac{R^{3-\nu}}{3-\nu} \sum_{k=1}^{3-\nu} \binom{3-\nu}{k} \frac{(r_{\lim}-R)^{k} - \left[H^{\mathrm{O}}(\Omega')\right]^{k}}{R^{k}} \mathrm{P_n}(\cos\psi(\Omega,\Omega'))\, \mathrm{d}\Omega'. \qquad (7.15)$$

The „*reference residual gravitational potential of atmospheric masses*" $\delta V^{a}_{\mathrm{ref}}(r,\Omega)$ is then obtained as the difference of the reference gravitational potential $V^{a}_{\mathrm{ref}}(r,\Omega)$ of atmospheric masses, see Eqn. (7.13), and the reference gravitational potential $V^{ca}_{\mathrm{ref}}(r,\Omega)$ of condensed atmospheric masses, see Eqn. (7.15) and (Novák, 2000),

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\, r > r_{\lim},\, \nu > 2 \wedge \nu \in \Im^{+}:$$

$$\delta V^{a}_{\mathrm{ref}}(r,\Omega) \cong -2\pi\, G\rho^{a}_{o} \sum_{n=1}^{\bar{n}} \frac{n}{2n+1} \left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{H}^{2}_{\mathrm{n,m}}(\Omega)\, \mathrm{Y_{n,m}}(\Omega) -$$

$$- 2\pi\, G \frac{\rho^{a}_{o}}{3R} \sum_{n=1}^{\bar{n}} \frac{n(n-2\nu+3)}{2n+1} \left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{H}^{3}_{\mathrm{n,m}}(\Omega)\, \mathrm{Y_{n,m}}(\Omega). \qquad (7.16)$$

7.3 Reference gravity potential in the Helmert space

The reference gravity potential $W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega)$ in the Helmert space in Eqn. (7.2) can be expressed by the following formula (Vaníček et al., 1995)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\, r > R: W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega) = \frac{GM}{r} - \sum_{n=2}^{\bar{n}} \left(\frac{a_o}{r}\right)^{n+1} \sum_{m=-n}^{n} \mathrm{W}^{\mathrm{H}}_{\mathrm{n,m}}\ \mathrm{Y_{n,m}}(\Omega). \qquad (7.17)$$

Since the summation in the expansion of Helmert's reference gravity potential $W^{\mathrm{H}}_{\mathrm{ref}}(r,\Omega)$ is finite, i.e., the validity of this expression is not limited to the outside of the Brillouin sphere (in the case of the topographical effect) and of the upper limit of atmosphere (in the case of the atmospheric effect), the series in Eqn. (7.17) can be used at the geoid to evaluate the reference gravity field in the Helmert space (Vaníček et al., 1995). If this surface is unknown, the appropriate approximation of the geoid by the reference ellipsoid ($\forall\ \Omega \in \Omega_{\mathrm{O}}$: $r_g(\Omega) \approx r_o(\Omega)$) can be applied (Vaníček et al., 1995)

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : r_g(\Omega) \approx r_o(\Omega) \cong a\left(1 - f \sin^2 \varphi\right). \tag{7.18}$$

Substituting the term (Vaníček et al., 1995)

$$\forall\ \ \Omega \in \Omega_{\mathrm{O}},\ n = 1, 2, ..., \bar{n} : \left[\frac{a_o}{r_g(\Omega)}\right]^{n+1} = 1 + (n+1) f \sin^2 \varphi - ..., \tag{7.19}$$

into Eqn. (7.17), „*Helmert's reference gravity potential*“ in the ellipsoidal approximation takes the following form (Vaníček et al., 1995)

$$\forall\ \Omega \in \Omega_{\mathrm{O}},\ r > R : W_{\mathrm{ref}}^{\mathrm{H}}\left(r_g(\Omega)\right) \approx \frac{GM}{r_o(\Omega)} - \sum_{n=2}^{\bar{n}} \left[1 + (n+1) f \sin^2 \varphi\right] \sum_{m=-n}^{n} \mathrm{W}_{\mathrm{n,m}}^{\mathrm{H}}\ \mathrm{Y}_{\mathrm{n,m}}(\Omega). \tag{7.20}$$

7.4 Reference gravity anomaly and reference spheroid in the Helmert space

According to the boundary condition (Heiskanen and Moritz, 1967), „*Helmert's reference gravity anomaly*“ can be expressed as follows

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : \Delta g_{\mathrm{ref}}^{\mathrm{H}}\left(r_g(\Omega)\right) \approx - \left.\frac{\partial T_{\mathrm{ref}}^{\mathrm{H}}(r, \Omega)}{\partial\, r}\right|_{r = r_o(\Omega)} + \frac{2}{R} T_{\mathrm{ref}}^{\mathrm{H}}\left(r_o(\Omega)\right), \tag{7.21}$$

where $T_{\mathrm{ref}}^{\mathrm{H}}\left(r_g(\Omega)\right) \cong T_{\mathrm{ref}}^{\mathrm{H}}\left(r_o(\Omega)\right) = W_{\mathrm{ref}}^{\mathrm{H}}\left(r_g(\Omega)\right) - U_o(\phi)$ is „*Helmert's reference disturbing potential*“. The „*reference spheroid*“ is given by the reference co-geoidal heights $N_{\mathrm{ref}}^{\mathrm{H}}(\Omega)$. Applying Bruns's spherical formula (Bruns, 1878) to Helmert's reference disturbing potential $T_{\mathrm{ref}}^{\mathrm{H}}\left(r_g(\Omega)\right)$, the reference co-geoidal height $N_{\mathrm{ref}}^{\mathrm{H}}(\Omega)$ can be expressed by the equation

$$\forall\ \Omega \in \Omega_{\mathrm{O}} : N_{\mathrm{ref}}^{\mathrm{H}}(\Omega) = \frac{T_{\mathrm{ref}}^{\mathrm{H}}\left(r_g(\Omega)\right)}{\gamma_o(\phi)}. \tag{7.22}$$

8. Stokes's boundary-value problem in the Helmert space

The equipotential boundary surface in the Helmert space, which is given by co-geoidal heights $N^{\mathrm{H}}(\Omega)$, can be evaluated from Helmert's gravity anomalies $\Delta g^{\mathrm{H}}(R, \Omega)$ referred to the reference sphere of radius R by applying the Stokes integral formula (Stokes, 1849) and the Bruns spherical formula (Bruns, 1878) into the following equation (Heiskanen and Moritz, 1967)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}(\Omega) = \frac{R}{4\pi \gamma_{\mathrm{o}}(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \Delta g^{\mathrm{H}}(R, \Omega')\, \mathrm{S}(\psi(\Omega, \Omega'))\, \mathrm{d}\Omega' . \tag{8.1}$$

The homogenous spherical Stokes function $\mathrm{S}(\psi(\Omega,\Omega'))$, see (Stokes, 1849), can be described in the following spectral and spatial form (Heiskanen and Moritz, 1967)

$\forall\, \Omega, \Omega' \in \Omega_{\mathrm{O}}$:

$$\mathrm{S}(\psi(\Omega,\Omega')) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1}\, \mathrm{P_n}(\cos\psi(\Omega,\Omega')) = 1 + \operatorname{cosec}\frac{\psi(\Omega,\Omega')}{2} - 6\sin\frac{\psi(\Omega,\Omega')}{2} -$$

$$- 5\cos\psi(\Omega,\Omega') - 3\cos\psi(\Omega,\Omega')\ln\left(\sin\frac{\psi(\Omega,\Omega')}{2} + \sin^2\frac{\psi(\Omega,\Omega')}{2}\right). \tag{8.2}$$

To evaluate the co-geoidal height $N^{\mathrm{H}}(\Omega)$ by a surface integration according to the Stokes integral in Eqn. (8.1), the gravity anomalies $\Delta g^{\mathrm{H}}(R,\Omega)$ have to be known over the entire Earth.

8.1 Spheroidal Stokes's function

In practice, the gravity anomalies over the entire Earth are not available. For this reason Vaníček and Kleusberg (1987) introduced the idea to separate the summation over n in the Stokes function in Eqn. (8.2) into low and high-degree parts:

$$\forall\, \Omega, \Omega' \in \Omega_{\mathrm{O}} : \mathrm{S}(\psi(\Omega,\Omega')) = \sum_{n=2}^{\bar{n}} \frac{2n+1}{n-1}\, \mathrm{P_n}(\cos\psi(\Omega,\Omega')) + \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{n-1}\, \mathrm{P_n}(\cos\psi(\Omega,\Omega')). \tag{8.3}$$

The second term on the right-hand side of Eqn. (8.3) represents the „*spheroidal Stokes function*“ $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega'))$, see (Vaníček and Kleusberg, 1987; Vaníček and Featherstone, 1998),

$$\forall\, \Omega, \Omega' \in \Omega_{\mathrm{O}} : \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi(\Omega,\Omega')) = \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{n-1}\, \mathrm{P_n}(\cos\psi(\Omega,\Omega')). \tag{8.4}$$

Substituting the decomposition of the Stokes spherical function $\mathrm{S}(\psi(\Omega,\Omega'))$ into Eqn. (8.1), the co-geoid can be split into the low and high-frequency part (Martinec, 1993)

$\forall\, \Omega \in \Omega_{\mathrm{O}}$:

$$N^{\mathrm{H}}(\Omega) = N^{\mathrm{H}}_{\mathrm{ref}}(\Omega) + N^{\mathrm{H}}_{n>\bar{n}}(\Omega) = \frac{R}{4\pi\, \gamma_{\mathrm{o}}(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \Delta g^{\mathrm{H}}(R,\Omega') \sum_{n=2}^{\bar{n}} \frac{2n+1}{n-1}\, \mathrm{P_n}(\cos\psi(\Omega,\Omega'))\, \mathrm{d}\Omega' +$$

$$+ \frac{R}{4\pi\, \gamma_{\mathrm{o}}(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}} \Delta g^{\mathrm{H}}(R,\Omega') \sum_{n=\bar{n}+1}^{\infty} \frac{2n+1}{n-1}\, \mathrm{P_n}(\cos\psi(\Omega,\Omega'))\, \mathrm{d}\Omega' . \tag{8.5}$$

The reference co-geoid (spheroid) of degree $\bar{n}$ is given by the reference co-geoidal heights

$N^{\mathrm{H}}_{\mathrm{ref}}(\Omega)$, and $N^{\mathrm{H}}_{n>\bar{n}}(\Omega)$ represents the high-frequency part of the co-geoid (Novák et al., 2001). According to this approach the reference spheroid determined from the satellite data is assumed (Vaníček and Kleusberg, 1987). The surface integration by the Stokes integral formula can be employed to compute the high-frequency part of the co-geoid only from terrestrial data.

8.2 Modified spheroidal Stokes's function

Values of the spheroidal Stokes function $\mathrm{S}_{\mathrm{n>\bar{n}}}(\psi(\Omega,\Omega'))$ diminish with the growing spherical distance $\psi(\Omega,\Omega')$. The integration domain Ω_{O} of Stokes's integral formula can be divided into the near-zone integration sub-domain $\Omega_{\psi_{\mathrm{o}}}$ (defined on the interval $\psi \in \langle 0,\psi_o \rangle$) and the far-zone integration sub-domain $\Omega_{\mathrm{O}} - \Omega_{\psi_{\mathrm{o}}}$ (on the interval $\psi \in \langle \psi_o, \pi \rangle$), see (Vaníček and Kleusberg, 1987):

$$\iint\limits_{\Omega\in\Omega_{\mathrm{O}}} \mathrm{d}\Omega = \iint\limits_{\Omega\in\Omega_{\psi_{\mathrm{o}}}} \mathrm{d}\Omega + \iint\limits_{\Omega\in\Omega_{\mathrm{O}}-\Omega_{\psi_{\mathrm{o}}}} \mathrm{d}\Omega \;. \tag{8.6}$$

The near-zone contribution to the high-frequency co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_{\mathrm{o}}}}(\Omega)$ is (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} :\, N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_{\mathrm{o}}}}(\Omega) = \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint\limits_{\Omega'\in\Omega_{\psi_{\mathrm{o}}}} \Delta g^{\mathrm{H}}(R,\Omega')\, \mathrm{S}_{\mathrm{n>\bar{n}}}(\psi(\Omega,\Omega'))\, \mathrm{d}\Omega' , \tag{8.7}$$

and far-zone contribution to the high-frequency co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega)$ is given by (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} :\, N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega) = \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint\limits_{\Omega'\in\Omega_{\mathrm{O}}-\Omega_{\psi_{\mathrm{o}}}} \Delta g^{\mathrm{H}}(R,\Omega')\, \mathrm{S}_{\mathrm{n>\bar{n}}}(\psi(\Omega,\Omega'))\, \mathrm{d}\Omega' . \tag{8.8}$$

According to Molodensky et al. (1960), Vaníček and Kleusberg (1987) proposed to modify the spheroidal Stokes function $\mathrm{S}_{\mathrm{n>\bar{n}}}(\psi(\Omega,\Omega'))$ so that the far-zone contribution (truncation error) $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega)$ is minimal in the least-squares sense. The „*modified spheroidal Stokes's function*" $\mathrm{S}_{\mathrm{n>\bar{n}}}(\psi_o,\psi(\Omega,\Omega'))$ can be expressed as (Vaníček and Kleusberg, 1987)

$$\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)=\begin{cases}0, & \psi\in\langle 0,\psi_o\rangle,\\ \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi(\Omega,\Omega')\right), & \psi\in\langle\psi_o,\pi\rangle,\end{cases}\tag{8.9}$$

and then expanded into the series of Legendre polynomials

$$\forall\,\psi\in\langle 0,\,\pi\rangle:\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)=\sum_{n=\bar{n}+1}^{\infty}\frac{2n+1}{2}\mathrm{Q}_\mathrm{n}\left(\psi_o,\psi(\Omega,\Omega')\right)\mathrm{P}_\mathrm{n}\left(\cos\psi(\Omega,\Omega')\right),\tag{8.10}$$

where $\mathrm{Q}_\mathrm{n}\left(\psi_o,\psi(\Omega,\Omega')\right)$ are „*truncation coefficients for the modified spheroidal Stokes function*“ $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)$, see (Molodensky et al., 1960). Multiplying Eqn. (8.10) by the Legendre polynomials $\mathrm{P}_\mathrm{m}\left(\cos\psi(\Omega,\Omega')\right)$,

$$\begin{aligned}&\forall\,\psi\in\langle 0,\,\pi\rangle:\\&\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)\mathrm{P}_\mathrm{m}\left(\cos\psi(\Omega,\Omega')\right)=\\&=\sum_{n=\bar{n}+1}^{\infty}\frac{2n+1}{2}\mathrm{Q}_\mathrm{n}\left(\psi_o,\psi(\Omega,\Omega')\right)\mathrm{P}_\mathrm{n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{P}_\mathrm{m}\left(\cos\psi(\Omega,\Omega')\right),\end{aligned}\tag{8.11}$$

and integrating the result over the interval $\psi\in\langle 0,\,\pi\rangle$, the following expression can be found

$$\begin{aligned}&\int_{\psi=0}^{\pi}\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)\mathrm{P}_\mathrm{m}\left(\cos\psi(\Omega,\Omega')\right)\sin\psi(\Omega,\Omega')\,\mathrm{d}\psi=\\&=\sum_{n=\bar{n}+1}^{\infty}\frac{2n+1}{2}\mathrm{Q}_\mathrm{n}\left(\psi_o,\psi(\Omega,\Omega')\right)\int_{\psi=0}^{\pi}\mathrm{P}_\mathrm{n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{P}_\mathrm{m}\left(\cos\psi(\Omega,\Omega')\right)\sin\psi(\Omega,\Omega')\,\mathrm{d}\psi\,.\end{aligned}\tag{8.12}$$

Using the orthogonality property of the Legendre polynomials (Hobson, 1931)

$$\forall\,\psi\in\langle 0,\,\pi\rangle,\,n\neq m:\int_{\psi=0}^{\pi}\mathrm{P}_\mathrm{n}\left(\cos\psi(\Omega,\Omega')\right)\mathrm{P}_\mathrm{m}\left(\cos\psi(\Omega,\Omega')\right)\sin\psi(\Omega,\Omega')\,\mathrm{d}\psi=0\,,\tag{8.13}$$

$$\forall\,\psi\in\langle 0,\,\pi\rangle,\,n=m:\int_{\psi=0}^{\pi}\left[\mathrm{P}_\mathrm{n}\left(\cos\psi(\Omega,\Omega')\right)\right]^2\sin\psi(\Omega,\Omega')\,\mathrm{d}\psi=\frac{2}{2n+1}\,,\tag{8.14}$$

and substituting for $\mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}\left(\psi_o,\psi(\Omega,\Omega')\right)$ from Eqn. (8.9), the truncation coefficients $\mathrm{Q}_\mathrm{n}\left(\psi_o,\psi(\Omega,\Omega')\right)$ of the modified spheroidal Stokes function become (Molodensky et al., 1960)

$\forall\,\Omega\in\Omega_\mathrm{O}$:

$$\mathrm{Q_n}\left(\psi_o, \psi(\Omega,\Omega')\right) = \int_{\psi=0}^{\pi} \mathrm{S_{n>\bar{n}}}\left(\psi_o, \psi(\Omega,\Omega')\right) \mathrm{P_n}\left(\cos\psi(\Omega,\Omega')\right) \sin\psi(\Omega,\Omega')\, \mathrm{d}\psi \,. \tag{8.15}$$

8.3 Near-zone contribution to the high-frequency co-geoid

Helmert's gravity anomaly referred to the co-geoid can be divided into the low-frequency (reference) gravity anomaly $\Delta g^{\mathrm{H}}_{n<\bar{n}}(R,\Omega) \equiv \Delta g^{\mathrm{H}}_{\mathrm{ref}}(R,\Omega)$ and the high-frequency (residual) gravity anomaly $\Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega)$. The low-frequency Helmert's gravity anomalies $\Delta g^{\mathrm{H}}_{\mathrm{ref}}(R,\Omega)$ are evaluated according to Eqn. (7.21). The high-frequency Helmert's gravity anomalies $\Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega)$ are evaluated by subtracting the reference gravity anomalies $\Delta g^{\mathrm{H}}_{\mathrm{ref}}(R,\Omega)$ from Helmert's gravity anomalies downward continued onto the co-geoid according to Eqn. (6.5).

Taking Eqn. (8.6) into the account, the near-zone contribution of the high-frequency Helmert gravity anomalies to the co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_o}(\Omega)$ can be described by (Novák, 2000)

$$\forall\, \Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_o}}(\Omega) = \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega' \in \Omega_{\psi_o}} \Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega')\, \mathrm{S_{n>\bar{n}}}\left(\psi_o, \psi(\Omega,\Omega')\right) \mathrm{d}\Omega' \,. \tag{8.16}$$

The Stokes integral is only weakly singular for the spherical distance $\psi = 0$ (Martinec, 1993). A classical method for treating a removable singularity consists of adding and subtracting the value of gravity anomaly at the singular point, see (Martinec, 1993),

$$\begin{aligned} &\forall\, \Omega \in \Omega_{\mathrm{O}} : \\ &N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\psi_o}}(\Omega) = \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega' \in \Omega_{\psi_o}} \left[\, \Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega') - \Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega) \,\right] \mathrm{S_{n>\bar{n}}}\left(\psi_o, \psi(\Omega,\Omega')\right) \mathrm{d}\Omega' + \\ &+ \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega) \iint_{\Omega' \in \Omega_{\psi_o}} \mathrm{S_{n>\bar{n}}}\left(\psi_o, \psi(\Omega,\Omega')\right) \mathrm{d}\Omega' \,. \end{aligned} \tag{8.17}$$

8.4 Far-zone contribution to the high-frequency co-geoid

The far-zone contribution of high-frequency Helmert's gravity anomalies $\Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega)$ to the co-geoidal height $N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_o}}(\Omega)$ is given by

$$\forall\,\Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega) = \frac{R}{4\pi\gamma_{\mathrm{o}}(\phi)} \iint_{\Omega' \in \Omega_{\mathrm{O}}-\Omega_{\psi_{\mathrm{o}}}} \Delta g^{\mathrm{H}}_{n>\bar{n}}(R,\Omega')\, \mathrm{S}_{\mathrm{n}>\bar{\mathrm{n}}}(\psi_o, \psi(\Omega,\Omega'))\, \mathrm{d}\Omega' . \tag{8.18}$$

If gravity anomalies are not available over the entire Earth, the numerical computation can be done by using the following equation (Novák, 2000)

$$\forall\,\Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}_{n>\bar{n},\Omega'_{\mathrm{O}}-\Omega'_{\psi_{\mathrm{o}}}}(\Omega) = \frac{R}{2} \sum_{n=\bar{n}+1,\ldots} \mathrm{Q}_{\mathrm{n}}(\psi_o, \psi(\Omega,\Omega')) \sum_{m=-n}^{n} \mathrm{T}^{\mathrm{H}}_{\mathrm{n,m}}\ \mathrm{Y}_{\mathrm{n,m}}(\Omega). \tag{8.19}$$

9. Primary indirect effect on the geoidal height

After evaluation of the Stokes boundary-value problem in the Helmert space, an equipotential surface in the Helmert space, i.e., the co-geoid, is obtained. To find the geoid in the real space, the primary indirect topographical and atmospheric effects on the geoidal height have to be evaluated (Vaníček and Martinec, 1994b). Helmert's disturbing gravity potential referred on the co-geoid (in the spherical approximation) reads

$$\forall\,\Omega \in \Omega_{\mathrm{O}} : T^{\mathrm{H}}(R,\Omega) = T(R,\Omega) - \delta V^{t}(R,\Omega) - \delta V^{a}(R,\Omega). \tag{9.1}$$

Applying Bruns's spherical formula (Bruns, 1878) to the disturbing gravity potential $T(R,\Omega)$ and Helmert's disturbing gravity potential $T^{\mathrm{H}}(R,\Omega)$:

$$\forall\,\Omega \in \Omega_{\mathrm{O}} : N(\Omega) = \frac{T(R,\Omega)}{\gamma_o(\phi)}, \tag{9.2}$$

$$\forall\,\Omega \in \Omega_{\mathrm{O}} : N^{\mathrm{H}}(\Omega) = \frac{T^{\mathrm{H}}(R,\Omega)}{\gamma_o(\phi)} = \frac{T(R,\Omega) - \delta V^{t}(R,\Omega) - \delta V^{a}(R,\Omega)}{\gamma_o(\phi)}, \tag{9.3}$$

the following relation between the geoidal height $N(\Omega)$ and the co-geoidal height $N^{\mathrm{H}}(\Omega)$ can be found (Martinec, 1993)

$\forall\,\Omega \in \Omega_{\mathrm{O}}$:

$$\delta N(\Omega) = N(\Omega) - N^{\mathrm{H}}(\Omega) = \frac{T(R,\Omega)}{\gamma_o(\phi)} - \frac{T^{\mathrm{H}}(R,\Omega)}{\gamma_o(\phi)} = \frac{\delta V^{t}(R,\Omega)}{\gamma_o(\phi)} + \frac{\delta V^{a}(R,\Omega)}{\gamma_o(\phi)}. \tag{9.4}$$

The first term on the right-hand side of Eqn. (9.4), i.e., $\delta V^{t}(R,\Omega)/\gamma_o(\phi)$, is the „*primary indirect topographical effect on the geoidal height*", and the second term $\delta V^{a}(R,\Omega)/\gamma_o(\phi)$ stands for the „*primary indirect atmospheric effect on the geoidal height*".

9.1 Primary indirect topographical effect

Considering the decomposition of the laterally varying topographical density $\rho(\Omega)$ into the mean and laterally varying anomalous topographical density, as described by Eqn. (4.24), and removing the weak singularity of Newton's integral, the gravitational potential $V^t(R,\Omega)$ of topographical masses (stipulated as being on the geoid) can be written as follows (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}:$$

$$V^t(R,\Omega) = 4\pi G\rho_o H^{\mathrm{O}}(\Omega)\left[R + \frac{1}{2}H^{\mathrm{O}}(\Omega)\right] +$$
$$+ G\rho_o \iint_{\Omega'\in\Omega_{\mathrm{O}}} \rho_o \int_{r'=R+H^{\mathrm{O}}(\Omega)}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[R,\psi(\Omega,\Omega'),r'\right] r'^2 \,\mathrm{d}r'\,\mathrm{d}\Omega' +$$
$$+ G \iint_{\Omega'\in\Omega_{\mathrm{O}}} \delta\rho(\Omega') \int_{r'=R}^{R+H^{\mathrm{O}}(\Omega')} l^{-1}\left[R,\psi(\Omega,\Omega'),r'\right] r'^2 \,\mathrm{d}r'\,\mathrm{d}\Omega', \tag{9.5}$$

where the first term on the right-hand side is the gravitational potential of the spherical Bouguer shell referred to the geoid, see Eqn. (4.17).

Similarly, the gravitational potential $V^{ct}(R,\Omega)$ of condensed topographical masses referred on the geoid can be described as (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}:$$

$$V^{ct}(R,\Omega) = 4\pi G\rho_o \frac{r_t^3(\Omega) - R^3}{3R} + G \iint_{\Omega'\in\Omega_{\mathrm{O}}} \rho_o \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3} l^{-1}\left(R,\psi(\Omega,\Omega'),R\right)\mathrm{d}\Omega' +$$
$$+ G \iint_{\Omega'\in\Omega_{\mathrm{O}}} \delta\rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3}\; l^{-1}\left(R,\psi(\Omega,\Omega'),R\right)\mathrm{d}\Omega', \tag{9.6}$$

where the first term on the right-hand side represents the gravitational potential of the spherical condensation layer.

Substituting the gravitational potential $V^t(R,\Omega)$ of topographical masses in Eqn. (9.5) and the gravitational potential $V^{ct}(R,\Omega)$ of condensed topographical masses in Eqn. (9.6) into the residual gravitational potential of topographical masses $\delta V^t(R,\Omega)$, the primary indirect topographical effect on the geoidal height takes the following form (Martinec, 1993)

$$\forall\, \Omega \in \Omega_{\mathrm{O}}:$$

$$\frac{\delta V^{t}(R,\Omega)}{\gamma_o(\phi)} = \frac{G}{\gamma_o(\phi)} 4\pi\rho_o \left[R\,H^{O}(\Omega) + \frac{\left[H^{O}(\Omega)\right]^2}{2} - \frac{r_t^3(\Omega) - R^3}{3R} \right] +$$
$$+ \frac{G}{\gamma_o(\phi)} \rho_o \iint_{\Omega' \in \Omega_O} \int_{r'=R+H^{O}(\Omega)}^{R+H^{O}(\Omega')} l^{-1}\left[R, \psi(\Omega,\Omega'), r'\right] r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' -$$
$$- \frac{G}{\gamma_o(\phi)} \rho_o \iint_{\Omega' \in \Omega_O} \frac{r_t^3(\Omega') - r_t^3(\Omega)}{3} \, l^{-1}\left(R, \psi(\Omega,\Omega'), R\right) \mathrm{d}\Omega' +$$
$$+ \frac{G}{\gamma_o(\phi)} \iint_{\Omega' \in \Omega_O} \delta\rho(\Omega') \int_{r'=R}^{R+H^{O}(\Omega')} l^{-1}\left[R, \psi(\Omega,\Omega'), r'\right] r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' -$$
$$- \frac{G}{\gamma_o(\phi)} \iint_{\Omega' \in \Omega_O} \delta\rho(\Omega') \frac{r_t^3(\Omega') - R^3}{3} \, l^{-1}\left(R, \psi(\Omega,\Omega'), R\right) \mathrm{d}\Omega' . \tag{9.7}$$

9.2 Primary indirect atmospheric effect

The primary indirect atmospheric effect on the geoidal height can be described in the following basic form (Novák, 2000)

$$\forall\, \Omega \in \Omega_O :$$
$$\frac{\delta V^{a}(R,\Omega)}{\gamma_o(\phi)} = \frac{V^{a}(R,\Omega)}{\gamma_o(\phi)} - \frac{V^{ca}(R,\Omega)}{\gamma_o(\phi)} =$$
$$= \frac{G}{\gamma_o(\phi)} \iint_{\Omega' \in \Omega_O} \int_{r'=R+H^{O}(\Omega')}^{r_{\lim}} \rho^{a}(r') \, l^{-1}\left[R, \psi(\Omega,\Omega'), r'\right] r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' -$$
$$- \frac{G}{\gamma_o(\phi)} \iint_{\Omega' \in \Omega_O} \int_{r'=R+H^{O}(\Omega')}^{r_{\lim}} \rho^{a}(r') \, r'^2 \, \mathrm{d}r' \, l^{-1}\left(R, \psi(\Omega,\Omega'), R\right) \mathrm{d}\Omega' . \tag{9.8}$$

10. Conclusions

To solve the geodetic boundary-value problem in the Helmert space, mean values of Helmert’s gravity anomalies are evaluated on the Earth’s surface. In the UNB approach, the mean values are considered for 5’x 5’ cells. It follows from Eqn. (3.9) that the mean values of Helmert’s gravity anomalies are functions of mean values of the free-air gravity anomalies, see Eqn. (3.8), ellipsoidal correction to the gravity disturbance, see Eqn. (2.10), ellipsoidal correction for the spherical approximation, see Eqn. (2.17), direct topographical and atmospheric effects, see Eqns. (4.26) and (5.16), secondary indirect topographical and atmospheric effects, see Eqns. (4.30) and (5.18), and geoid-quasigeoid correction to the boundary-value problem given by the fifth term on the right-hand side of Eqn. (3.9).

In the case of the ellipsoidal corrections to the gravity disturbance and for the spherical approximation, discrete values of Helmert's gravity anomalies computed from the geopotential model at the mid-points of corresponding cells can be considered as mean values, because they are smooth and change the geoid only by a few centimeters (as it can be seen from numerical results shown by Vaníček et al. (1999)). Similarly, the direct atmospheric effect and secondary indirect topographical effects can be evaluated as discrete values in the regular grid of 5'x 5' (as it follows from the numerical results in Novák (2000)). The mean values of the geoid-quasigeoid correction to the boundary-value problem are sufficiently (with an error < 10 μgal) computed for the mean orthometric heights of corresponding cells. The secondary indirect atmospheric effect is negligible (Novák, 2000).

Since the free-air gravity anomalies are not suitable for interpolation (Heiskanen and Moritz, 1967), their mean values are computed from the mean complete Bouguer gravity anomalies by subtracting average values of the gravimetric terrain correction and the gravitational attraction of the Bouguer plate with the mean topographical density and mean orthometric height (Janák and Vaníček, 2002). Mean values of the complete Bouguer gravity anomalies are given by averaging a certain number of discrete values, which are predicted on the regular grid from the complete Bouguer gravity anomalies at the observation points.

Mean values of the direct topographical effect have to be averaged from a sufficient number of discrete values. The number of discrete values needed for the precise evaluation of mean values of the direct topographical effect depends on the terrain roughness. The relation between the terrain roughness and the number of discrete values was investigated (at the Canadian Rocky Mountains) by Janák et al. (2001). In some areas, hundreds of discrete values (for one cell of size 5'x 5') must be computed to obtain sufficient accuracy.

To compute the effects of topographical masses, integration is carried out over the laterally varying topographical densities. When the geoid is to be determined with high accuracy (< 1 cm), the effect of lake water must also be considered. Numerical values for the lake Superior showed that the correction to the geoidal height due to the direct topographical effect on gravitational attraction lies within –1.1 and 1.3 cm, and the correction to the primary indirect topographical effect on the geoidal height is within –0.2 and 0.0 cm (Martinec et al., 1995). On the other hand, the effect of the laterally varying anomalous topographical density can cause changes of the geoid up to 10 cm (in Canada), see (Martinec, 1993; Huang et al., 2001; Huang, 2002), so that at least the laterally varying model of topographical density has to be considered.

Solving Dirichlet's boundary-value problem, the mean Helmert gravity anomalies are downward continued to the geoid by applying the discrete Poisson integral equation, see Eqn. (6.3). The Fredholm integral equation of the first kind (generic form of Poisson's integral equation) is known to be an unstable problem due to the fact that a comparatively smooth gravity anomaly on the Earth's surface is used to obtain a rougher gravity anomaly on the geoid. Solving the downward continuation for the 5'x 5' grid of Helmert's gravity anomalies, the ill effect of the instability might be partly reduced. Heck (1993) realized that space without topography is more suitable for the downward continuation than the Helmert space. For this reason, only the effect of topographical masses on the gravitational attraction can be subtracted from the gravity anomalies on the earth surface. The gravitation attraction of condensed topographical masses is then added to gravity anomalies downward continued onto the geoid.

The reference gravity anomalies and the spheroid in the Helmert space are evaluated from the satellite geopotential coefficients up to degree 20 according to Eqns. (7.21) and (7.22).

To solve the Stokes boundary-value problem in the modification for higher than the second-degree reference field (Vaníček and Sjöberg, 1991), the Stokes integration is employed for numerical integration over the 6° spherical cap, see Eqn. (8.17). The far-zone contribution is evaluated from the combined geopotential model. Usually EGM-96 up to degree 120 of the geopotential coefficients (Novák, 2000) is used according to Eqn. (8.19).

To obtain the geoid, the co-geoid (given by the discrete co-geoidal heights) is finally transformed into the real space by evaluation of discrete values of the primary indirect topographical and atmospheric effects. The primary indirect topographical effect can be computed by Eqn. (9.7) while the primary indirect atmospheric effect given by Eqn. (9.8) can be considered constant (equal to –0.6 cm), see (Sjöberg, 1998; Novák, 2000).

Evaluating the topographical and atmospheric effects on the gravitational potential and attraction, the integration domain is split into the near and far-zone integration sub-domains, where the near zone can be given by the 3° spherical cap, i.e., $\psi \in \left\langle 0, 3^{\circ} \right\rangle$. The near-zone contributions are then evaluated by numerical integration over the sufficiently dense grid of heights from the digital terrain model (especially numerical integration of the topographical effect and condensed topographical effect requires high density of elevation data (1″ or 3″) at the intermediate area surrounding the computation point. The spectral forms of Newton's integrals for evaluation of the far-zone contributions from the global elevation model were formulated by Novák (2000).

The actual accuracy of geoid determination is limited first of all by accuracy and spatial distribution of terrestrial gravity observations and orthometric heights. Other important attributes are the correctness of theoretical formulation and accuracy of numerical solutions.

Main factors limiting the theory of geoid determination by the UNB approach are the approximation of the actual topographical density by the laterally varying topographical density, resolution of gravity data for the downward continuation and primary indirect topographical effect, and spherical approximation of the geoid in the case of evaluation of topographical effects.

Computing the topographical and atmospheric effects on the gravitational potential and attraction, the geoid is approximated by the reference sphere of the geocentric radius $R \approx r_g(\Omega)$. This approximation yields a relative error 3×10^{-3} at most which then causes errors of 6 mm at most in the geoidal heights (Martinec, 1993). Since the density distribution of topographical masses between the geoid and Earth's surface is not available, the errors of geoid determination from the approximation of actual topographical density $\rho(r,\Omega)$ by the laterally varying topographical density $\rho(\Omega)$ are difficult to predict. Considering that the effect of laterally varying anomalous topographical density can cause changes of the geoid up to 10 cm (Martinec, 1993; Huang et al., 2001; Huang, 2002), the vertical variation of topographical density may cause changes of the geoid at most a few centimeters.

The surface density $\sigma(\Omega)$ of condensed topographical masses in the definition, see Eqn. (4.8), is chosen according to the principle of mass-conservation condensation (Wichiencharoen, 1982; Martinec, 1993), i.e., the mass of the condensation layer is equal to the mass of lateral topographical masses. Under this assumption, the disturbing gravity potential $T^{\mathrm{H}}(r,\Omega)$ in the Helmert space has no spherical harmonic of degree zero but it contains spherical harmonics of the first degree (because the so-called Hörmander's condition is not satisfied, $\lim_{r \to \infty} r^2 T^{\mathrm{H}}(r,\Omega) \neq 0$). It means, that the centre of the Earth's masses is shifted from the origin of the co-ordinate system. The magnitude of this shift represents 2 cm at most in each co-ordinate component and can precisely be computed (Martinec, 1993). The accuracy of numerical solution mainly depends on the interpolation of free-air gravity anomalies, evaluation of the near-zone contribution to the direct topographical effect, and accuracy of the Poisson integral equation in the case of 5'x 5' data.

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